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On Subclass of Meromorphic Univalent Functions Defined by Multiplier Transformation

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ABSTRACT

In this paper, we will investigate and discuss a new class of meromorphic univalent functions defined by multiplier transformation which is R(c, ®, y, 〈), as well as study the coefficient estimates and growth theorems, and then another line in this work, upon to get the close under the convex linear combination.

KEYWORDS: meromorphic univalent function; multiplier transformation; subordination theorem; close under convex linear combination.

الخلاصة

في البحث ، سوف نتحرى ونناقش فئة جديدة من الدوال أحادية الشكل التي تم تحديدها بواسطة التحويل المضاعف وهو (R(c, ®, y, 〈 وكذلك دراسة تقديرات المعامل ونظريات النمو ، والخط الآخر في هذا العمل ، للحصول على تركيبة خطية تحديبة.

INTRODUCTION

Recently appeared more studied about meromorphic univalent functions defined on some types operators ,such as in 2011 Atshan and Joudah [1] studied the class of meromorphic univalent functions with some geometric properties have been got ,in 2018 Shabeeb [8] introduced a class S(〈,®), of meromorphic univalent functions defined by Ruschweyh derivative and got geometric some properties.

Let G be the class of functions of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k$$

$$f(z) = z^{-1} + \sum \square \sum \square \sum \square \qquad (1)$$

which are analytic and meromoorphic univalent in the perforated unit disc

$U^* = \{z \ \mathbb{C} : 0 < |z| < 1\}$.

Suppose R be a subclass of G of functions of the form.

$$f(z) = z^{-1} + \sum_{k-1}^{\infty} a_k z^k \quad f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \qquad (a_k \geq$$

0, k N) (2)

Definition/ A function f R be given by (2) then the class R(c, ®, y, 〈) is defined by

$$\left| \frac{\dfrac{z(L_1(c,\beta) \ \ f(z))'}{(L_1(c,\beta) f(z))} + 1}{(1+2y) - \dfrac{4zy(L_1(c,\beta) \ \ f(z))'}{(L_1(c,\beta) f(z))}} \right| < \alpha \qquad (3)$$

For 0<y<1, 0<〈<1

Several authors studied geometric properties of this function subclass for other classes, Mille [6]

Where L_1(c, ®) is a multiplier transformation studied by Kho and Kim [3], Liue and Srivastave[5]. And then:

$$L_1(c,B)\, f(z) = z^{-1} + \sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^c a_k z^k \;, \beta \geq 0, (4)$$

have different styles to introduce the written text. For example, the introduction of a Functional Specification consists of information that the whole document is yet to explain. If a user guide is written, the introduction is about the product. In a report, the introduction gives a summary about the report contents.

MAIN RESULTS.

<u>Theorem (2.1)/</u> If f R then f R(c, ®, y, 〈) if and only if.

$$\sum_{k=1}^{\infty} \left(\frac{k+\beta}{1+\beta}\right)^c \Big((k+1) + \alpha\big(-1+y(4k-2)\big)\Big) a_k \leq \alpha(1+6y) \ (5)$$

Where 0<y<1, 0<〈<1

The result is sharp for the function.

$$f(z) = z^{-1} + \frac{\alpha(1+6y)}{\left(\frac{k+\beta}{1+\beta}\right)^c \big((k+1)+a(-1+y(4k-2))\big)} z^k \quad (6)$$

Proof/ Suppose that the inequality's (5) holds true and |z| =1.Then from(3), We've got:

|z(L1(c, ®) f(z))' + (L1(c, ®) f(z))|

-〈|(1+2y) (L1(c, ®) f(z))- 4zy(L1(c, ®) f(z))'|

$$= \left|\sum_{k=1}^{\infty}(k+1) \left(\frac{k+\beta}{1+\beta}\right)^c a_k z^k\right|$$

$$-\alpha\left|(1+6y)z^{-1} - \sum_{k=1}^{\infty}(-1+y(4k-2))\left(\frac{k+\beta}{1+\beta}\right)^c a_k z^k\right| (7)$$

$$\leq \sum_{k=1}^{\infty} \quad (k+1)\left(\frac{k+\beta}{1+\beta}\right)^c a_k - \alpha(1+6y)$$

$$+\sum_{k=1}^{\infty} \quad \alpha(-1 \ +y(4k-2))\left(\frac{k+\beta}{1+\beta}\right)^c a_k$$

$$\sum_{k=1}^{\infty} \quad \left(\frac{k+\beta}{1+\beta}\right)^c ((k+1) \quad +\alpha(-1+y(4k-2)) -\alpha(1+6y) \leq 0$$

by hypothesis .Hence f R (c, ®, y, 〈) then from (3) we have

$$\left|\frac{z\,(L_1(c,\beta)f\ (z\,))' + 1(L_1(c,\beta)f\ (z\,))}{(1+2y\,)(L_1(c,\beta)f\ (z\,)) - 4zy\,(L_1(c,\beta)f\ (z\,))'}\right| < \alpha$$

Since Re(Z) ≤ |Z| .for all(z∈U*)

We have.

$$\mathrm{Re}\left[\frac{\sum_{k=1}^{\infty} \quad (k+1)\left(\frac{k+\beta}{1+\beta}\right)^c a_k z^k}{(1+6y)z^{-1} - \sum_{k=1}^{\infty} \quad (-1+y(4k-2))\left(\frac{k+\beta}{1+\beta}\right)^c a_k z^k}\right] \alpha, (8)$$

We opt for the value of z on the real axis so that ($\frac{z(L_1(c,\beta)\, f(\,z\,))'}{(L_1(c,\beta)\, f(\,z\,))}$ *is real*).when liquidating the denominator in (8) and letting Re Z → 1^- ,we obtain.

$$\sum_{k=1}^{\infty} (k+1) \left(\frac{k+\beta}{1+\beta}\right)^c a_k \leq -\sum_{k=1}^{\infty} \alpha(-1+y(4k-2)) \left(\frac{k+\beta}{1+\beta}\right)^c a_k + \alpha(1+6y)$$

We can write (8)

$$\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^c (k+1) + \alpha(-1+y(4k-2)))a_k \ \alpha(1+6y)$$

Eventually, sharpness follows, if we take.

$$f(z) = z^{-1} + \frac{\alpha(1+6y)}{\left(\frac{k+\beta}{1+\beta}\right)^c \big((k+1)+a(-1+y(4k-2))\big)} z^k$$

Corollary (2.1) / let f R(c, ®, y, 〈) then

$$a_k \leq \frac{\alpha(1+6y)}{\left(\frac{k+\beta}{1+\beta}\right)^c ((k+1)+\alpha \quad (-1+y(4k-2)))} \quad (9)$$

k=1, 2, …

Next we get the following distortion and growth theorems for the class R(c, ®, y, 〈).

<u>Theorem (2.2)/</u> Let f R(c, ®, y, 〈) then

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$$\frac{1}{r}-\frac{\alpha(1+6y)}{(2+\alpha(-1+2y))}r \le |L_1(c,\beta)f(z)|$$

$$\le \frac{1}{r}+\frac{\alpha(1+6y)}{(2+\alpha(-1+2y))}r \qquad (10)$$

(|z|= r<1)

The result is sharp of the next function.

$$f(z) \;= z^{-1}+\frac{\alpha(1+6y)}{(1)^{c}(2+\alpha(-1+2y))}z \qquad (11)$$

Proof/ Let f R(c, ®, y, 〈) then by theorem (2.1)

We get.

$$(1)^{C}(2+\alpha(-1+2y))\sum_{k=1}^{\infty}a_k \le \sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{C}((k+1)+\alpha(-1+y(4k-2))a_k$$ Or

$$\le \alpha(1+6y)$$

$$\sum_{k=1}^{\infty}a_k \le \frac{\alpha(1+6y)}{(1)^{c}(2+\alpha(-1+2y))} \qquad (12)$$

Hence

$$|L_1(c,\beta)f(z)| \le \frac{1}{|z|}+\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{c}a_k|z|^{k}$$

$$\le \frac{1}{|z|}+(1)^{c}|z|\sum_{k=1}^{\infty}a_k$$

$$\le \frac{1}{r}+\frac{\alpha(1+6y)}{(2+\alpha(-1+2y))}r \qquad (13)$$

Similarly

$$|L_1(c,\beta)f(z)| \ge \frac{1}{|z|}-\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{c}a_k|z|^{k}$$

$$\ge \frac{1}{|z|}-(1)^{c}|z|\sum_{k=1}^{\infty}a_k$$

$$\ge \frac{1}{r}-\frac{\alpha(1+6y)}{(2+\alpha(-1+2y))}r \qquad (14)$$

From (13) and (14), we get (10).

Theorem (2.3)/ Let f R(c, ®, y, 〈) then

$$\frac{1}{r^{2}}-\frac{\alpha(1+6y)}{(2+\alpha(-1+2y))} \le |L_1(c,\beta)f(z)'|$$

$$\le \frac{1}{r^{2}}+\frac{\alpha(1+6y)}{(2+\alpha(-1+2y))} \qquad (15)$$

(|z| = r<1)

The result is sharp of the next function:

$$f(z)=z^{-1}+\frac{\alpha(1+6y)}{(1)^{c}(2+\alpha(-1+2y))}z \qquad (16)$$

Proof/ Let f R(c, ®, y, 〈) then by theorem (2.1), we get

$$(1)^{C}(2+\alpha(-1+2y))\sum_{k=1}^{\infty}a_k \le \sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{C}((k+1)+\alpha(-1+y(4k-2))a_k$$ Or

$$\le \alpha(1+6y)$$

$$\sum_{k=1}^{\infty}a_k \le \frac{\alpha(1+6y)}{(1)^{c}(2+\alpha(-1+2y))} \qquad (17)$$

Hence

$$|L_1(c,\beta)f(z)'| \le \frac{1}{|z|^{2}}+\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{c}ka_k|z|^{k-1}$$

$$\le \frac{1}{|z|^{2}}+\;(1)^{c}\sum_{k=1}^{\infty}a_k$$

$$\le \frac{1}{r^{2}}+\frac{\alpha(1+6y)}{(2+\alpha(-1+2y))} \qquad (18)$$

Similarly

$$|L_1(c,\beta)f(z)'| \ge \frac{1}{|z|^{2}}-\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{c}ka_k|z|^{k-1}$$

$$\ge \frac{1}{|z|^{2}}\;-(1)^{c}\sum_{k=1}^{\infty}a_k$$

$$\ge \frac{1}{r^{2}}-\frac{\alpha(1+6y)}{(2+\alpha(-1+2y))} \qquad (19)$$

From (18) and (19), we get (15).

Theorem (2.4)/ The class R(c, ®, y, 〈) is close under convex linear combination.

Proof/ Let *f*1 and *f*2 belong to the class R(c, ®, y, 〈) for 0 ≤⌊ ≤ 1. We must clarify that.

⌊f1(z)+ (1-⌊)f2(z) R(c, ®, y, 〈)

And so we have :

$$z^{-1}+\sum_{k=1}^{\infty}[\lambda a_{k,1}+(1-\lambda)a_{k,2}]z^{k}$$

Then:

$$\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{c}((k+1)+\alpha(-1+y(4k-2)))(\lambda a_{k,1}+(1-\lambda)a_{k,2})$$

$$=\lambda\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{c}((k+1)+\alpha(-1+y(4k-2)))a_{k,1}$$

$$+(1-\lambda)\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1+\beta}\right)^{c}((k+1)+\alpha(-1+y(4k-2)))a_{k,2}$$

$$\leq\lambda\alpha(1+6y)+(1-\lambda)\alpha(1+6y)$$

$$=\alpha(1+6y)$$

Then by theorem (2.1) .We have h(z) R(c, ®, y, 〈).

Theorem (2.5)/ Let

$$fi(z)=z^{-1}+\sum_{k=1}^{\infty}a_{k,i}z^{k}\in R(c,\beta,y,\alpha),\, i\in\{1,2,...,j\}$$

and 0 < Si < 1

Such that $\sum_{i=1}^{j}S_i=1$

The function X defined

$$X=\sum_{i=1}^{j}S_i f_i(z)\in R(c,\beta,y,\alpha)$$

Proof/ By theorem (2.1) for every i {1,2,…,j}. We have

$$\sum_{k=1}^{\infty}\frac{\left(\frac{k+\beta}{1+\beta}\right)^{c}((k+1)+\alpha(-1+y(4k-2)))}{\alpha(1+6y)}a_{k,i}\leq 1$$

Since

$$X(z)=\sum_{i=1}^{j}S_i(z^{-1}+\sum_{k=1}^{\infty}a_{k,i}z^{k})$$

$$=z^{-1}+\sum_{k=1}^{\infty}\left(\sum_{i=1}^{j}S_i a_{k,i}\right)z^{k}$$

So: $$\sum_{k=1}^{\infty}\frac{\left(\frac{k+\beta}{1+\beta}\right)^{c}((k+1)+\alpha(-1+y(4k-2)))}{\alpha(1+6y)}\left(\sum_{i=1}^{j}S_i a_{k,i}\right)$$

$$=\sum_{i=1}^{j}S_i\left(\sum_{k=1}^{\infty}\frac{\left(\frac{k+\beta}{1+\beta}\right)^{c}((k+1)+\alpha(-1+y(4k-2)))}{\alpha(1+6y)}a_{k,i}\right)$$

$$\leq\sum_{i=1}^{j}S_i=1$$

Hence x R(c, ®, y, 〈).

Theorem (2.6)/ Let $f\in$ R(c, β, y, α) then f is univalent meromorphic convex of order θ (0 ≤ θ < 1) in the disc |z|<R.

Where:

$$R\ \ inf_k\left[\frac{\left(\frac{k+\beta}{1+\beta}\right)^{c}(1-\theta)(((k+1)+\alpha(-1+y(4k-2)))}{(k(k-\theta+2))(\alpha(1+6y))}\right]^{\frac{1}{k-1}}$$

Proof/ It is enough to show that.

$$\left|\frac{zf''(z)}{f(z)}+2\right|\leq 1-\theta \quad (20)$$

$$for\ |z|<R$$

But.

$$\left|\frac{zf''(z)}{f'(z)}+2\right|=\left|\frac{zf''(z)+2f'(z)}{f'(z)}\right|$$

$$\leq\frac{\sum_{k=1}^{\infty}\ k(k+1)a_k|z|^{k-1}}{1-\sum_{k=1}^{\infty}\ ka_k|z|^{k-1}}$$

by (20) we have.

$$\frac{\sum_{k=1}^{\infty}\ k(k+1)a_k|z|^{k-1}}{1-\sum_{k=1}^{\infty}\ ka_k|z|^{k-1}}\leq 1-\theta$$

Or

$$\sum_{k=1}^{\infty}\frac{k(k-\theta+2)}{1-\theta}a_k|z|^{k-1}\leq 1 \quad (21)$$

Since f ∈ R(c, β, y, α), We have.

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$$\sum_{k=1}^{\infty}\frac{\left(\frac{k+\beta}{1+\beta}\right)^{c}((k+1)+\alpha(-1+y(4k-2)))}{\alpha(1+6y)}$$

Hence (21) will be true if.

$$\frac{k(k-\theta+2)}{1-\theta}|z|^{k-1} \leq \frac{\left(\frac{k+\beta}{1+\beta}\right)^{c}((k+1)+\alpha(-1+y(4k-2))}{\alpha(1+6y)}$$

Or equivalently

$$\text{if.}|z| \leq \left[\frac{\left(\frac{k+\beta}{1+\beta}\right)^{c}(1-\theta)(((k+1)+\alpha(-1+y(4k-2)))}{(k(k-\theta+2))(\alpha(1+6y))}\right]^{\frac{1}{k-1}}$$

closed under the convex linear combination.

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