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Finding Fixed Points for Set-Valued Mappings by Graph Concepts

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ABSTRACT

The researchers have presented some theorems of the fixed points of single-valued mappings by defining known contractive conditions on those points in the same path from a given graph. Here, this procedure will be modified and used to find fixed points of order-preserve mappings in a complete partially ordered g-metric space.

KEYWORDS: Fixed point, Directed graph, g-metric space, Set-valued mappings.

الخلاصة

لقد قدم الباحثون بعض مبرهنات من النقاط الصامدة لتطبيقات أحادية القيمة من خلال تحديد شروط انكماشية تعرف على تلك النقاط في نفس المسار من رسم بياني معين. هناء سوف يتم تعديل هذا الإجراء واستخدامها لإيجاد نقاط صامدة لتطبيقات تحافظ على الترتيب في الفضاء g- الكامل المرتب جزئيا.

INTRODUCTION

For set-valued mappings, Nadler [1] presented one of the most important research on fixed points in complete metric space. Then, fixed point theorems for set-valued mappings were established in different directions due to Reich[2], Many other results can see in [3-8] In 2005, Mustafa [9] introduced g- metric spaces, as, a generalization of a metric space (X, d). Subsequently, many fixed point results on such spaces appeared in [10-12]. Recently Jachymski [13] established a result of single-valued mapping in metric spaces with a graph instead of partial ordering. Beg and Butt [4-5] obtained sufficient conditions about the existence of fixed points by a graph. This article aims to employ previous ideas to present fixed points and common fixed points for set-valued in $g-$ metric spaces. These results relate to the content of the references [4-5, 13]. We begin with the following definition

Definition (1.1) [8]:

Let $\mathcal{M}$ be a nonempty set and ω: $\mathcal{M}^3 \rightarrow [0,\infty)$be a satisfying the following condition:

1-$\omega(p,q,e)$=0 if and only if $p=q=e$.
2-0< $\omega(p,p,q), \forall p,q \in \mathcal{M}$ with $p \neq q$.
3- $\omega(p,p,q) \leq \omega(p,q,e)$ for all $p,q,e \in \mathcal{M}$ with q $\neq e$.
4- $\omega(p,q,e) = \omega(p,e,q) = \cdots$,(symmetryin all three variables).
5- $\omega(p,q,e) \leq \omega(p,a,a) + \omega(a,q,e)$for all, $q,e,a \in \mathcal{M}$.
then the function ω is called generalized metric on $\mathcal{M}$ and the pair $(\mathcal{M},\omega)$ is called a g $-$metric space.

Example (1.2) [9]:

$\mathcal{M}$ $=R^+$, with usual distance $d(p,q) =\mid p-q\mid$, for all p,q in $\mathcal{M}$. Define ω:$\mathcal{M}^3 \rightarrow R^+$ ω $(p,q,e) = |p-q| + |q-e| + |e-p|$, for all $p,q,e \in \mathcal{M}$. Then $\omega$is a g $-$metric on $\mathcal{M}$.

Definition (1.3) [11]:

Let $(\mathcal{M}$,ω) be a g-metric space, then ω is called symmetric if $\omega(p,q,q) = \omega(p,p,q)$ for all ,q, $\in$ $\mathcal{M}$.

Example (1.4) [9]**:**
Let $\mathcal{M} = \{p, q\}$and$\omega(p,p,p) = \omega(q,q,q) = 0, \omega(p,p,q) = 1, \omega(p,q,q) = 2$ and by symmetry expand ω to all of $\mathcal{M}\, x\, \mathcal{M}\, x\, \mathcal{M}$. Then ω is a g −metric, but $\omega(p,q,q) \neq \omega(p,p,q)$.

Proposition (1.5) [12]**:**
Let $(\mathcal{M}, \omega\ )$ be a g −metric space, then the following are equivalent:
1- ($\mathcal{M}$,ω) is symmetric.
2-$\omega(p,q,q) \leq \omega(p,q,a), \forall p,q,a\ \in \mathcal{M}$.
3-
$\omega(p,q,e) \leq$
$\omega(p,q,a) + \omega(e,p,b), \forall p,q,e,a,b \in \mathcal{M}$.

Definition (1.6) [11-8]**:**
Let ($\mathcal{M}$, ω)be a g –metric space and $\{r_j\}$ be a sequence of points of $\mathcal{M}$, if there exist $L \in \mathbb{N}$ $\in >0\ for\ j,i,l \geq L$ then the sequence $\{r_j\}$ is said to be
i) ω - convergent to r if ω (r, r_j, r_i) <∊ for all $i\ ,j\ \geq L$ That is $\lim_{i,j\to\infty} \omega\ (r, r_j, r_i)$ =0 as $i\ ,j \to \infty$.
ii)ω – Cauchy if $\omega(r_j, r_i, r_l)$ <∊ for all $i\ ,j\ ,\ l \geq L$.That is $\omega(r_j, r_i, r_l) \to 0$ as $i\ ,j\ ,\ l \to \infty$.
iii) Ag–metric space ($\mathcal{M}$, ω) is complete if every ω-Cauchy sequence is ω-convergent in ($\mathcal{M}$, ω).

Proposition (1.7) [11]**:**
Let $(\mathcal{M}, \omega)$ be a g-metric space the following statements are equivalent
i)$\{r_j\}$ is ω-convergent to r, if and only if $\omega(r_j, r_j, r) \to 0, as j \to \infty$.
ii)$\omega(r_j, r, r) \to 0, as\ j \to \infty$. if and only if $\omega(r_j, r_i, r) \to 0, as\ j, i \to \infty$.

Remark (1.8) [9]**:**
Every g –metric ($\mathcal{M}$, ω) on $\mathcal{M}$ defines a metric d_ω on $\mathcal{M}$ given by
$d_\omega\ (p,q\ ) =\ \omega(p,q,q) + \omega(q,p,p)$ for all $p, q \in \mathcal{M}$ and
$\omega(p,q,e) = max\{\ |\ p - q\ \ |, |\ q - e\ \ |, |\ e - p\ \ |\}$.

Proposition (1.9) [9]**:**
Let ($\mathcal{M}$,ω) be ag −metric space, then for any p, q, e, and $a \in \mathcal{M}$ is following that
1. $If\ \omega(\ p\ ,q,e)\ = 0\ than\ p = q = e.$
2. $\omega(\ p\ ,q,e) \leq\ \omega(\ p\ ,p,q\ ) +\ \omega(\ q\ ,q,e\ )$.
3. $\omega(\ p\ ,q,q) \leq 2\ \omega(\ q\ ,p,p\ )$.
4. $\omega(\ p\ ,q,e\ ) \leq\ \omega(\ p\ ,a,e\ ) + \omega(\ a\ ,q,e\ )$.
5. $\omega(\ p\ ,q,e\ ) \leq\ 2/3\ (\ \omega\ p\ ,q,a\ ) + \omega(\ p\ ,a,e\ ) +\ \omega(\ a\ ,q,e\ ))$.
6. $\omega(\ p\ ,q,e\ ) \leq\ (\omega(\ p\ ,a,a\ ) + \omega(\ q\ ,a,a\ ) + \omega(\ e\ ,a,a\ ))$.

Below, if ($\mathcal{M}$,ω) is a g-metric space, $2^{\mathcal{M}} = \{A: \emptyset \neq A \subset \mathcal{M}\}$ and $CB(\mathcal{M}) = \{A: \emptyset \neq A \subset \mathcal{M}, A$ is closed & bounded$\}$ and $K(\mathcal{M}) = \{A: \emptyset \neq A \subset \mathcal{M}, A$ is compact$\}$,
and Ω={The Hausdorff}.

Definition (1.10) [1]**:**
The point p in $\mathcal{M}$ is called a fixed point of the set-valued mapping $S\colon \mathcal{M} \to 2^{\mathcal{M}}$ if $p \in Sp$ and p is a fixed point of a single mapping $S\colon \mathcal{M} \to \mathcal{M}$ if $p = Sp$.

Definition (1.11) [1]**:**
The mapping $H\colon \mathcal{M}^3 \ \to R^+$ is called the Hausdorff g −distance on CB($\mathcal{M}$), if
$\Omega(A,B,C) = max\{sup_{p\in A}\ \omega(p,B,C), sup_{p\in B}\ \omega(p,C,A), sup_{p\in C}\ \omega(p,A,B)\}$,
where $\omega(p,B,C) = d_\omega(p,B) + d_\omega(B,C) + d_\omega(p,C)$, $d_\omega(p,B) = inf\{d_\omega(p,q), q \in B\}$, $d_\omega(A,B) = \inf\{d_\omega(a,b), a \in A, b \in B and\ A, B, C \in \mathrm{CB}(\mathcal{M}),\}$.

Lemma (1.12) [1]**:**
i) If$A, B \in\ CB(\mathcal{M})$ with $\Omega(A,B,B)$< ε then $\forall a \in A\ \exists b \in B$ such that $\omega(a,b,b) <$ ε.
ii) If$A, B \in\ CB(\mathcal{M})$ and $a \in A$, then $\forall$ε $> 0, \exists\ b \in B$ such that $\omega(a,b,b) \leq \Omega(A,B,B) +$ ε.

Lemma (1.13) [11]**:**
i) If$A\ \in\ CB(\mathcal{M})$ and $B \in\ K(\mathcal{M})$ then $\forall\ a \in A$, $\exists\ b \in B$ such that:$\omega(a,b,b) \leq \Omega(A,B,B)$.
ii) Let $\{A_j\}$ be a sequence in $CB(\mathcal{M})$ and $\lim_{j\to\infty} \Omega(A_{j,} A, A) = 0$ for $A \in CB(\mathcal{M})$.If$p_j \in A_j$ and $\lim_{j\to\infty} \omega(p_j\ , p, p) = 0$, then $p \in A$.

Definition (1.14) [4-5]**:**
Let G_r be a graph with finite vertices denoted by $V(G_r)$ and finite edges $E(G_r)$ of different pairs of different elements of $V(G_r)$. Also, ${G_r}^{-1}$ denotes the converse of $G_r($, which is obtained by reversing the direction of its edges.

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Definition (1.15) [4-5]**:**
A graph $G_r^{\sim}$ is called directed if its edges are symmetric, then $E(G_r^{\sim}) \coloneqq E(G_r) \cup E(G_r^{-1})$.

Definition (1.16) [5-4]**:**
we say that H is a subgraph of G_r if $V(H) \subseteq V(G_r)$ and $E(H) \subseteq E(G_r)$.

Definition (1.17) [4-5]**:**
If each edge in G_r has an associated weight function $W: E(G_r) \to R$ then G_r is called A weighted graph.

Definition (1.18) [4-5]**:**
Let $p,\ q \in V(G_r)$. A path in G_r from p to q of length $j\ (j \in NU\{0\})$is a sequence $(p_i)_{i=0}^{j} \subseteq V(G_r) \ni p_0 = p, p_j = q$ and $(p_{i-1}, p_i) \in E(G_r)$, $i = 1,2,\ldots,j$.

Definition (1.19) [4-5]**:**
The length of the path is the number of elements in $E(G_r)$.

Definition (1.20) [4-5]**:**
If there is a path between any two vertices of G_r then G_r is called connected otherwise it is disconnected. Moreover, $G_r$is weakly connected if $G_r^{\sim}$ is connected.

Let G_{r_p} be the component of G_r, consisting of all edges and vertices which are contained in some path in$G_r$beginning at p. Assume that$G$ is such that $E(G_r)$ is symmetric, then The equivalence class $[p]_{G_r}$ defined on $V(G_r)$ by the rule R (uRv if there is a path from u to v) is $V(G_{r_p})= [p]_{G_r}$. See[13].

Jachymski [13] proved some fixed point results for the G_r-contraction mapping in a metric space endowed with a graph, and he stated the following results,

Definition (1.21) [13]**:**
Let $\mathcal{M}$ be a complete metric space. A single-valued mapping $S: \mathcal{M} \to \mathcal{M}$ is a Banach G_r-contraction if $(p,q) \in E(G_r)$ implies $(Sp, Sq) \in E(G_r)$, and $\forall (p,\ q) \in E(G_r)$ $\exists K \in (0,1) \ni \omega(Sp, Sq) < k\omega(p,q)$.

property A: for any sequence $(p_j)_{j\in N}$ in $\mathcal{M}$, if$p_j \to p$ and $(p_j, p_{j+1}) \in E(G_r)$ for $j \in N$, then $(p_j, p) \in E(G_r)$.

By using Banach G_r-contraction, Jachymski proved that:

Theorem (1.22) [13]**:**
Let $\mathcal{M}$ be a complete metric space with property A: for any sequence $(p_j)_{j\in N}$ in $\mathcal{M}$, if$p_j \to p$ and $(p_j, p_{j+1}) \in E(G_r)$ for $j \in N$, then there is a subsequence $(p_{kj})_{j\in N}$ with $(p_{kj}, p) \in E(G_r)$ for $j \in N$. Let $S: \mathcal{M} \to \mathcal{M}$ be a G_r-contraction and $\mathcal{M}_S := \{p \in \mathcal{M}: (p, Sp) \in E(G_r)\}$.
Then the following hold:

1. card Fix S = card$\{[p]_{G_r^{\sim}}: p \in \mathcal{M}_p\}$.
2. Fix $S \neq \emptyset$ if and only if $\mathcal{M}_S \neq \emptyset$.
3. S has a unique fixed point iff there exists $p_0 \in \mathcal{M}_p$ such that$\mathcal{M}_S \subseteq [p_0]_{G_r^{\sim}}$.
4. For any $p \in \mathcal{M}_S$, $S|[p_0]_{G_r^{\sim}}$ is a Picard operator.
5. If $\mathcal{M}_S \neq \emptyset$ and G_r is a weakly connected, then S is a Picard operator.

Beg and Butt [10] presented a version of Jachymski's Theorem for set-valued mappings as the following:

Definition (1.23) [13]**:**
Let $\mathcal{M}$ be a complete metric space. The mapping: $\mathcal{M} \to CB(\mathcal{M})$ is said to be a G_r-contraction if $\exists k \in (0,1) \ni \Omega\ (Sp\ Sq) < k(p,q)$ $\forall (p,q) \in E(G_r)$ and if $u \in Sp$ and $\in Sq$ $\ni \omega\ (u,v) < k\omega\ (p,q) + \alpha, \forall\ \alpha > 0$ then $(u,v) \in E(G_r)$.

Theorem (1.24) [13]**:**
Let $\mathcal{M}$be a complete metric space with property (A). Let$S: \mathcal{M} \to CB(\mathcal{M})$ be a G_r-contraction and $\mathcal{M}_S := \{p \in \mathcal{M}: (p,u) \in E(G_r)$ for some $u \in Sp\}$, then the following hold:

1. For any $p \in \mathcal{M}_S$, $S|[p]_{G_r^{\sim}}$has a fixed point.
2. If $\mathcal{M}s \neq \emptyset\ and\ G_r$ is weakly connected, then S has a fixed point in $\mathcal{M}$.
3. If $S' := U\{[p]_{G_r^{\sim}}: p \in \mathcal{M}_S\}$, then $S|_{p'}$ has a fixed point.
4. If $S \subseteq E(G_r)$ then S has a fixed point.
5. Fix$S \neq \emptyset \Leftrightarrow \mathcal{M}_S \neq \emptyset$.

MAIN RESULTS

Let $(\mathcal{M}, \omega)$ is a complete$g -$ metric space and G_r is a directed and weighted graph with $E(G_r)$ is symmetric such that $E(G_r)$ contains all loops, i.e., $\Delta \subseteq E(G_r)$, where Δ denote the diagonal of the Cartesian product $\mathcal{M} \times \mathcal{M}$.

Definition (2.1):

Let $S: \mathcal{M} \to CB(\mathcal{M})$ be a set-valued mapping. S is called a G_r-contraction if S preserves edges of G_r, i.e.,

$\forall p, q \in \mathcal{M}, (p,q) \in E(G_r) \Rightarrow (Sp, Sq) \in E(G_r)$

And $\exists k \in (0,1) \ni \Omega(Sp, Sq, Se) \leq k\omega(p,q,e), \forall p, q$ and e belong to the same path. Metaphorically, it can be the expression like the following: p, q and e belong to (G_r) .

Definition (2.2):

The mapping $S: \mathcal{M} \to CB(\mathcal{M})$ is said to be a G_r-contraction if there exists a $k \in (0,1)$ such that $\Omega\,(Sp\, Sq, Se) \leq k(p,q,e)$ for all $(p,q,e) \in E(G_r)$.

Definition (2.3):

Let $(\mathcal{M}, \omega)$ be a $g-$ metric space and $S, H, T: \mathcal{M} \to CB(\mathcal{M})$. The mappings S, H, T are said to be G_r -contractive if there exists $k \in (0,1)$ such that $(p \neq q \neq e), (p,q,e) \in \mathcal{M}$ then $\Omega(Sp, Hq, Te) < k\omega(p,q,e)$, and $u \in Sp$ and $v \in Hq, w \in Te$ with $\omega\,(u, v, w) \leq k\omega(p,q,e)$ then $(u,v,w) \in E(G_r)$.

The symmetry of Ω and ω implies the following:

Proposition (2.4):

If $S: \mathcal{M} \to CB(\mathcal{M})$ is a G_r-contraction then S is also a $G_r{}^{-1} -$ contraction.

The following property is needed

Property (B):

If $u \in Sp$, $v \in Sq$ and $w \in Se \ni \omega\,(u,v,e) < k\omega\,(p,q,e) + \alpha, \forall \alpha > 0$ then u, v and w belong to the path of length 2.

Theorem (2.5):

Suppose that the triple $(\mathcal{M}, \omega, G_r)$ has the properties (A-B). Let $S: \mathcal{M} \to CB(\mathcal{M})$ be a G_r-contraction and $\mathcal{M}_S := \{p \in \mathcal{M}: (p,u,v) \in E(G_r)$ for some $u \in Sp, v \in Su\}$.

Then the following hold:

1. For any $p \in \mathcal{M}_S$, $S\big|[p]_{G_r^{\sim}}$ has a fixed point.

2. If $\mathcal{M}s \neq \emptyset\ and\ G_r$ is weakly connected, then S Fix$S \neq \emptyset$□ $\mathcal{M}$.

3. If $S' := U\{[p]_{G_r^{\sim}}: p \in \mathcal{M}_S\}$, then Fix $S\,|_{p'} \neq . \emptyset$

4. If $S \subseteq E(G_r)$ then Fix $S \neq \emptyset$.

5. Fix$S \neq \emptyset \Leftrightarrow \mathcal{M}_S \neq \emptyset$.

Proof: 1. Let$p_0 \in \mathcal{M}_S$then there exists $p_1 \in Sp_0, p_2 \in Sp_1$ such that p_0, p_1, p_2 belong to the same path, metaphorically, $p_0, p_1, p_2 \in E(G_r)$. Since S is G_r-contraction, we have

$\Omega\,(Sp_0, Sp_1, Sp_2) \leq k\omega(p_0, p_1, p_2)$.

Using Lemma1.12,and property (B), we have the existence of an $p_3 \in E(G_r)$ such that

$\omega(p_1, p_2, p_3) \leq \Omega(Sp_0, Sp_1, Sp_2) + k \leq k\,\omega(p_0, p_1, p_2) + k$. (1)

Again let $p_1 \in Sp_0, p_2 \in Sp_1, p_3 \in Sp_2$, such that p_1, p_2, p_3 belong to the same path, $p_1, p_2, p_3 \in E(G_r)$. Since S is G_r-contraction, we have

$\Omega(Sp_1, Sp_2, Sp_3) \leq k\omega(p_0, p_1, p_2)$

Also, by property (B), we have the existence of an $p_4 \in E(G_r)$ such that

$\omega(p_2, p_3, p_4) \leq \Omega(Sp_1, Sp_2, Sp_3) + k^2$ (2)

Using (1) in (2), we obtain

$\omega(p_2, p_3, p_4) \leq k^2\,\omega(p_0, p_1, p_2) + 2k^2$ (3)

Continuing in this way we have $p_{j+1} \in E(G_r)$ and

$\omega(p_j, p_{j+1}, p_{j+1}) \leq k^j\,\omega(p_0, p_1, p_2) + jk^j$ (4)

Now, we prove that $\{p_j\}$ is $g -$ Cauchy sequence in $\mathcal{M}$.

$\sum_{j=0}^{\infty} \omega\,(p_j, p_{j+1}, p_{j+2}) \leq \omega(p_0, p_1, p_2) \sum_{j=0}^{\infty} k_j + \sum_{j=0}^{\infty} k_j < \infty$.

Thus $\{p_j\}$ is a $g-$ Cauchy sequence. By completeness, it converges to p in $\mathcal{M}$.

The next step is devoted to showing that p is a fixed point of S. By applying property (A) and G_r-contractivaty of S, we have $\Omega\ (Sp_j, Sp, Sp) \leq k\omega(p_j, p, p)$. Since $p_{j+1} \in Sp_j$and$p_j \to p$, therefore by Lemma 1.13, $p \in Sp$. So, $(p_j, p) \in E(G_r)$for $j \in N$, we infer that $(p_0, p_1, \ldots. p_j, p)$ is a path in G_r and so $p \in [p_0]_{G_r^{\sim}}$.

2. Since$\mathcal{M}_S \neq \emptyset$, $\exists p_0 \in \mathcal{M}_S$. And G_r is weakly connected, thus $[p_0]_{G_r^{\sim}} = \mathcal{M}$ and by part 1, S has a fixed point.

3. From 1 and 2, it holds.

4. $S \subseteq E(G_r) \Rightarrow$ for all $\in \mathcal{M}$ $u \in Sp$ and p, u, belong to the same path, metaphorically, $p, u, \in E(G_r)$ so$\mathcal{M}_S = \mathcal{M}$and by 2 and 3, S has a fixed point

5. Let Fix$S \neq \emptyset \Rightarrow \exists p \in$Fix$S \ni p \in Sp$.As $\Delta \subseteq E(G_r)$; so, $(p,p,p) \in E(G_r) \Rightarrow p \in \mathcal{M}_S$. So $\mathcal{M}_S$

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$\neq \emptyset$. Conversely, if $\mathcal{M}_S \neq \emptyset$ then Fix$S \neq \emptyset$ follows from 2 and 3.

Corollary (2.6):

Let $(\mathcal{M}, \omega, g)$ have the property A and G_r is weakly connected then every G_r-contraction $S : \mathcal{M} \to CB(\mathcal{M}) \ni p_0, p_1,$ belong to the same path, metaphorically, $p_0, p_1, \in E(G_r)$. For some $p_1 \in Sp_0$ has a fixed point.

Theorem (2.7):

Let $S, H, T: \mathcal{M} \to CB(\mathcal{M})$ be G_r -contractive with properties A, B. Set $\mathcal{M}_S := \{p \in \mathcal{M}: (p, u, v) \in E(G_r)$ for some $u \in Sp, v \in Su\}$. Then the following hold:

1. For any $p \in Sp, S, H, T\big|_{[p]_{G_r}}$ have a common fixed point.
2. If $\mathcal{M}_S \neq \emptyset$ and G_r is weakly connected, then F, H, T have a common fixed point in $\mathcal{M}$.
3. If $\mathcal{M}' := U\{[p]_{G_r}: p \in \mathcal{M}S\}$, then $Fix\ F \cap Fix\ H \cap FixT\ |_{p'} \neq \emptyset$.
4. If $S \subseteq E(G_r)$ then $Fix\ F \cap Fix\ H \cap FixT \neq \emptyset$

Proof.

1. Let $p_0 \in \mathcal{M}_S$ then there exists $p_1 \in Sp_0$ and $p_2 \in Hp_1$, such that $p_0, p_1, , p_2$ belong to the same path of length 2, metaphorically, $p_0, p_1, , p_2 \in E(G_r)$. Since S, H, T are G_r-contractive, we have $\Omega\ (Sp_0, Hp_1, Tp_2) \leq k\omega(p_0, p_1, p_2)$.
Using Lemma 1.12, and property (B), we have the existence of $p_3 \in Tp_2$ such that
$\Omega\ (p_1, p_2, p_3) < k\omega\ (p_0, p_1, p_2)$
(5)
Again let $p_1 \in Sp_0, p_2 \in Sp_1, p_3 \in Sp_2$, such that p_1, p_2, p_3 belong to the same path, $p_1, p_2, p_3 \in E(G_r)$. Since S, H, T is G_r-contraction and since $E(G_r)$ is symmetric, we have

$$\Omega\ (Sp_2, Hp_1, Tp_0) < k\omega(p_1, p_2, p_3) < k^2\omega(p_0, p_1, p_2).$$

Lemma 1.12gives the existence of $p_4 \in Sp_3$ such that

$$\omega(p_2, p_3, p_4) < k^2\ \omega(p_0, p_1, p_2) \qquad (6)$$

Continuing in this way we have $p_{2j+1} \in Sp_{2j}$ and $p_{2j+2} \in Hp_{2j+1}$, and $p_{2j+3} \in Tp_{2j+2}$, $j = 0, 1, 2..$ Also $p_{2j+1}, p_{2j+2}, p_{2j+3}$ belong to the same path of length 2, metaphorically, $p_{2j+1}, p_{2j+2}, p_{2j+3} \in E(G_r)$ such that

$$\omega(p_j, p_{j+1}, p_{j+2}) < k^j\ \omega(p_0, p_1, p_2) \qquad (7)$$

Next, we will show that $\{p_j\}$ is a g-Cauchy sequence in $\mathcal{M}$. Let $i > j$. Then

$$\begin{aligned}\omega(p_j, p_i, p_i) &\leq \omega(p_j, p_{j+1}, p_{j+1}) \\ &\quad + \omega(p_{j+1}, p_{j+2}, p_{i+2}) \\ &\quad + \cdots \omega(p_{i-1}, p_i, p_i) \\ &< [k^j + k^{j+1} + \cdots + k^{i-1}]\omega(p_0, p_1, p_1) \\ &= k^j[1 + k^j + \cdots + k^i]\omega(p_0, p_1, p_1) \\ &= k^j[1 - k^{i-j}/1 - k]\omega(p_0, p_1, p_1).\end{aligned}$$

Because $k \in (0, 1), 1 - k^{i-j} < 1$.
Therefore $\omega(p_j, p_i, p_i) \to 0$ as $j \to \infty \Rightarrow \{p_j\}$ is a g − Cauchy sequence, ⇒converges to $p \in \mathcal{M}$.
To show that $p \in Sp \cap \mathrm{H}p \cap Tp$. For j even: By (A-B), p_j, p belong to the same path, $p_j, p \in E(G_r)$. Therefore using G_r − contractivity, we have

$$\Omega\ (Sp_j, Hp, Tp) < k\omega(p_j, p, p).$$

Since $p_{j+1} \in Sp_j, and\ p_j \to p.$ by Lemma 1.13 $\Rightarrow Hp, p \in Tp$. For j odd:
As $p_j, p,$ belong to the same path $p_j, p \in E(G_r)$, $\Omega(Sp, Hp, Tp) < k\omega(p, p_j, p_j)$
Hence, the same arguments as above $p \in Sp$.
Next as p_j , p_{j+1} belong to the same path p_j , $p_{j+1} \in E(G_r)$ also p_j, p belong to the same path $p_j, p, \in E(G_r)$, for $j \in N$. We infer that $(p_0, p_1, \ldots . p_j, p)$ is a path in G_r and so $p \in [p_0]_{G_r}$.
2. Since $\mathcal{M}_S \neq \emptyset$, $\Rightarrow \exists\ p_0 \in \mathcal{M}_S$ and since G_r is weakly connected then $[p_0]_{G_r} = \mathcal{M}$ and by 1, mappings S and H, T have a common fixed point in $\mathcal{M}$.
3. It follows from parts 1 and 2.
4. $S \subseteq C\ E(G_r) \Rightarrow$ all $p \in \mathcal{M}$ be such that there exists some $u \in Sp$ belong to the same path $p, u \in E(G_r)$ so $\mathcal{M}_S = \mathcal{M}$ and by 2 and 3.S, H, T have a fixed point.

Remark (2.8):

Replace $\mathcal{M}_S$ by $\mathcal{M}_H := \{p \in \mathcal{M}: (p, u, v) \in E(G_r)$ for some $u \in Hp, v \in Hu\}$ in conditions 1-3 of Theorem 2.7, the conclusion remains true. That is if $\mathcal{M}_S U \mathcal{M}_H U \mathcal{M}_T$ then getting Fix$S$ ∩Fix$H$ ∩ Fix$T \neq \emptyset$ which follows easily from 1-3. Similarly, in condition 4 we can replace $S \subseteq E(G_r)$ by $H \subseteq E(G_r)$.

Corollary (2.9):

Let the triple $(\mathcal{M}, \omega, g)$ have the property (A). If G_r is weakly connected then G_r-contractive mappings $S, H, T : \mathcal{M} \to CB(\mathcal{M})$ such that (p_0, p_1, p_2) for some $p_1 \in Sp_0$has a common fixed point

Corollary (2.10):

Let $(\mathcal{M}, \omega)$ be a ε-chainable complete $g-$ metric space for someε > 0. Let S, $H, T: \mathcal{M} \to CB(\mathcal{M})$be such that there exists $k \in (0, 1)$ with $0 < \omega(p, q, e) < \epsilon = \Omega(Sp, Hq, Te) < k\omega(p, q, e)$. Then S and H, T have a common fixed point.

Proof.

$E(G_r) = \{(p, q) \in \mathcal{M} \times \mathcal{M} \ \ 0 < \omega(p, q, e) < \epsilon\}$ (8)

The ϵ-chainability of $(\mathcal{M}, \omega)$ means G_r is connected.

If $p, q, e,$ belong to the same path, $p, q, e \in E(G_r)$, then

$\Omega(Sp, Hq, Te) < k\omega(p, q, e) < k\epsilon < \epsilon$

and using Lemma 1.12, for each $u \in Sp$ we have the existence of $v \in Hq$ and $w \in Te$ such that $\omega(u, v, w) < \epsilon$ which implies $u, v, w,$ belong to the same path $u, v, w \in E(G_r)$. Hence S and H , T are G_r-contractive.

Also $(\mathcal{M}, \omega, g)$ has property (A). Indeed if $p_j \to p$, then $\omega(p_j, p, p) < \epsilon$ for sufficiently j. therefore $p_j, p,$ belong to the same path $p_j, p, \in E(G_r)$. So, by Theorem 2.7 (2); S and H, T have a common fixed point.

Theorem (2.11):

Let $S, H, T : \mathcal{M} \to CB(\mathcal{M})$be G_r -contractive mappings and properties A, B hold. Set$\mathcal{M}_S := \{p \in \mathcal{M}: (p, u, v) \in E(G_r)$ for some$u \in Sp, v \in Su\}$. Then the following hold:

1. For any $p \in Sp, S, H, T|_{[G_r]}$ has a common fixed point.
2. If $\mathcal{M}_S \neq \emptyset$ and G_r is weakly connected, then $\emptyset \neq Fix\ F \cap Fix\ H \cap FixT$ □ $\mathcal{M}$.
3. If $\mathcal{M}' := \cup\{[p]_{G_r}: p \in \mathcal{M}_S\}, Fix\ F \cap Fix\ H \cap FixT\ |_{p'} \neq \emptyset$.
4. If $S \subseteq E(G_r)$ then $Fix\ F \cap Fix\ H \cap FixT \neq \emptyset$.
5. If $\mathcal{M}_S \neq \emptyset$then Fix$S \neq \emptyset$.

Proof. The parts 1-4 can be proved by putting $S = H$ in Theorem 2.7 and 5 obtained from the Remark 2.8 We observe that the symmetric of $E(G_r)$ is not needed in Theorem 2.11.

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