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The Formulas of Möbius-Bretschneider and Möbius-Cagnoli in the Poincaré Disc Model of Hyperbolic Geometry

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ABSTRACT

In this paper we present two gyroarea formulas (Möbius-Bretschneider's formula and Möbius-Cagnoli's formula) for Möbius gyroquadrilaterals in the Poincaré disc model of hyperbolic geometry.

KEYWORDS: Hyperbolic triangle; hyperbolic quadrilateral; hyperbolic Bretschneider's formula; hyperbolic Cagnoli's formula

INTRODUCTION

In Euclidean geometry, if the lengths of a triangle are known, then it is possible to calculate the area of the triangle with the Heron's formula, since the Heron's formula connects the area of the Euclidean triangle to its side lengths. Heron's formula states that the area of the triangle ABC whose sides have lengths a, b, c is

$$\Delta(ABC) = \sqrt{S(S-a)(S-b)(S-C)}$$

where S is the semi-perimeter of the triangle, that is $S = (a+b+c)/2$. Unlike other triangle area formulas, there is no need to calculate angles or other distances in the triangle first. Similar to the Heron's formula, the area of a cyclic quadrilateral can be found with its semi-perimeter and side lengths by Brahmagupta's formula. More precisely, the area of the cyclic quadrilateral $ABCD$ whose sides have lengths a, b, c, d is

$$\Omega(ABCD) = \sqrt{(S-a)(S-b)(S-c)(S-d)},$$

where S is the semi-perimeter of the quadrilateral, that is $S = (a+b+c+d)/2$. Similar to Heron's formula there is no need to calculate angles or other distances in the quadrilateral first. However, if the quadrilateral is not cyclic, then the side lengths are not sufficient to get the area of the quadrilateral. To calculate the area of a non-cylic quadrilateral, besides the side lengths of the quadrilateral, the sum of the angles in the two opposite vertices should be known. Carl Anton Bretschneider, the German mathematician who lived between 1808-1878 stated that the area of the quadrilateral $ABCD$ with side lengths a, b, c, d and opposite angles A, C is

$$\Omega(ABCD) = \sqrt{(S-a)(S-b)(S-c)(S-d) - abcdcos^2\left(\frac{A+C}{2}\right)},$$

or equivalently

$$\Omega(ABCD) = \sqrt{(S-a)(S-b)(S-c)(S-d) - abcdsin^2 K}$$

where S is the semi-perimeter of the quadrilateral and $K = (A-B+C-D)/4$. The counterparts of these formulas in hyperbolic geometry have been studied by some researchers [1]-[4]. In this paper we try to present the counterpart of the Bretschneider's formula in the Poincaré disc model of hyperbolic geometry. In our proofs, instead of classical hyperbolic distance, we use gyrodistance function which is not a metric.

THE FORMULAS OF MÖBIUS-BRETSCHNEIDER AND MÖBIUS-CAGNOLI IN MÖBIUS GYROVECTOR SPACE $(\mathbb{R}_s^2, \oplus, \otimes)$

Hyperbolic geometry is a non-Euclidean geometry that rejects the validity of Euclid's fifth postulate. Euclid's fifth postulate states that if two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. In a sense, Euclid's fifth postulate says that two parallel lines will never meet. Euclid's fifth postulate cannot be proven as a theorem, although this was attempted by many researchers. Euclid himself used only the first four postulates (absolute geometry) for the first 28 propositions of the Elements, but was forced to invoke the parallel postulate on the 29th. In 1823, Janos Bolyai and Nicolai Lobachevsky independently realized that entirely self-consistent "non- Euclidean geometries" could be created in which the parallel postulate did not hold. In hyperbolic geometry, through a point not on a given line there are at least two lines parallel to the given line. The principles of hyperbolic geometry, however, admit the other four Euclidean postulates. Although there are many common features between Euclidean geometry and hyperbolic geometry, both geometries have their own different features. It is well known that there are many principal hyperbolic geometry models, for instance Poincare´ upper-half plane model, Poincare´ disc model, Beltrami-Klein model, Weierstrass model, etc. In this paper we choose the Poincare´ disc model of hyperbolic geometry for our results. This model is defined on the complex unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The points of this model are the points of $\mathbb{D}$ and the lines (hyperbolic lines) of this model are circular arcs whose ends are perpendicular to the boundary of $\mathbb{D}$ (and diameters are also permitted).

Two arcs which do not meet correspond to parallel rays, arcs which meet orthogonally correspond to perpendicular lines, and arcs which meet on the boundary are a pair of limits rays. The angles between two hyperbolic lines are the usual Euclidean angles between Euclidean tangents to the circular arcs. In hyperbolic geometry, the angle sum of a hyperbolic triangle is less than π. More generally, the angle sum of an n-sided hyperbolic polygon is less than $(n-2)\pi$.

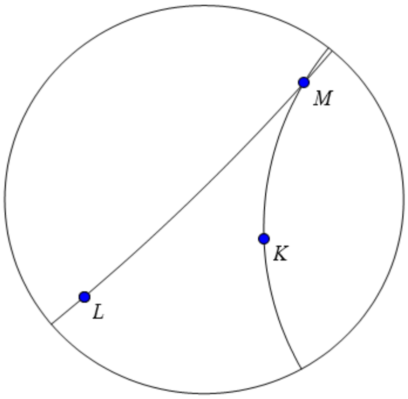

Figure 1. A hyperbolic line passing through the points K and M is a circular arc that intersect the disc $\mathbb{D}$ orthogonally. The hyperbolic lines passing through the center of disc are also correspond to chords of the disc.

The classical hyperbolic distance between $z, w \in \mathbb{D}$ is defined by

$$sinh \frac{d_H(z,w)}{2} = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)} \quad (1)$$

and the triangle inequality is provided. Hence $(\mathbb{D}, d_H)$ is a metric space. Unlike Euclidean geometry, the following famous theorem allows to find the area of a hyperbolic triangle whose angles are known.

Theorem 1. (Hyperbolic Girard's Theorem) Let ABC be a hyperbolic triangle with internal angles α, β and γ. Then the hyperbolic area of ABC is

$$\Gamma(ABC) = \pi - (\alpha + \beta + \gamma).$$

The hyperbolic area $\Gamma(A_1 A_2 \cdots A_n)$ of an n-sided hyperbolic polygon $A_1 A_2 \cdots A_n$ with internal hyperbolic angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ is

$$\Gamma(A_1 A_2 \cdots A_n) = (n-2)\pi - (\alpha_1 + \alpha_2 + \cdots + \alpha_n).$$

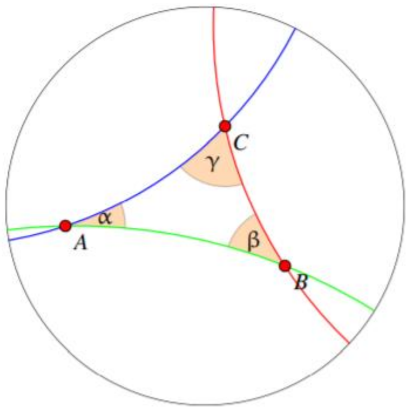

Figure 1. A hyperbolic triangle in the unit disc $\mathbb{D}$.

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Instead of three angles, the area of a hyperbolic triangle whose three sides are known can be found by the formula below.

Theorem 2. (Hyperbolic Heron's Formula, [1]). The hyperbolic area of the hyperbolic triangle ABC with side lengths a, b, c and semi-perimeter $S = (a+b+c)/2$ is given by

$$tan\frac{\Gamma(ABC)}{4} = \sqrt{tanh\frac{S}{2} sinh\frac{S-a}{2} sinh\frac{S-b}{2} sinh\frac{S-c}{2}}.$$

Instead of four angles, the area of a hyperbolic quadrilateral whose four sides are known can be found by the formula below.

Theorem 3. (Hyperbolic Bretschneider's Formula [3]). The hyperbolic area of the hyperbolic quadrilateral $ABCD$ with side lengths a, b, c, d, angles A, B, C, D and semi-perimeter $S = (a+b+c+d)/2$ is given by

$$sin^2\frac{\Gamma(ABCD)}{4} = \frac{sinh\frac{S-a}{2} sinh\frac{S-b}{2} sinh\frac{S-c}{2} sinh\frac{S-d}{2}}{cosh\frac{a}{2} cosh\frac{b}{2} cosh\frac{c}{2} cosh\frac{d}{2}} - tanh\frac{a}{2} tanh\frac{b}{2} tanh\frac{c}{2} tanh\frac{d}{2} sin^2 K$$

where $K = (A - B + C - D)/4$.

A Möbius transformation of the extended complex plane $\mathbb{C} \cup \{\infty\}$ is a rational function of the form

$$w = \frac{az+b}{cz+d}$$

of one complex variable z; here the coefficients a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. Möbius transformations are named in honor of August Ferdinand Möbius; they are also variously named homographic transformations, bilinear transformations or fractional linear transformations. The set of all Möbius transformations forms a group under composition. Möbius transformations preserve the measures of the angles with orientation. Euclidean transformations, Euclidean rotations, inversions $z \mapsto \frac{1}{z}$ and similarities $(z \mapsto az + b,\ a \neq 0)$ are well known Möbius transformations. The most general Möbius transformation of the complex unit disc $\mathbb{D}$ in the complex plane to itself

$$z \mapsto e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}(z_0 \oplus z)$$

defines the Möbius addition in the disc, which allows the Möbius transformation of the disc to be viewed as a Möbius left translation [4], [5]

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

where $\theta \in \mathbb{R}$, $z_0 \in \mathbb{D}$, and $\overline{z_0}$ is the complex conjugate of z_0. A left Möbius translation is also called a left gyrotranslation [4]. It is known that the Möbius addition "$\oplus$" is analogous to the common vector addition "+" in Euclidean plane geometry. Möbius addition $\oplus$ is neither commutative nor associative. By defining the gyrator

$$gyr : \mathbb{D} \times \mathbb{D} \to Aut(\mathbb{D}, \oplus),$$

$$gyr[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + b\bar{a}}$$

where $Aut(\mathbb{D}, \oplus)$ is the automorphism group of the Möbius groupoid $(\mathbb{D}, \oplus)$, the following group-like properties of $\mathbb{D}$ can be verified by straightforward algebra for all $a, b, c \in \mathbb{D}$:

G1. $a \oplus b = gyr[a,b](b \oplus a)$, (*Gyrocommutative Law*)
G2. $a \oplus (b \oplus c) = (a \oplus b)gyr[a,b]c$, (*Left Gyroassociative Law*)
G3. $(a \oplus b) \oplus c = a \oplus b\ gyr[b,a]c)$, (*Right Gyroassociative Law*)
G4. $gyr[a,b] = gyr[a \oplus b, b]$, (*Left Loop Property*)
G5. $gyr[a,b] = gyr[a, b \oplus a]$, (*Right Loop Property*)

The Möbius gyrodistance function in $\mathbb{D}$ is

$$d_M(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|$$

which is closely related to classical hyperbolic distance (1) as follows:

$$tanh\frac{d_H(z,w)}{2} = \left|\frac{z-w}{1-\bar{z}w}\right| = d_M(z,w) \qquad (2)$$

For more details, we refer to readers [4]. While the classical hyperbolic distance provides triangle inequality, the Möbius gyrodistance function does not provide; but the Möbius gyrodistance function satisfies the Möbius triangle inequality

$$d_M(a,c) = d_M(a,b) \oplus d_M(b,c)$$

for all $a,b,c \in \mathbb{D}$. Clearly $(\mathbb{D}, d_H)$ is a metric space but $(\mathbb{D}, d_M)$ is not. Identifying vectors in the Euclidean plane $\mathbb{R}^2$ with complex numbers in $\mathbb{C}$, we have

$$\mathbb{R}^2 \ni \boldsymbol{u} = (u_1, u_2) \leftrightarrow u_1 + iu_2 = u \in \mathbb{C}.$$

The inner product and the norm in $\mathbb{R}^2$ then become

$$\boldsymbol{uv} \leftrightarrow \frac{\bar{u}v + u\bar{v}}{2}, \quad \|\boldsymbol{u}\| \rightarrow |u|.$$

If the elements of the complex unit disc mapped to the elements of open unit disc $\mathbb{R}_1^2 = \{u \in \mathbb{R}^2 : \|u\| < 1\}$ then the Möbius addition $u \oplus v$ in $(\mathbb{D}, \oplus)$ takes the form

$$\boldsymbol{u} \oplus \boldsymbol{v} = \frac{(1 + 2\boldsymbol{uv} + \|\boldsymbol{v}\|^2)\boldsymbol{u} + (1 - \boldsymbol{v}^2)\boldsymbol{v}}{1 + 2\boldsymbol{uv} + \|\boldsymbol{v}\|^2\|\boldsymbol{u}\|^2}$$

In the open disc $\mathbb{R}_s^2 = \{u \in \mathbb{R}^2 : \|u\| < s\}$, the Möbius addition is defined by

$$\boldsymbol{u} \oplus \boldsymbol{v} = \frac{\left(1 + \frac{2}{s^2}\boldsymbol{uv} + \frac{1}{s^2}\|\boldsymbol{v}\|^2\right)\boldsymbol{u} + \left(1 - \frac{1}{s^2}\boldsymbol{v}^2\right)\boldsymbol{v}}{1 + \frac{2}{s^2}\boldsymbol{uv} + \frac{1}{s^4}\|\boldsymbol{v}\|^2\|\boldsymbol{u}\|^2}$$

In the limit of large s, $s \to \infty$, $\mathbb{R}_s^2$ expands to the whole its space $\mathbb{R}^2$ and Möbius addition $\oplus$ reduces the vector addition + in $\mathbb{R}^2$.

A Möbius gyroline in $(\mathbb{R}_s^2, \oplus)$ (also in $(\mathbb{D}, \oplus)$) that passes through the points a and b can be expreesed by

$$L = a \oplus (\ominus a \oplus b) \otimes t, \quad t \in R$$

where $\otimes$ is Möbius gyroscalar product defined by

$$r \otimes a = s \tanh(r \tanh^{-1}\left(\frac{\|a\|}{s}\right)\frac{a}{\|a\|}$$

$r \in \mathbb{R}$, $a \in \mathbb{R}_s^2$. In literature the gyroalgebraic structure $(\mathbb{R}_s^2, \oplus, \otimes)$ (and $(\mathbb{D}, \oplus, \otimes)$) is called as Möbius gyrovector space. Interestingly, a Möbius gyroline that passes through points a and b is actually the classical hyperbolic line that passes through points a and b. Naturally, a Möbius gyropolygon in $(\mathbb{R}_s^2, \oplus, \otimes)$ (and in $(\mathbb{D}, \oplus, \otimes)$) is also classical hyperbolic polygon in the Poincaré disk model of hyperbolic geometry with the same interior angles and same vertices. The gyroarea $\Delta(ABC)$ of a Möbius gyrotriangle ABC with gyroangles α, β, γ is

$$\Delta(ABC) = \frac{1}{2} \tan\left(\frac{\pi - (\alpha + \beta + \gamma)}{2}\right)$$

and the gyroarea $\Delta(ABCD)$ of a Möbius gyroquadrilateral $ABCD$ with gyroangles $\alpha, \beta, \gamma, \theta$ is

$$\Delta(ABCD) = \frac{1}{2} \tan\left(\frac{2\pi - (\alpha + \beta + \gamma + \theta)}{2}\right).$$

A.A. Ungar obtained the counterpart of Heron's formula as follows:

Theorem 4. ([4]). Let ABC be a gyrotriangle in $(\mathbb{R}_s^2, \oplus, \otimes)$ with vertices A, B, C corresponding gyroangles α, β, γ and side gyrolengths a, b, c. The gyroarea of ABC is given by Möbius-Heron's formula

$$\Delta(ABC) = \frac{s^2}{2} \frac{\sqrt{\left(\frac{a}{s}+\frac{b}{s}+\frac{c}{s}+\frac{abc}{sss}\right)\left(\frac{-a}{s}+\frac{b}{s}+\frac{c}{s}-\frac{abc}{sss}\right)\left(\frac{a}{s}-\frac{b}{s}+\frac{c}{s}-\frac{abc}{sss}\right)\left(\frac{a}{s}+\frac{b}{s}-\frac{c}{s}-\frac{abc}{sss}\right)}}{2+\left(\frac{a}{2}\right)^2\left(\frac{b}{2}\right)^2\left(\frac{c}{2}\right)^2-\left(\frac{a}{2}\right)^2-\left(\frac{b}{2}\right)^2-\left(\frac{c}{2}\right)^2}$$

Remarkably, in the Euclidean limit $s \to \infty$, when the open disc $\mathbb{R}_s^2$ expands to $\mathbb{R}^2$, Möbius-Heron's formula reduces to Heron formula of the triangle area in $\mathbb{R}^2$. Now we try to obtain Möbius-Bretschneider's formula in $(\mathbb{D}, \oplus, \otimes)$ with the help of the relation in (2). In addition to (2), there is a relation between the gyroarea $\Delta(ABCD)$ of the Möbius gyroquadrilateral $ABCD$ and the classical hyperbolic area $\Gamma(ABCD)$ of the same quadrilateral as follows:

$$\Delta(ABCD) = \frac{1}{2} \tan\frac{\Gamma(ABCD)}{2}. \qquad (3)$$

Assume $d_H(A,B) = a$, $d_H(B,C) = b$, $d_H(C,D) = c$, $d_H(D,A) = d$. Then, by (2), we get

$$a = 2\tanh^{-1} d_M(A,B), b = 2\tanh^{-1} d_M(B,C)$$

$$c = 2\tanh^{-1} d_M(C,D), d = 2\tanh^{-1} d_M(D,A).$$

Thus the semi-perimeter $S = (a + b + c + d)/2$ can be written by

$$S = \tanh^{-1}x + \tanh^{-1}y + \tanh^{-1}z + \tanh^{-1}w,$$

where $x = d_M(A,B)$, $y = d_M(B,C)$, $z = d_M(C,D)$, $w = d_M(D,A)$. Hence the hyperbolic Bretschneider's formula in *Theorem 3* which is

$$\sin^2\frac{\Gamma(ABCD)}{4}$$

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$$=\frac{\sinh\frac{S-a}{2}\sinh\frac{S-b}{2}\sinh\frac{S-c}{2}\sinh\frac{S-d}{2}}{\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}}$$
$$-\tanh\frac{a}{2}\tanh\frac{b}{2}\tanh\frac{c}{2}\tanh\frac{d}{2}\sin^2 K$$

takes the form

$$\sin^2\frac{\tan^{-1}(2\cdot\Delta(ABCD))}{2}=\frac{sinhS_x\cdot sinhS_y\cdot sinhS_z\cdot sinhS_w}{\cosh(\tanh^{-1}x)\cdot\cosh(\tanh^{-1}y)\cosh(\tanh^{-1}z)\cdot\cosh(\tanh^{-1}xw)}-xyzw\sin^2K,$$

where $S_x=S-\tanh^{-1}x,\ S_y=S-\tanh^{-1}y,\ S_z=S-\tanh^{-1}z,\ S_w=S-\tanh^{-1}w$. Thus we get

$$\sin\frac{\tan^{-1}(2\cdot\Delta(ABCD))}{2}=\sqrt{\frac{sinhS_x\cdot sinhS_y\cdot sinhS_z\cdot sinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-xyzw\sin^2K}$$

where, $\gamma_x=\frac{1}{\sqrt{1-x^2}}$, $\gamma_y=\frac{1}{\sqrt{1-y^2}}$, $\gamma_z=\frac{1}{\sqrt{1-z^2}}$, $\gamma_w=\frac{1}{\sqrt{1-w^2}}$ and this implies

$$\Delta(ABCD)=\frac{1}{2}\tan\left(2\sin^{-1}\sqrt{\frac{sinhS_x\cdot sinhS_y\cdot sinhS_z\cdot sinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-xyzw\sin^2K}\right)$$

Since

$\frac{1}{2}\tan(2\sin^{-1}A)=\frac{1}{2}\frac{2\tan(\sin^{-1}A)}{1-(\tan(\sin^{-1}A))^2}=\frac{A\sqrt{1-A^2}}{1-2A^2}$, we get

$$\Delta(ABCD)=\sqrt{\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-xyzw\sin^2K}\cdot\sqrt{1-\left(\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-xyzw\sin^2K\right)}\cdot\frac{1}{1-2\left(\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-xyzw\sin^2K\right)}.$$

Clearly,

$$sinhS_x=\sinh(\tanh^{-1}y+\tanh^{-1}z+\tanh^{-1}w)$$

and this yields

$$sinhS_x=\frac{(x+y+z+w)-x-wyz}{\sqrt{(1-yz)(1-w)-(y+z)(1+w)}\sqrt{(1-yz)(1+w)+(y+z)(1-w)}}$$

$$sinhS_y=\frac{(x+y+z+w)-y-xwz}{\sqrt{(1-zw)(1-x)-(z+w)(1+x)}\sqrt{(1-zw)(1+x)+(z+w)(1-x)}},$$

$$sinhS_z=\frac{(x+y+z+w)-z-xyw}{\sqrt{(1-wx)(1-y)-(w+x)(1+y)}\sqrt{(1-wx)(1+y)+(w+x)(1-y)}},$$

$$sinhS_w=\frac{(x+y+z+w)-w-xyz}{\sqrt{(1-xy)(1-z)-(x+y)(1+z)}\sqrt{(1-xy)(1+z)+(x+y)(1-z)}}$$

Hence we are ready to present the Möbius-Bretschneider's formula in $\mathbb{D}$ as follows:

Theorem 5. (Möbius-Bretschneider's Formu-la). The gyroarea of the Möbius gyroqu-adrilateral $ABCD$ in $\mathbb{D}$ with side gyrolengths x,y,z,w, and with gyroangles A,B,C,D is given by

$$\Delta(ABCD)=\sqrt{\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-xyzw\sin^2K}\cdot\sqrt{1-\left(\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-xyzw\sin^2K\right)}\cdot\frac{1}{1-2\left(\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-xyzw\sin^2K\right)}$$

where $K=(A-B+C-D)/4$.

In $(\mathbb{R}_s^2,\oplus,\otimes)$ it is known that the gyroarea of a Möbius gyroquadrilateral $ABCD$ with gyroangles A,B,C,D is

$$\Delta(ABCD)=\frac{s^2}{2}\tan\left(\frac{2\pi-(A+B+C+D)}{2}\right).$$

Hence the gyroarea of the Möbius gyroquadrilateral $ABCD$ with side gyrolengths x,y,z,w, and with gyroangles A,B,C,D is given by

$$\Delta(ABCD)=s^2\sqrt{\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-\frac{x}{s}\frac{y}{s}\frac{z}{s}\frac{w}{s}\sin^2K}\cdot\sqrt{1-\left(\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-\frac{x}{s}\frac{y}{s}\frac{z}{s}\frac{w}{s}\sin^2K\right)}\cdot\frac{1}{1-2\left(\frac{sinhS_xsinhS_ysinhS_zsinhS_w}{\gamma_x\gamma_y\gamma_z\gamma_w}-\frac{x}{s}\frac{y}{s}\frac{z}{s}\frac{w}{s}\sin^2K\right)}$$

where $K=(A-B+C-D)/4$,

$$sinhS_x=\frac{\left(\frac{x}{s}+\frac{y}{s}+\frac{z}{s}+\frac{w}{s}\right)-\frac{x}{s}-\frac{wyz}{sss}}{\sqrt{\left(1-\frac{yz}{ss}\right)\left(1-\frac{w}{s}\right)-\left(\frac{y}{s}+\frac{z}{s}\right)\left(1+\frac{w}{s}\right)}\sqrt{\left(1-\frac{yz}{ss}\right)\left(1+\frac{w}{s}\right)+\left(\frac{y}{s}+\frac{z}{s}\right)\left(1-\frac{w}{s}\right)}},$$

$sinhS_y$

$$= \frac{\left(\frac{x}{s}+\frac{y}{s}+\frac{z}{s}+\frac{w}{s}\right)-\frac{y}{s}-\frac{x}{s}wz}{\sqrt{\left(1-\frac{zw}{s\,s}\right)\left(1-\frac{x}{s}\right)-\left(\frac{z}{s}+\frac{w}{s}\right)\left(1+\frac{x}{s}\right)}\sqrt{\left(1-\frac{zw}{s\,s}\right)\left(1+\frac{x}{s}\right)+\left(\frac{z}{s}+\frac{w}{s}\right)\left(1-\frac{x}{s}\right)}},$$

$sinhS_z$

$$= \frac{\left(\frac{x}{s}+\frac{y}{s}+\frac{z}{s}+\frac{w}{s}\right)-\frac{z}{s}-\frac{xyw}{s\,s\,s}}{\sqrt{\left(1-\frac{wx}{s\,s}\right)\left(1-\frac{y}{s}\right)-\left(\frac{w}{s}+\frac{x}{s}\right)\left(1+\frac{y}{s}\right)\sqrt{\left(1-\frac{wx}{s\,s}\right)\left(1+\frac{y}{s}\right)+\left(\frac{w}{s}+\frac{x}{s}\right)\left(1-\frac{y}{s}\right)}}},$$

$sinhS_w$

$$= \frac{\left(\frac{x}{s}+\frac{y}{s}+\frac{z}{s}+\frac{w}{s}\right)-\frac{w}{s}-\frac{xyz}{s\,s\,s}}{\sqrt{\left(1-\frac{xy}{s\,s}\right)\left(1-\frac{z}{s}\right)-\left(\frac{x}{s}+\frac{y}{s}\right)\left(1+\frac{z}{s}\right)\sqrt{\left(1-\frac{xy}{s\,s}\right)\left(1+\frac{z}{s}\right)+\left(\frac{x}{s}+\frac{y}{s}\right)\left(1-\frac{z}{s}\right)}}},$$

and $\gamma_x = \frac{1}{\sqrt{1-\left(\frac{x}{s}\right)^2}}$, $\gamma_y = \frac{1}{\sqrt{1-\left(\frac{y}{s}\right)^2}}$, $\gamma_y = \frac{1}{\sqrt{1-\left(\frac{z}{s}\right)^2}}$, $\gamma_w = \frac{1}{\sqrt{1-\left(\frac{w}{s}\right)^2}}$. Remarkably, in the Euclidean limit $s \to \infty$, of large s, when the open disc $\mathbb{R}_s^2$ expands to $\mathbb{R}^2$, Möbius-Bretschneider's formula reduces to Bretschneider's formula in $\mathbb{R}^2$.

Just like in Euclidean geometry, there are many triangle area formulas for hyperbolic triangles and these formulas are used to get the area formulas of hyperbolic quadrilaterals. Cagnoli's theorem states that in hyperbolic plane, the hyperbolic area of the hyperbolic triangle ABC with side lengths a, b, c and the opposite angles A, B, C is

$$sin\frac{\Gamma(ABC)}{2} = \frac{sinh\frac{b}{2}sinh\frac{c}{2}sinA}{cosh\frac{a}{2}}$$

or equivalently,

$$\Gamma(ABC) = 2sin^{-1}\left(\frac{sinh\frac{b}{2}sinh\frac{c}{2}sinA}{cosh\frac{a}{2}}\right)$$

For more details, we refer to readers [6]. Let $ABCD$ be a hyperbolic quadrilateral in the hyperbolic plane $\mathbb{D}$ and P be the common points of the hyperbolic diagonals AC and BD. Now we assume $\angle APB = \alpha$, $d_H(P,A) = e$, $d_H(P,B) = f$, $d_H(P,C) = g$, $d_H(P,D) = k$, $d_H(A,B) = a$, $d_H(B,C) = b$, $d_H(C,D) = c$, $d_H(D,A) = d$. Clearly the hyperbolic diagonals AC and BD divides $ABCD$ to four hyperbolic triangles APD, DPC, CPB, BPA satisfying

$$\Gamma(ABCD) = \Gamma(APB) + \Gamma(BPC) + \Gamma(CPD) + \Gamma(DPA) \qquad (4)$$

which implies

$$\Gamma(ABCD) = 2sin^{-1}\left(\frac{sinh\frac{e}{2}sinh\frac{f}{2}sin\alpha}{cosh\frac{a}{2}}\right) + 2sin^{-1}\left(\frac{sinh\frac{f}{2}sinh\frac{g}{2}sin\alpha}{cosh\frac{b}{2}}\right) + 2sin^{-1}\left(\frac{sinh\frac{g}{2}sinh\frac{k}{2}sin\alpha}{cosh\frac{c}{2}}\right) + 2sin^{-1}\left(\frac{sinh\frac{k}{2}sinh\frac{e}{2}sin\alpha}{cosh\frac{d}{2}}\right). \qquad (5)$$

Notice that (4) is not provided for the Möbius gyroquadrilateral $ABCD$ and the Möbius gyrotriangles APD, DPC, CPB, BPA. As with the Möbius-Bretschneider's formula, a new formula for the Möbius gyroquadrilaterals can be obtained by (2) and (3). Indeed, by using the relations

$$\Delta(ABCD) = \frac{1}{2}tan\frac{\Gamma(ABCD)}{2}$$
$$\Delta(APB) = \frac{1}{2}tan\frac{\Gamma(APB)}{2}$$
$$\Delta(BPC) = \frac{1}{2}tan\frac{\Gamma(BPC)}{2}$$
$$\Delta(CPD) = \frac{1}{2}tan\frac{\Gamma(CPD)}{2}$$
$$\Delta(DPA) = \frac{1}{2}tan\frac{\Gamma(DPA)}{2}$$

and

$$a = 2tanh^{-1}d_M(A,B), b = 2tanh^{-1}d_M(B,C)$$
$$c = 2tanh^{-1}d_M(C,D), d = 2tanh^{-1}d_M(D,A)$$
$$e' = d_M(A,P), f' = d_M(B,P), g' = d_M(C,P)$$
$$k' = d_M(D,P), a' = d_M(A,B), b' = d_M(B,C)$$
$$c' = d_M(C,D), d' = d_M(D,A)$$

(5) takes the form

$$tan^{-1}(2\cdot\Delta(ABCD)) = sin^{-1}\frac{\gamma_{e'}e'\gamma_{f'}f'sin\alpha}{\gamma_{a'}} + sin^{-1}\frac{\gamma_{f'}f'\gamma_{g'}g'sin\alpha}{\gamma_{b'}} + sin^{-1}\frac{\gamma_{g'}g'\gamma_{k'}k'sin\alpha}{\gamma_{c'}} + sin^{-1}\frac{\gamma_{k'}k'\gamma_{e'}e'sin\alpha}{\gamma_{d'}}$$

where $\gamma_{e'} = \frac{1}{\sqrt{1-e'^2}}$, $\gamma_{f'} = \frac{1}{\sqrt{1-f'^2}}$, $\gamma_{g'} = \frac{1}{\sqrt{1-g'^2}}$, $\gamma_{k'} = \frac{1}{\sqrt{1-k'^2}}$. Thus we get

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$$\Delta(ABCD) = \frac{1}{2}\tan\left(\sin^{-1}\frac{\gamma_{e'}e'\gamma_{f'}f'\sin\alpha}{\gamma_{a'}} + \sin^{-1}\frac{\gamma_{f'}f'\gamma_{g'}g'\sin\alpha}{\gamma_{b'}} + \sin^{-1}\frac{\gamma_{g'}g'\gamma_{k'}k'\sin\alpha}{\gamma_{c'}} + \sin^{-1}\frac{\gamma_{k'}k'\gamma_{e'}e'\sin\alpha}{\gamma_{d'}}\right).$$

After simple calculations with trigonometric identities, one can easily get

$$\Delta(ABCD) = \frac{\sin\alpha}{2}\frac{\left(1-\sin^2\alpha\frac{\gamma_{e'}e'\gamma_{f'}f'}{\gamma_{a'}}\frac{\gamma_{g'}g'\gamma_{k'}k'}{\gamma_{c'}}\right)\cdot\left(\frac{\gamma_{f'}f'\gamma_{g'}g'}{\gamma_b}+\frac{\gamma_{k'}k'\gamma_{e'}e'}{\gamma_{d'}}\right)+\left(1-\sin^2\alpha\frac{\gamma_{f'}f'\gamma_{g'}g'}{\gamma_{b'}}\frac{\gamma_{k'}k'\gamma_{e'}e'}{\gamma_{d'}}\right)\cdot\left(\frac{\gamma_{e'}e'\gamma_{f'}f'}{\gamma_{a'}}+\frac{\gamma_g g'\gamma_{k'}k'}{\gamma_{c'}}\right)}{\left(1-\sin^2\alpha\frac{\gamma_g g'\gamma_{k'}k'}{\gamma_{c'}}\frac{\gamma_{k'}k'\gamma_{e'}e'}{\gamma_{d'}}\right)\cdot\left(1-\sin^2\alpha\frac{\gamma_{e'}e'\gamma_{f'}f'}{\gamma_{a'}}\frac{\gamma_{f'}f'\gamma_{g'}g'}{\gamma_b}\right)-\left(\frac{\gamma_{g'}g'\gamma_{k'}k'}{\gamma_{c'}}+\frac{\gamma_{k'}k'\gamma_{e'}e'}{\gamma_{d'}}\right)\cdot\left(\frac{\gamma_{e'}e'\gamma_{f'}f'}{\gamma_{a'}}+\frac{\gamma_{f'}f'\gamma_{g'}g'}{\gamma_b}\right)\sin^2\alpha}$$

Hence we obtain the following theorem:

Theorem 6. (Möbius-Cagnoli's formula). If $ABCD$ is a Möbius gyroquadrilateral in $\mathbb{D}$ with $d_M(P,A)=e$, $d_M(P,B)=f$, $d_M(P,C)=g$, $d_M(P,D)=k$ where P is the common points of the diagonals AC and BD, then the gyroarea $\Delta(ABCD)$ of $ABCD$ is

$$\Delta(ABCD) = \frac{\sin\alpha}{2}\frac{\left(1-\sin^2\alpha\frac{\gamma_e e\gamma_f f}{\gamma_a}\frac{\gamma_g g\gamma_k k}{\gamma_c}\right)\left(\frac{\gamma_f f\gamma_g g}{\gamma_b}+\frac{\gamma_k k\gamma_e e}{\gamma_d}\right)+\left(1-\sin^2\alpha\frac{\gamma_f f\gamma_g g}{\gamma_b}\frac{\gamma_k k\gamma_e e}{\gamma_d}\right)\left(\frac{\gamma_e e\gamma_f f}{\gamma_a}+\frac{\gamma_g g\gamma_k k}{\gamma_c}\right)}{\left(1-\sin^2\alpha\frac{\gamma_g g\gamma_k k}{\gamma_c}\frac{\gamma_k k\gamma_e e}{\gamma_d}\right)\left(1-\sin^2\alpha\frac{\gamma_e e\gamma_f f}{\gamma_a}\frac{\gamma_f f\gamma_g g}{\gamma_b}\right)-\left(\frac{\gamma_g g\gamma_k k}{\gamma_c}+\frac{\gamma_k k\gamma_e e}{\gamma_d}\right)\left(\frac{\gamma_e e\gamma_f f}{\gamma_a}+\frac{\gamma_f f\gamma_g g}{\gamma_b}\right)\sin^2\alpha}$$

where $\angle APB = \alpha$.

In $(\mathbb{R}_s^2,\oplus,\otimes)$, the Möbius-Cagnoli's formula takes the formula

$$\Delta(ABCD) = \frac{s^2\sin\alpha}{2}\frac{\left(1-\sin^2\alpha\frac{\gamma_e e\gamma_f f}{\gamma_a}\frac{\gamma_g g\gamma_k k}{\gamma_c}\right)\left(\frac{\gamma_f f\gamma_g g}{\gamma_b}+\frac{\gamma_k k\gamma_e e}{\gamma_d}\right)+\left(1-\sin^2\alpha\frac{\gamma_f f\gamma_g g}{\gamma_b}\frac{\gamma_k k\gamma_e e}{\gamma_d}\right)\left(\frac{\gamma_e e\gamma_f f}{\gamma_a}+\frac{\gamma_g g\gamma_k k}{\gamma_c}\right)}{\left(1-\sin^2\alpha\frac{\gamma_g g\gamma_k k}{\gamma_c}\frac{\gamma_k k\gamma_e e}{\gamma_d}\right)\left(1-\sin^2\alpha\frac{\gamma_e e\gamma_f f}{\gamma_a}\frac{\gamma_f f\gamma_g g}{\gamma_b}\right)-\left(\frac{\gamma_g g\gamma_k k}{\gamma_c}+\frac{\gamma_k k\gamma_e e}{\gamma_d}\right)\left(\frac{\gamma_e e\gamma_f f}{\gamma_a}+\frac{\gamma_f f\gamma_g g}{\gamma_b}\right)\sin^2\alpha}$$

where $\gamma_e = \frac{1}{\sqrt{1-\frac{e^2}{s^2}}}$, $\gamma_f = \frac{1}{\sqrt{1-\frac{f^2}{s^2}}}$, $\gamma_g = \frac{1}{\sqrt{1-\frac{g^2}{s^2}}}$, $\gamma_k = \frac{1}{\sqrt{1-\frac{k^2}{s^2}}}$.

In the Euclidean limit $s \to \infty$, when the open disc $\mathbb{R}_s^2$ expands to $\mathbb{R}^2$, Möbius-Cagnoli's formula reduces $\frac{1}{2}|AC||BD|\sin\alpha$, where α is the measure of the angle between AC and BD.

CONCLUSIONS

The counterparts of the hyperbolic Bretschneider's theorem (that gives the hyperbolic area of a certain hyperbolic quadrilateral if the side lengths and the interior angles are known), and the hyperbolic Cagnoli's theorem (that gives the hyperbolic area of a certain hyperbolic quadrilateral if the distances of the vertices to the common point of the diagonals and the measure of the angle between the diagonals are known) are provided for a Möbius gyroquadrilateral in $(\mathbb{R}_s^2,\oplus,\otimes)$ by using the same geometric information.

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