Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 32, Issue 1, 2021 DOI: http://doi.org/10.23851/mjs.v32i1.920

Research Article Open Access

On (P*-N) Quasi Normal Operators of Order "n" in Hilbert Space

Salim dawood M.*, Jaafer Hmood Eidi

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, IRAQ.
*Correspondent contact: dr_salim2015@yahoo.com

Article Info

Received
16/10/2020

Accepted
27/12/2020

Published
20/02/2021

ABSTRACT

Through this paper, we submitted some types of quasi normal operator is called be (k*-N)-quasi normal operator of order n defined on a Hilbert space H, this concept is generalized of some kinds of quasi normal operator appear recently form most researchers in the field of functional analysis, with some properties and characterization of this operator as well as, some basic operation such as addition and multiplication of these operators had been given, finally the relationships of this operator proved with some examples to illustrate conversely and introduce the sufficient conditions to satisfied this case with other types had been studied.

KEYWORDS: Operator; quasi normal; (K-N)-quasi normal operator; quasi normal operator of order (n).

الخلاصة

من خلال هذا البحث سنقدم بعض انواع المؤثرات شبه السوية وتسمى شبه السوية-(k*-N)- من الرتبة n والمعرفة على فضاء هلبرت H هذا المفهوم هوة تعميم لبعض انواع المؤثرات شبه السوية والتي ضهرت حديثا من اغلب الباحثين في حقل التحليل الدالي مع بعض الخواص والتميزات لهذا المؤثر بالاضافة الى بعض العمليات الاساسية مثل الجمع والضرب لهذه المؤثرات قد اعطيت, واخيرا العلاقة لهذا المؤثر اعطيت مع بعض الامثلة التوضيحية والعكسية مع تقديم الشروط الكافية لتحقيق الحالات مع الانواع الاخرى قد درست.

INTRODUCTION

In the field of the functional analysis some students focus almost their works about some types of operators especially quasi normal operator such as, [1] is the first one introduced the concept of normal operator in 1953, in addition [2] continue by study some properties of quasi normal operator, but [3] presented another properties of quasi normal operator in Hilbert space and in [4] introduce new types of quasi normal operator is said to be n-power quasi normal operators. In, [5] properties of (N) quasi normal operators had been given more studied in this field continue also, introduced the condition to make some operation satisfies of quasi normal operator, Also by [6] given more general operator it's called (K-N)-quasi normal operator this generalized in to (K-N)-quasi-n-normal operator.

SOME BASIC DEFINITIONS

An bounded linear operator $A: H \rightarrow K$ is called quasi normal operator if satisfy [2]

$$A(A^*A) = (A^*A)A$$

A bounded linear operator $A: H \rightarrow H$ is called n-power quasi normal operator if satisfy [4]

$$A^*(A^*A) = (A^*A)A^*$$

An bounded linear operator $A: H \rightarrow K$ is called quasi normal operator of order n if satisfy:

$$A(A^*A)^n = (A^*A)^n A$$

where n is positive integer number

A bounded linear operator A:H→K is called(N)-quasi normal operator if satisfy [9]

$$A(A^*A) = N(A^*A)A$$
$$A^k(A^*A) = N(A^*A)A^k$$

i. A bounded linear operator A:H→K is called (K-N) quasi normal operator if satisfy [5]

A bounded linear operator A:H→K is called quasi- (n) normal if satisfy [8]

$$A^k(A^*A^n) = N(A^*A^n)A^k$$

Where n, k is positive integer's number

A bounded linear operator A:H→K is called [K-N] quasi (n)-normal operator if [6]

$$A^k(A^*A^n)^k = N(A^*A^n)^k A^k,$$

Where n, k is positive integer's number

An bounded linear operator A:H→K is called K^* quasi normal operator of order(n) if satisfy[6],

$$(A^*)^k(A^*A)^n = (A^*A)^n(A^*)^k$$

Where k is a positive integer number.

Now, we introduce the definition of $(P^* - N)$ quasi (n) - normal operator.

$(P^* - N)$- QUASI NORMAL OPERATORS OF ORDER (N)

Here we introduce the new type of quasi normal operator, which is $(P^* - N)$– quasi normal operators of order (n), this concept is modified to definition appear in [6].

Definition

Let $A: H \to H$ is a bounded linear operator is called $(P^* - N)$ – quasi normal operator of order (n) if

$$(A^*)^P(A^*A)^n = N(A^*A)^n(A^*)^P$$

where P is a positive integer number.

Theorem

Let $A: H \to H$ is$(P^* - N)$ –quasi normal operator of order (n) then A^m is also $(P^* - N)$ –quasi normal operator of order (n), where $m \geq 1$, is a positive number.

Proof

Let A is $(P^* - N)$ –quasi normal of order (n)

Assume that, A^m is also $(P^* - N)$ –quasi normal operator of (n). By mathematical induction:

Since A be $(P^* - N)$ –quasi normal operator of order (n)

The result true for $m = 1$

$(A^*)^P(A^*A)^n = N(A^*A)^n(A^*)$

(1)

Assumption that the result true when $(m - H)$

$((A^*)^P(A^*A)^n)^H = (N(A^*A)^n(A^*)^P)^H$

(2)

Next, prove the result for, $m = H + 1$

That:$((A^*)^P(A^*A)^n)^{H+1} = (N(A^*A)^n(A^*)^P)^{H+1}$

$((A^*)^P(A^*A)^n)^{H+1}=$

$((A^*)^P(A^*A)^n)^H(A^*)^P(A^*A)^n$

$=(N(A^*A)^n(A^*)^P)^H N(A^*A)^n(A^*)$

$= (N(A^*A)^n(A^*)^P)^{H+1}$

Thus, the result is true for $m = H + 1$

Therefor A^m is $(P^* - N)$ –quasi normal operator of (n) for all m where $m \geq 1$

Remark

It is clear that, Let A_1 and A_2 be two $(P^* - N)$ –quasi normal of order (n) then $A_1 + A_2$ is need not to be $(P^* - N)$ –quasi normal of order (n).To illustrate that consider

Let $A_1 = [3\ 3\ 3i\ 3\ ]$ and $A_2 = [2\ 2i\ -2i\ 2\ ]$ then $A_1$and $A_2$are$(P^* - N)$ –quasi normal of order (n), but $A_1 + A_2$ and is not $(P^* - N)$ – quasi normal of order (n).

Now, the theorem 3.4 gives the condition to make remark (3.3) is true.

Theorem

Let A_1 and A_2 be two $(P^* - N)$ –quasi normal of order (n), from Hilbert space H to H, such that $A_1^*A_2^* = A_1A_2 = A_1^*A_2 = 0$ Then $A_1 + A_2$ $(P^* - N)$ –quasi normal operator of order (n).

Proof

$$((A_1 + A_2)^*)^P\big((A_1 + A_2)^*(A_1 + A_2)\big)^n$$

$$= (A_1^* + A_2^*)^P((A_1^* + A_2^*)^n(A_1 + A_2)^n)$$

$=((A_1^*)^P + P(A_1^*)^{P-1}A_2^* + \cdots + (A_2^*)^P)$

$=((A_1^*)^k + (A_2^*)^k)\big(((A_1^*)^n + (A_2^*)^n)(A_1^n + A_2^n)\big)$

$=((A_1^*)^P + (A_2^*)^P)((A_1^*)^nA_1^n +$

$(A_1^*)^nA_{2+}^n(A_2^*)^nA_1^n + (A_2^*)^nA_2^n)$

$=((A_1^*)^k + (A_2^*)^k)((A_1^*A_1)^n + (A_1^*A_2)^n +$

$(A_2^*A_1)^n + (A_2^*A_2)^n)$

$= ((A_1^*)^P(A_1^*)^nA_1^n$

$+ (A_1^*)^P(A_2^*)^nA_2^n+(A_2^*)^P(A_1^*)^nA_1^n$

$+ (A_2^*)^P(A_2^*)^nA_2^n)$

$=((A_1^*)^P(A_1^*)^nA_1^n + (A_2^*)^P(A_2^*)^nA_2^n)$

Since A_1 and A_2 are $(P^* - N)$ -quasi normal operator of order (n)

$((A_1 + A_2)^*)^P\big((A_1 + A_2)^*(A_1 + A_2)\big)^n =$

$N((A_1^*)^nA_1^n(A_1^*)^P) + N((A_2^*)^nA_2^n(A_2^*)^P)=$

$N((A_1^*)^nA_1^n(A_1^*)^P + (A_2^*)^nA_2^n(A_2^*)^P)$

Hence $A_1 + A_2$ is

$$((A_1 + A_2)^*)^P\big((A_1 + A_2)^*(A_1 + A_2)\big)^n$$

ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 32, Issue 1, 2021 DOI: http://doi.org/10.23851/mjs.v32i1.920

$=((A_1^*)^P(A_1^*)^nA_1^n+(A_1^*)^P(A_2^*)^nA_2^n$
$+(A_2^*)^P(A_1^*)^nA_1^n+(A_2^*)^P(A_2^*)^nA_2^n)$
$=(A_1^*)^P(A_1^*)^nA_1^n+(A_2^*)^P(A_2^*)^nA_2^n$

Since A_1 and A_2 are (P^*-N)-quasi normal operator of order (n)

$$((A_1+A_2)^*)^P\big((A_1+A_2)^*(A_1+A_2)\big)^n$$
$$=N((A_1^*)^nA_1^n(A_1^*)^P)+N((A_2^*)^nA_2^n(A_2^*)^P)$$
$$=N((A_1^*)^nA_1^n(A_1^*)^P+(A_2^*)^nA_2^n(A_2^*)^P)$$

Hence A_1+A_2 is (P^*-N)-quasi normal operator of order (n).

Remark

It is clear that, Let A_1 and A_2 be two (P^*-N) – quasi normal operators of order (n)
Then A_1A_2 are not necessary two (P^*-N)-quasinormal operators of order (n).
Next, we give condition in order to remark (3.5)
It's true by the following theorem**.**

Theorem

Let $A_1: H\to H$ be (K^*-N)-quasi normal operator of order (n) and $A_2: H\to H$ be K^*-quasi normal operator of order (n) then the product A_1A_2 is(P^*-N)-quasi normal operator of order (n) if satisfy the following conditions

$$A_1A_2 = A_2A_1$$
$$A_2^*A_1 = A_1A_2^*$$

Proof

$$((A_1A_2)^*)^P\big((A_1A_2)^*(A_1A_2)\big)^n$$
$$=((A_2A_1)^*)^P\big((A_2A_1)^*(A_1A_2)\big)^n$$
$$=((A_2A_1)^*)^P(((A_2A_1)^*)^n(A_1A_2)^n)$$
$$=(A_1^*A_2^*)^P((A_1^*A_2^*)^n(A_1A_2)^n)$$
$$=((A_1^*)^P(A_2^*)^P)\big(((A_1^*)^n(A_2^*)^n)(A_1^nA_2^n)\big)$$
$$=(A_1^*)^P((A_2^*)^P(A_1^*)^n)((A_2^*)^nA_1^n)A_2^n$$
$$=(A_1^*)^P((A_1^*)^n(A_2^*)^P)(A_1^n(A_2^*)^n)A_2^n$$
$$=((A_1^*)^P(A_1^*)^n)(A_1^n(A_2^*)^P)((A_2^*)^nA_2^n)$$
$$=\big((A_1^*)^P(A_1^*)^n((A_2^*)^PA_1^n)((A_2^*)^nA_2^n)\big)$$
$$=((A_1^*)^P(A_1^*)^nA_1^n)((A_2^*)^P(A_2^*)^nA_2^n)$$
$$=(N(A_1^*)^nA_1^n(A_1^*)^P)((A_2^*)^nA_2^n(A_2^*)^P)$$
$$=N((A_1^*)^nA_1^n)((A_2^*)^n(A_1^*)^P)(A_2^n(A_2^*)^P)$$
$$=N(A_1^*)^n(A_1^n(A_2^*)^n)((A_1^*)^PA_2^n)(A_2^*)^P$$
$$=N(A_1^*)^n((A_2^*)^nA_1^n)(A_2^n(A_1^*)^P)(A_2^*)^P$$
$$=N(A_1^*A_2^*)^n(A_1^nA_2^n)((A_1^*)^P(A_2^*)^P)$$
$$=N((A_2A_1)^*)^n(A_1A_2)^n\big((A_1^*)(A_2^*)\big)^P$$
$$=N((A_2A_1)^*)^n(A_1A_2)^n((A_2A_1)^*)^P$$
$$=N((A_1A_2)^*(A_1A_2)^n)((A_1A_2)^*)^P$$
$$\to$$
$$((A_1A_2)^*)^P\big((A_1A_2)^*(A_1A_2)\big)^n=N\big((A_1A_2)^*(A_1A_2)\big)^n((A_1A_2)^*)^P$$

Hence, the product A_1A_2 is(P^*-N)-quasi Normal operator of order (n)

3.7 Theorem

Let $A: H\to H$ is(P^*-N)-quasi normal at order (n) then

1)
Is (P^*-N)-quasi normal operator of order (n), where $\lambda\in R$.

2)
(P^*-N)-quasi normal operator of order (n), such that $N=\left(\frac{N}{M}\right)$, where (M) is closed subspace.

Proof

1) $$((\lambda A)^*)^P\big((\lambda A)^*(\lambda A)\big)^n$$
$$=(\lambda A^*)^P\big((\lambda A)^*(\lambda A)\big)^n$$
$$=\lambda^P(A^*)^P(\lambda A^*)^n(\lambda^nA^n)$$
$$=\lambda^P\lambda^n\lambda^n(A^*)^P(A^*)^n(A^n)$$
$$=\lambda^P\lambda^n\lambda^nN(A^*A)^n(A^*)^p$$
$$=N\lambda^n(A^*)^n\lambda^nA^n(\lambda^PA^*)^P$$
$$=N((\lambda A)^*)^n(\lambda A)^n((\lambda A)^*)^P$$

Hence λA is (P^*-N)-quasi normal operator of order (n)

2) $$\left(\left(\frac{A}{M}\right)^*\right)^P\left(\left(\frac{A}{M}\right)^*\left(\frac{A}{M}\right)\right)^n=\left(\frac{(A^*)^P}{M}\right)\left(\frac{(A^*A)^n}{M}\right)$$
$$=\left(\frac{(A^*)^P(A^*A)^n}{M}\right)$$
$$=\left(\frac{N(A^*A)^n(A^*)^P}{M}\right)$$
$$=\left(\frac{N}{M}\frac{(A^*A)^n}{M}\frac{(A^*)^P}{M}\right)$$
$$=\left(N\frac{(A^*A)^n}{M}\frac{(A^*)^P}{M}\right)$$
$$=\left(N\left(\frac{(A^*)^n}{M}\frac{(A)^n}{M}\right)\frac{(A^*)^P}{M}\right)$$

Then $(\frac{A}{M})$ is (P^*-N)-quasi normal Operator of order (n).

CONCLUSIONS

The new type of quasi normal operator which is (k*-N)- quasi normal operator, have been given in this search, with some basic important theorems of properties and operations about this concept introduced, also using mathematical induction to prove some theorems in this work.

REFERENCES

[1] Brown, A. (1953). On a class of operators. Proceedings of the American Mathematical Society, 4(5), 723-728. https://doi.org/10.1090/S0002-9939-1953- 0059483-2

[2] Bala, A. (1977). A note on quasi-normal operators. Indian Journal of Pure and Applied Mathematics, 8, 463-465.

[3] [3] Bernau, S. J., & Smithies, F. (1963, October). A note on normal operators. In Mathematical Proceedings of the Cambridge Philosophical Society (Vol. 59, No. 4, pp. 727-729). Cambridge University Press. https://doi.org/10.1017/S0305004100003728

[4] [4] Mahmoud, S. A. O. A., & Ahmed, O. B. S. (2019). ON THE CLASSES OF (n, m)–POWER D–NORMAL AND (n, m)–POWER D–QUASI–NORMAL OPERATORS. Operatorsand Matrices, 13(3), 705-732. https://doi.org/10.7153/oam-2019-13-51

[5] [5] Ahmad, N., & JOUF, S. A. O. A. M. (2020). On the class of k-quasi-(n, m)-power normal operators. Hacettepe Journal of Mathematics and Statistics, 1-16. DOI : 10.15672/HJMS.xx

[6] Muhsin S. D., Khalaf A. M.(2017). On the class of (k-N) quasi-N-normal operators on Hilbert. Iraqi Journal of science, 58(4B), 2172-2176. https://doi.org/10.24996/ijs.2017.58.4B.2