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Analytic Approach for Solving System of Fractional Differential Equations

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ABSTRACT

In this paper, Sumudu transformation (ST) of Caputo fractional derivative formulae are derived for linear fractional differential systems. This formula is applied with Mittage-Leffler function for certain homogenous and nonhomogenous fractional differential systems with nonzero initial conditions. Stability is discussed by means of the system's distinctive equation.

KEYWORDS: Caputo derivatives; Sumudu transform; Mittage-Leffler function; asymptotically stable.

الخلاصة

في هذا البحث محول سومودو لصيغة مشتقة كابوتو اشتق لحل انظمة المعادلات التفاضلية الكسرية. هذه الصيغة طبقت بأستخدام دالة ميتاك-لفير للانظمة المتجانسة و اللامتجانسة بشوط ابتدائية لاتساوي صفر. الاستقرارية نوقشت بواسطة نظام متجانس.

INTRODUCTION

Applications of fractional derivative in the present day includes fluid flow, dynamical process, electrical networks, probability and statistics, control theory and so on, [1]. ST first defined in 1993, which used to solve engineering control problems see [2]. However, ST solved fractional ordering differential equations and graph two-dimensional solutions. As shown in [3], the Taylor collection method was derived for solving fractional differential equations based on taking the truncated Taylor expansions of the vector-function solution. In [4] analytical solutions presented for systems of fractional differential equations using the differential transform method.

As in [5], several sufficient criteria were established to ensure the Mittage-Leffler stability and asymptotic stability for the differential system of fractional order. In [6] study properties of stability, Mittage-Leffler stability, Lipchitz stability and comparison results of stability.

Preliminaries

Some important preliminaries of fractional calculus are given here.

Definition (2.1), [7]

Consider a set A defined as:

$$A = \left\{ f(t) | \exists\, M, \tau_1, \tau_2 > 0, |f(t)| \le M e^{\frac{|t|}{\tau_j}}\ if\ t \in (-1)^j X[0,\infty) \right\}$$

For all real t ≥ 0, and $f(t) \in A$, The Sumudu transform of f(t) is denoted by

$S[f(t)] = F(u)$, and it's defined as

$S[f(t)](u) = \int_0^\infty e^{-t} f(ut)dt,\ u \in (\tau_1, \tau_2)$ (1)

Definition (2.2), [8]

The Caputo fractional differential operator D_t^v of order v is:

$D_t^v f(t) = \frac{1}{\Gamma(n-v)} \int_0^t (t-\tau)^{n-v-1} f^{(n)}(\tau) d\tau$, (2)

for $n-1 < v < n$, $n \in \mathbb{N}$, t > 0.

Definition (2.3), [8]

The Mittage Leffler function $E_v(Z)$ with $v > 0$, is define by the following series:

$$E_v(Z) = \sum_{n=0}^{\infty} \frac{Z^v}{\Gamma(nv+1)}, v > 0, Z \in C$$

Definition (2.4), [1]

Mittage-Leffler functions of one and two parameters are defined respectively:

$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \alpha > 0, x \in C$

$E_{\alpha\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \alpha > 0, \beta > 0, x \in C$

Definition (2.5), [1]

The one parameters of Mittage-leffer function of the matrix $\mathcal{A} \in M_n$ (M_n square matrix of order nxn) is defined for $\alpha > 0$

$E_\alpha(\mathcal{A}) = \sum_{k=0}^{\infty} \frac{\mathcal{A}^k}{\Gamma(\alpha k+1)}$,

$$\boldsymbol{E_\alpha(\mathcal{A}t^\alpha) = \sum_{k=0}^{\infty} \frac{\mathcal{A}^k t^{\alpha k}}{\Gamma(\alpha k+1)}}, \quad (3)$$

Remark (2.6)

If $\mathcal{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a square Matrix of order 2×2, the Matrix Mittage-leffler function $E_\alpha(\mathcal{A}X^\alpha)$ is given by:

$$E_\alpha(\mathcal{A}X^\alpha) = \begin{bmatrix} E_\alpha\left(\lambda_1 X^\alpha + \frac{a-\lambda_1}{\lambda_1-\lambda_2}E\right) & X^{1-\alpha}\frac{b}{\lambda_1-\lambda_2}E \\ X^{1-\alpha}\frac{c}{\lambda_1-\lambda_2}E & X^{1-\alpha}\left(e^{\lambda x} + \frac{d-\lambda_1}{\lambda_1-\lambda_2}E\right) \end{bmatrix} \quad (4)$$

where $E = E_\alpha(\lambda_1 X^\alpha) - E_\alpha(\lambda_2 X^\alpha)$ and $\lambda_1 \neq \lambda_2$ are the eigenvalues of $\mathcal{A}$.

The Method;

Method is derived by ST of Mittage-Leffler function for solving certain type of fractional differential equations.

Lemma (3.1), [9]:

Let $\alpha > 0, \beta > 0, \lambda \epsilon R$ and $u^{-\alpha} > |\lambda|$ then:

$$\mathbb{S}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\right]u = \left[\frac{u^{\beta-1}}{1-\lambda u^\alpha}\right] \quad (5)$$

where $E_{\alpha,\beta}$ is Mittage-Leffler function in two parameters.

Theorem (3.2), [9]:

Let $n \in N$ and $\alpha > 0$ be such that $n-1 < \alpha < n$ and $F(u)$ be the ST of the function $f(t)$ then the ST Caputo α derivative of $f(t)$ is given by:

$$S[D^\alpha f(t)]u = u^{-\alpha}F(u) - \sum_{k=o}^{n-1} u^{k-\alpha}\left[f^{(k)}(0)\right] \quad (6)$$

Example (3.3)

Take into account the initial value problem (I.V. Problem) for a homogenous fractional differential equation

$D^\alpha f(t) + af(t) = 0$, $0 < \alpha < 1$,

$f(0) = c$

where a and c are constants, applying ST on both sides, hence

$S(D^\alpha f(t))(u) + aS(f(t))(u) = 0$

$u^{-\alpha}F(u) - f(0)u^{-\alpha} + aF(u) = 0$

$(u^{-\alpha} + a)F(u) = f(0)u^{-\alpha}$

since $f(0) = c$ then $F(u) = \frac{u^{-\alpha}c}{(u^{-\alpha}+a)} = \frac{c}{1+au^\alpha}$

by eq.(6) replacing β=1

$F(u) = S\left[E_{\alpha,1}(-at^\alpha)(u)\right]c$

Taking inverse ST, we get

$f(t) = cE_\alpha(-at^\alpha)$

Now, we will generalize lemma(3.1) to solve a homogenous linear fractional system of order $0 < \alpha < 1$

Theorem (3.4)

Let $\mathcal{A} \in M_n$ be a scalar matrix , $\eta \in M_{n,1}$ be a scaler vector and $y(t) \in M_{n,1}$ be unknown vector. The exact solution homogenous linear fractional system of $0 < \alpha < 1$

$$D^\alpha y(t) = \mathcal{A}y(t), y(0) = \eta \quad (7)$$

is given by:

$$y(t) = E_\alpha(\mathcal{A}t^\alpha).\eta \quad (8)$$

Where $E_\alpha(\mathcal{A}t^\alpha)$ is the matrix Mittage-Leffler function.

Proof

Taking Sumudu transformation to both sides of eq. (7) and use the Sumudu transformation of the Caputo derivative to get

$u^{-\alpha}Y(u) - u^{-\alpha}y(0) = \mathcal{A}Y(u)$

$(u^{-\alpha}I - \mathcal{A})Y(u) = u^{-\alpha}\eta$

$$Y(u) = \frac{u^{-\alpha}\eta}{u^{-\alpha}I - \mathcal{A}} = \frac{\eta}{I - u^\alpha \mathcal{A}}$$

by lemma (3.1)

$y(t) = S[E_{\alpha,1}(\mathcal{A}t^\alpha)(u)].\eta$

taking inverse of Sumudu transform we get eq. (8)

$y(t) = E_\alpha(\mathcal{A}t^\alpha).\eta$

Example (3.5)

Let the I.V. Problem for a fractional differential system of order $0 < \alpha < 1$,

$$y(t) = \mathcal{A}y(t) \quad ,y(0) = \eta \quad (9)$$

where $\mathcal{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $and\ \eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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The eigenvalues of $\mathcal{A}$ are $\lambda_1 = 1 + i\ , i$
$\lambda_2 = 1 - i$

$$E_\alpha(\mathcal{A}t^\alpha) = \begin{bmatrix} \mathrm{E}_\alpha(1+\mathrm{i})\mathrm{t}^\alpha + \frac{-1}{2}(\mathrm{E}_\alpha(1+\mathrm{i})\mathrm{t}^\alpha) - \mathrm{E}_\alpha\left((1-\mathrm{i})\mathrm{t}^\alpha\right) & \mathrm{t}^{1-\alpha}\frac{1}{2\mathrm{i}}(\mathrm{E}_\alpha(1+\mathrm{i})\mathrm{t}^\alpha) - \mathrm{E}_\alpha((1-\mathrm{i})\mathrm{t}^\alpha) \\ \mathrm{t}^{1-\alpha}\frac{-1}{2\mathrm{i}}(\mathrm{E}_\alpha(1+\mathrm{i})\mathrm{t}^\alpha) - \mathrm{E}_\alpha((1-\mathrm{i})\mathrm{t}^\alpha) & \mathrm{t}^{1-\alpha}(\mathrm{e}^{(1+\mathrm{i})\mathrm{t}} + \frac{-1}{2}(\mathrm{E}_\alpha(1+\mathrm{i})\mathrm{t}^\alpha) - \mathrm{E}_\alpha((1-\mathrm{i})\mathrm{t}^\alpha) \end{bmatrix}$$

$y(t) = \begin{bmatrix} \mathrm{y}_1(\mathrm{t}) \\ \mathrm{y}_2(\mathrm{t}) \end{bmatrix}$ and $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hence:

$\mathrm{y}_1(\mathrm{t}) = \mathrm{t}^{1-\alpha}\frac{-1}{2\mathrm{i}}(\mathrm{E}_\alpha(1+\mathrm{i})\mathrm{t}^\alpha) - \mathrm{E}_\alpha((1-\mathrm{i})\mathrm{t}^\alpha)$

$\mathrm{y}_2(\mathrm{t}) = \mathrm{t}^{1-\alpha}(\mathrm{e}^{(1+\mathrm{i})\mathrm{t}} + \frac{-1}{2}(\mathrm{E}_\alpha(1+\mathrm{i})\mathrm{t}^\alpha) - \mathrm{E}_\alpha((1-\mathrm{i})\mathrm{t}^\alpha)$

To solve nonhomogeneous linear fractional system of order $0 < \alpha < 1$, we first introduce the Sumudu transform convolution theorms.

Theorem (3.5)[10]

let $W_1(t)$ and $W_2(t)$ functions in the set of functions A having Sumudu transforms $F(u)$ and $G(u)$ respectively. Then the ST of the convolutions of $W_1(t)$ and $W_2(t)$, where

$(W_1 * W_2)(t) = \int_0^\infty W_1(t)W_2(t-\tau)d\tau$

Defined by:

$\mathbb{S}\left((W_1 * W_2)(t)\right) = uF(u)G(u).$

Now, we will generalize lemma(3.1) , theorem(3.4) and theorem(3.5) to solve nonhomogenous linear fractionals system of order $0 < \alpha < 1$

Theorem (3.6)

Let $\mathcal{A} \in M_n$ be a scalar matrix, $\eta \in M_{n,1}$ be a scaler vector $W_1(t) \in M_{n,1}$ and $y(t) \in M_{n,1}$ be unknown vector. The exact solution nonhomogenous linear fractional systems of $0 < \alpha < 1$,

$D^\alpha y(t) = \mathcal{A}y(t) + W_1(t)\ ,\ y(0) = \eta$ (10)

is given by:

$y(t) = E_\alpha(\mathcal{A}t^\alpha)\eta + \int_0^t (t-s)^\alpha E_\alpha(\mathcal{A}(t-s)^\alpha\, W_1(s)ds$ (11)

where $E_\alpha(\mathcal{A}t^\alpha)$ is the matrix Mittage-Leffler function.

Proof

taking ST to both sides of eq.(10),

$u^{-\alpha}Y(u) - u^{-\alpha}y(0) = \mathcal{A}Y(u) + F(u)$ (12)

$$Y(u) = \frac{u^{-\alpha}\eta}{u^{-\alpha}I - \mathcal{A}} + \frac{F(u)}{u^{-\alpha}I - \mathcal{A}}$$

Appling the inverse ST to both sides of eq. (12), we have

Using eq. (8) and eq.(4), we have

$$S^{-1}\{Y(u)\} = S^{-1}\left\{\frac{u^{-\alpha}\eta}{u^{-\alpha}I - \mathcal{A}}\right\} + S^{-1}\{F(u)\} * S^{-1}\{(u^{-\alpha}I - \mathcal{A})^{-1}\}$$

By substituting the Sumudu transform of the Mittage-Leffler function lemma (3.1) and theorem (3.6) we get the solution as in eq. (11).

Example (3.6)

Consider the initial value problem for a nonhomogeneous fractional differential system of order $0 < \alpha < 1$

$D^\alpha y(t) = \mathcal{A}y(t) + W_1(t),\quad y(0) = \eta$
$0 < \alpha < 1$

where $\mathcal{A} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \eta \begin{bmatrix} 1 \\ -2 \end{bmatrix}, f(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

hence by eq.(11):

$$y(t) = \left(E_\alpha \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} t^\alpha\right)\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \int_0^t ((t-s)^\alpha)(E_\alpha \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}(t-s)^\alpha) \begin{bmatrix} \sin s \\ \cos s \end{bmatrix} ds$$

Then

$$y_1(t) = E_\alpha(-2t^\alpha) + \int_0^t (t-s)^\alpha + E_\alpha(-2(t-s)^\alpha) \quad \sin s\ ds$$

$$y_2(t) = -2E_\alpha(3t^\alpha) + \int_0^t (t-s)^\alpha + E_\alpha(3(t-s)^\alpha) \quad \cos s\ ds$$

STABILITY ANALYSIS

Stability of the linear fractional differential system defined by the Caputo's derivative $0 < \alpha < 1$ is discussed here acording to two theorems

Theorem (4.1), [11]

The system eq.(7) is a symptotically stable if and only if the eigenvalues $\lambda(\mathcal{A})$ of the matrix $\mathcal{A}$ satisfy, $\frac{Cos(\lambda(\mathcal{A}))}{\|\lambda(\mathcal{A})\|} < 1 - \alpha$

Theorem (4.2), [12]

The system eq.(7) is a symptotically stable if and only if $|\arg(spec(\mathcal{A}))| > \alpha\frac{\pi}{2}$, where spec($\mathcal{A}$) is the spectrum of $\mathcal{A}$.

Now discuss the stability of the linear system given in example (3.5) as follows:

Since $\mathcal{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, and $\lambda = 1 \mp i$

By theorem (4.1),

$cos\lambda = \frac{1}{2}$ and $\|\lambda\| = \sqrt{2}$

then $\frac{Cos\lambda}{\|\lambda\|} = \frac{1}{2} < 1 - \alpha$

Hence $\alpha < 1/2$, there values of α get the system is asymptotically stable where

$0 < \alpha < 1.$

By theorem (4.2):

$|\arg(spec(\mathcal{A}))| = |\theta| = 0.785$

$0.785 > \alpha \frac{\pi}{2}$

hence $\alpha < 0.5$, then the system (10) is asymptotically stable when $\alpha < 0.5$.

CONCLUSIONS

In this work, we studied and proved the ST operational transform method as shown in theorems, which are important in solving certain homogenous and non-homogenous fractional differential systems associating the Caputo fractional derivatives.

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