

# A-Contraction Mappings of Integral Type in $n$ -Normed Spaces

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## ABSTRACT

In this article, A-contraction type mappings in integral case are defined on a complete  $n$ -normed spaces and the existence of some fixed point theorems are proved in the complete  $n$ -normed spaces and given some results on Picard operator.

**Keywords:**  $n$ -Normed spaces, Picard iteration sequence, contractive condition, integral mapping, Primary 46B20; Secondary 46C05, 46C15, 46B99

## الخلاصة

في هذه المقالة، يتم تعريف تطبيقات نوع A-انكماشية في حالة التكامل على فضاءات  $n$ -المعيارية الكاملة وأثبتت وجود بعض مبرهنات نقطة الصامدة في فضاءات  $n$ -المعيارية الكاملة واعطاء لبعض النتائج حول مؤثر بيكار.

## INTRODUCTION

Gähler presented the first developing for the theory of 2-normed spaces while that of  $n$ -normed spaces can be found in [1]. For every real normed space  $X$  of dimension  $d \geq n$ , Gähler showed that  $X$  can be viewed as an  $n$ -normed space by using the Gähler  $n$ -norm, which is denoted by  $\|., \dots, .\|_G$ . This  $n$ -norm is defined by using the set of bounded linear functionals on  $X$ . Since then, many researchers have studied operators and functionals on  $n$ -normed space  $X$  (see [2,3,4,5,6,7,8,9,10,11,12]). In [5], Pangalea and Gunawan introduced the concept of  $n$ -dual spaces and proved that for every real normed space  $X$  of dimension  $d \geq n$ , there are two  $n$ -dual spaces associated to  $X$ . The first is the  $n$ -dual space of  $(X, \|\cdot\|)$ , and the other is the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$ . In case  $X$  is the  $l^p$  space for some  $1 \leq p < \infty$ , Pangalea and Gunawan have investigated and given the relationship between both  $n$ -dual spaces [5]. Gunawan (see, [12]) investigated the  $n$ -dual space of  $(X, \|\cdot\|)$  by using the  $(n-1)$ -dual space of  $(X, \|\cdot\|)$ . Then, focused on a real normed space  $X$  of dimension  $d \geq n$  which satisfies a property (G) and discussed the

relationship between the  $n$ -dual space of  $(X, \|\cdot\|)$  and the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$ . An observation that both the  $n$ -dual space of  $(X, \|\cdot\|)$  and the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$  are Banach spaces. Soenjaya [6] proved an extension of the Open Mapping Theorem to the case of  $n$ -Banach space. Also, introduced the notions of  $n$ -bounded and  $n$ -continuous operator and proved some important results introduced the notions of  $n$ -bounded and  $n$ -continuous operator and proved some important results. Gunawan (see [11]) offered a definition of  $n$ -inner products which is simpler than (but equivalent to) the one formulated by Misiak [8]. Also proved the Cauchy-Schwarz inequality and gave a necessary and sufficient condition for the equality. Dutta and Mazaheri [9], introduced the idea of constructing sequence spaces with elements in an  $n$ -norm space in comparison with the spaces  $c_0$ ,  $c$ ,  $\infty$  and the Orlicz space and extended the notion of  $n$ -norm to such spaces. Also, stated and defined some statements about the  $n$ -best approximation in  $n$ -normed spaces. Gunawan and Mashadi [10], offered a simple way to derive an  $(n-1)$ -norm from the  $n$ -norm and realized that any  $n$ -normed

space is an  $(n-1)$ -normed space. Also showed that, in certain cases, the  $(n-1)$ -norm can be derived from the  $n$ -norm in such a way that the convergence and completeness in the  $n$ -norm is equivalent to those in the derived  $(n-1)$ -norm. Using this fact, proved a fixed point theorem for some  $n$ -Banach spaces. In 2015, K. Matsuzaki [13] showed uniform convexity's generalization in the geometric group theory and proved the fixed point property. Normal structure plays essential role in some problems of fixed point theory. Matsuzaki's results applied to prove fixed point theorems for nonexpansive type mappings in normal  $n$ -normed spaces. Meng and Song [14] extended the definitions of normal structure and geometric properties in  $n$ -normed spaces. Fixed point theorems for nonexpansive mappings are proved via the normal structure condition in  $n$ -normed spaces.

**Definition (1.1)** [2]: Let  $H$  be a real linear space with  $\dim H \geq n, n \in N$  and  $\|., \dots, .\|: H^n \rightarrow [0, \infty)$  be a function. Then  $(H, \|., \dots, .\|)$  is called a linear  $n$ -normed space, if for all  $w_1, \dots, w_n \in H$ , and  $\alpha \in R$ .

(N<sub>1</sub>)  $\|w_1, \dots, w_n\| = 0 \Leftrightarrow w_1, \dots, w_n$  are linearly dependent.

(N<sub>2</sub>)  $\|w_1, \dots, w_n\|$  is invariant under permutation.

(N<sub>3</sub>)  $\|w_1, \dots, w_{n-1}, \alpha w_n\| = |\alpha| \|w_1, \dots, w_{n-1}, w_n\|, \forall \alpha \in R$

(N<sub>4</sub>)  $\|w_1, \dots, w_{n-1}, u + v\| \leq \|w_1, \dots, w_{n-1}, u\| + \|w_1, \dots, w_{n-1}, v\|, \forall u, v \in R$ .

A usual example of an  $n$ -normed space is the following

**Example (1.2)** [15]: As an example of an  $n$ -normed space, we may take  $H = R^n$ , equipped with the Euclidean  $n$ -norm

$$\|w_1, w_2, \dots, w_n\|_E = |\det(w_{ij})| = \text{abs} \left( \begin{vmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{vmatrix} \right)$$

where  $w_i = (w_{i1}, \dots, w_{in}) \in R^n$  for each  $i = 1, 2, \dots, n$ .

$\|w_1, w_2, \dots, w_n\|_E$  = the volume of the  $n$ -dimensional parallelepiped spanned by the vectors  $w_1, w_2, \dots, w_n$ , in  $H$ . In special case  $n = 1$ , (If  $H$

$= R$ ,  $R$  is the set of real numbers)  $\|w\|_E$  = the absolute value of  $w, |w|$ .

**Remark (1.3)** [16]: In an  $n$ -normed space  $(H, \|., \dots, .\|)$ , we have

$$1-\|w_1, \dots, w_n\| \geq 0,$$

$$2-\|w_1, \dots, w_{n-1}, w_n\| = \|w_1, \dots, w_{n-1}, w_n + \alpha_1 w_1 + \dots + \alpha_n w_n\|, \forall w_1, \dots, w_n \in H \quad \text{and} \quad \alpha_1, \dots, \alpha_n \in R.$$

**Definition (1.4)** [15]: A sequence  $\{u_n\}$  in a  $n$ -normed space  $(H, \|., \dots, .\|)$  is said to be a Cauchy sequence if  $\lim_{n,m \rightarrow \infty} \|w_1, \dots, w_{n-1}, u_n - u_m\| = 0$ .

**Definition (1.5)** [15]: A sequence  $\{u_n\}$  in a  $n$ -normed space  $(H, \|., \dots, .\|)$  is said to be convergent if there is a point  $u$  in  $H$  such that  $\lim_{n \rightarrow \infty} \|w_1, \dots, w_{n-1}, u_n - u\| = 0$ , if  $\{u_n\}$  converges to  $u$  we write  $u_n \rightarrow u$  as  $n \rightarrow \infty$ , for every  $w_1, \dots, w_{n-1}$  in  $H$ .

**Definition (1.6)** [16]: A  $n$ -normed space is said to be complete if every Cauchy sequence is convergent to an element of  $H$ . A complete  $n$ -normed space  $H$  is called  $n$ -Banach space.

We give some topological concepts in  $n$ -normed space.

**Definition (1.7)** [15]: Let  $(H, \|., \dots, .\|)$  be a linear  $n$ -normed space,  $G$  be a subset of  $H$  then the closure of  $G$  is  $\bar{G} = \{u \in H; \text{there is a sequence } u_n \text{ of } G: u_n \rightarrow u\}$ . We say  $G$  is sequentially closed if  $G = \bar{G}$ .

**Theorem (1.8)** [11]: Let  $(H, \|., \dots, .\|)$  be a linear  $n$ -normed space,  $B$  be a nonempty subset of  $H$  and  $u \in B$ , then  $B$  is said to be  $u$ -bounded if there exist some  $M > 0, \|w_1, \dots, w_{n-1}, u\| \leq M, \forall u \in B$ . If for all  $u \in B, B$  is  $u$ -bounded then  $B$  is called a bounded set.

**Definition (1.9)** [11,17]: Let  $(H, \|., \dots, .\|)$  be a linear  $n$ -normed space. Then the mapping  $T: H \rightarrow H$  is said to be an  $A$ -contraction if there exists  $h \in [0, 1)$ ,

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq h \|w_1, \dots, w_{n-1}, u - v\|, \forall w_1, \dots, w_{n-1}, u, v \in H.$$

**Definition (1.10)** [12]: Let  $(H, \|., \dots, .\|)$  be a  $n$ -normed space. A mapping  $T: H \rightarrow H$  is called

Picard mapping if  $\exists u^* \in H \ni F_T = \{u^*\}$  and  $T^n(u_0) \rightarrow u^*, \forall u_0 \in H$ , where  $F_T$  is the set of fixed point of  $T$ .

In the following, we recall the fixed point theorem for a contractive mappings on a closed and bounded subset concerning

$$W = \{w_1, \dots, w_n\}.$$

**Theorem (1.11)** [11]: Let  $(H, \|\cdot, \dots, \cdot\|)$  be a  $n$ -Banach space and  $B \subset H$  be a nonempty closed and bounded with respect to  $W$ . If  $T : B \rightarrow B$  is contraction mapping with respect to  $W$ , then  $T$  has a unique fixed point in  $B$ .

Branciari [19], in 2002, studied the existence of fixed point for mapping with a general integral contractive condition of type in a complete metric space and proved the following:

**Theorem (1.12)** [19]. Let  $(H, d)$  be a complete metric space,  $h \in (0; 1)$  and let  $T : H \rightarrow H$  be a mapping such that for all  $u, v \in H$ ,

$$\int_0^{d(Tu, Tv)} \mu(t) dt \leq h \int_0^{d(u, v)} \mu(t) dt. \quad (1)$$

where  $\mu : [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0, +\infty)$ , non-negative, and such that for each  $\varepsilon > 0, \int_0^\varepsilon \mu(t) dt > 0$ , then  $T$  has a unique fixed point  $k \in H$  such that for each  $u \in H, \lim_{n \rightarrow \infty} T^n u = k$ .

Next, some research works have been executed to extending contractive conditions of integral type for distinct contractive mappings satisfying appropriate properties, see, Rhoades [20], Tarsh [17], Lazam and Abed [12]. Inspired and motivated by these consequent works, similarly, some results are presented in complete  $n$ -normed spaces which in turn generalize fixed point results. Also, the existence of fixed point of mapping [21] in the setting on  $n$ -normed spaces are analyze and existence. We will follow previous work to obtain similar results in the  $n$ -spaces

## MAIN RESULTS

A class of contraction mappings, called  $A$ -contraction, introduced by Akram et al. [22]. Below, we reform in  $n$ -normed spaces as the following:

Let  $A = \{\alpha: R_+^3 \rightarrow R_+ \text{ is a function satisfies condtions } N_1 \text{ and } N_2\}$ , where

(z<sub>1</sub>)  $\alpha$  is continuous in three variables under the Euclidean norm on  $R^3$ .

(z<sub>2</sub>) If  $c \leq \alpha(c, d, d)$  or  $c \leq \alpha(d, c, d)$  or  $c \leq \alpha(d, d, c)$  for all  $c, d$  then  $\exists h \in [0, 1) \ni c \leq hd$ .

**Definition (2.1):** A self mapping  $T$  on a  $n$ -normed space  $H$  is said to be  $A$ -contraction, if  $\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq \alpha[\|w_1, \dots, w_{n-1}, u - v\|, \|w_1, \dots, w_{n-1}, u - Tu\|,$

$\|w_1, \dots, w_{n-1}, v - Tv\|]$ , for all  $u, v \in H$  and some  $\alpha \in A$ .

**Example (2.2):** Let  $(H, \|\cdot, \dots, \cdot\|)$  be a  $n$ -normed space and  $T: H \rightarrow H$ ,  $T$  is  $A$ -contractions if  $T$  satisfies one of the following:

i) Kannan's type condition [18]:

$$\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq s[\|w_1, \dots, w_{n-1}, u - Tu\| + \|w_1, \dots, w_{n-1}, v - Tv\|], \quad s \in [0, \frac{1}{2}).$$

ii)  $\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq$

$$h \sqrt{\|w_1, \dots, w_{n-1}, u - Tu\| \|w_1, \dots, w_{n-1}, v - Tv\|},$$

$h \in [0, \frac{1}{2}).$

Firstly, we need to prove that  $T$  is Kannan's type whenever it satisfies the condition in (ii)

Let  $T: H \rightarrow H$  be as in (ii). Then there exists some  $h \in [0, 1)$  this condition. By the geometric mean of two positive real numbers  $p, q$  always precedes their arithmetic mean, that is  $\sqrt{pq} \leq \frac{p+q}{2}$ . So that  $h\sqrt{pq} \leq h\frac{p+q}{2}, \forall h \in [0, 1)$ . Hence with

$$p = \|w_1, \dots, w_{n-1}, Tu - u\|, q = \|w_1, \dots, w_{n-1}, v - Tv\|, \text{ we have}$$

$$h\sqrt{\|w_1, \dots, w_{n-1}, Tu - u\| \|w_1, \dots, w_{n-1}, Tv - v\|} \leq s\{\|w_1, \dots, w_{n-1}, Tu - u\| + \|w_1, \dots, w_{n-1}, Tv - v\|\}$$

For all  $s \in [0, \frac{1}{2})$  and for all  $u, v \in H$ . This, together with (ii), implies to

$$\frac{\|w_1, \dots, w_{n-1}, Tu - Tv\| \leq}{h\sqrt{\|w_1, \dots, w_{n-1}, Tu - u\| \|w_1, \dots, w_{n-1}, Tv - v\|}} \leq s[\|w_1, \dots, w_{n-1}, Tu - u\| + \|w_1, \dots, w_{n-1}, Tv - v\|].$$

Let  $T : H \rightarrow H$  be a Kannan's type, therefore,  $\exists s \in [0, \frac{1}{2})$  such that (i) holds for all  $u, v \in H$ . Keeping one such  $s$  fixed, we define a map  $\beta : R_+^3 \rightarrow R_+$  as  $\beta(g, p, q) = s(p + q), \forall g, p, q \in R_+$ .  $\beta$  is continuous since addition and multiplication of reals are continuous.

**Case I:** If  $g \leq \beta(g, p, p) = s(p + p)$  then  $g \leq kp$  with  $k = 2s \in [0, 1)$ .

**Case II:** If  $g \leq \beta(p, g, p) = s(g + p)$  then  $g \leq s(g + p)$  gives  $g \leq \frac{s}{1-s}p = kp$  with  $k = \frac{s}{1-s} \in [0, 1)$ .

**Case III:** If  $g \leq \beta(p, p, g)$  we have  $g \leq kp$  with  $k = \frac{s}{1-s} \in [0, 1)$ . So, in any case  $g \leq kp$  for some  $k \in [0, 1)$ . Hence  $\beta \in A$ .

Now, by taking

$$g = \|w_1, \dots, w_{n-1}, u - v\|, p = \|w_1, \dots, w_{n-1}, Tu - u\| \quad \text{and} \quad q = \|w_1, \dots, w_{n-1}, Tv - v\|$$

and using the inequality in (i), we get that

$$\begin{aligned} \|w_1, \dots, w_{n-1}, Tu - Tv\| &\leq \delta(\|w_1, \dots, w_{n-1}, Tu - u\| + \|w_1, \dots, w_{n-1}, Tv - v\|) \\ &= \beta(\|w_1, \dots, w_{n-1}, u - v\|, \|w_1, \dots, w_{n-1}, Tu - u\|, \|w_1, \dots, w_{n-1}, Tv - v\|), \forall u, v \in H. \end{aligned}$$

$\forall u, v \in H$ .

This shows that  $T$  is an A-contraction whenever  $T$  be as in (i) or (ii).

The summability has an active role in the next result.

**Definition (2.3):** A Lebesgue-integrable function  $\mu : [0, +\infty) \rightarrow [0, +\infty)$  is summable if  $\int |\mu(t)|^2 dt < \infty$  for each compact subset of  $[0, +\infty)$ .

**Theorem (2.4):** Let  $(H, \|\cdot, \dots, \cdot\|)$   $n$ -Banach space,  $T: H \rightarrow H$  and

$$\int_0^{\|w_1, \dots, w_{n-1}, Tu - Tv\|} \mu(t) dt \leq \alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, u - v\|} \mu(t) dt, \int_0^{\|w_1, \dots, w_{n-1}, u - Tu\|} \mu(t) dt, \int_0^{\|w_1, \dots, w_{n-1}, v - Tv\|} \mu(t) dt \right) \quad (2)$$

For all  $u, v \in H$  and some  $\alpha \in A$ , where  $\mu : [0, +\infty) \rightarrow [0, +\infty)$  is a summable such that

$$\text{For all } \varepsilon > 0, \int_0^\varepsilon \mu(t) dt > 0 \quad (3)$$

Then  $T$  is Picard operator.

**Proof:** Let  $u_0 \in H$ , define a sequence  $u_{n+1} = Tu_n$ . For each  $n \in N, n \geq 1$ , from (2) we get

$$\begin{aligned} \int_0^{\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\|} \mu(t) dt &= \int_0^{\|w_1, \dots, w_{n-1}, Tu_{n-1} - Tu_n\|} \mu(t) dt \\ &\leq \alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, u_{n-1} - u_n\|} \mu(t) dt, \int_0^{\|w_1, \dots, w_{n-1}, u_{n-1} - Tu_{n-1}\|} \mu(t) dt, \int_0^{\|w_1, \dots, w_{n-1}, u_n - Tu_n\|} \mu(t) dt \right) \\ &\leq \alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, u_{n-1} - u_n\|} \mu(t) dt, \int_0^{\|w_1, \dots, w_{n-1}, u_{n-1} - u_n\|} \mu(t) dt, \int_0^{\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\|} \mu(t) dt \right) \end{aligned}$$

By the condition  $N_2$  of function  $\alpha$  then

$$\int_0^{\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\|} \mu(t) dt \leq h \int_0^{\|w_1, \dots, w_{n-1}, u_{n-1} - u_n\|} \mu(t) dt \quad (4)$$

for some  $h \in [0, 1)$  as  $\alpha \in A$ . In this fashion, one can obtain

$$\begin{aligned} \int_0^{\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\|} \mu(t) dt &\leq h \int_0^{\|w_1, \dots, w_{n-1}, u_{n-1} - u_n\|} \mu(t) dt \\ &\leq h^2 \int_0^{\|w_1, \dots, w_{n-1}, u_{n-2} - u_{n-1}\|} \mu(t) dt \leq \dots \leq h^n \int_0^{\|w_1, \dots, w_{n-1}, u_0 - u_1\|} \mu(t) dt \end{aligned}$$

Taking limit as  $n \rightarrow \infty \Rightarrow \lim_{n \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\|} \mu(t) dt = 0,$

as  $h \in [0,1)$ .

Which from (3) implies that,  
 $\lim_{n \rightarrow \infty} \|w_1, \dots, w_{n-1}, u_n - u_{n+1}\| = 0$ .  
 (5)

Now, to show that  $\{u_n\}$  is a Cauchy sequence. If it is not. Then  $\exists \varepsilon > 0$  and subsequences  $\{m_q\}$  and  $\{n_q\} \ni m_q < n_q < m_{q+1}$  with  
 $\|w_1, \dots, w_{n-1}, u_{m_q} - u_{n_q}\| \geq \varepsilon$ ,

$$\|w_1, \dots, w_{n-1}, u_{m_q} - u_{n_{q-1}}\| < \varepsilon. \quad (6)$$

Now

$$\begin{aligned} & \|w_1, \dots, w_{n-1}, u_{m_{q-1}} - u_{n_{q-1}}\| \\ & \leq \|w_1, \dots, w_{n-1}, u_{m_{q-1}} - u_{m_q}\| \\ & \quad + \|w_1, \dots, w_{n-1}, u_{m_q} - u_{n_{q-1}}\| \\ & < \|w_1, \dots, w_{n-1}, u_{m_{q-1}} - u_{m_q}\| + \varepsilon \end{aligned} \quad (7)$$

From (5) and (7) getting that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, u_{m_{q-1}} - u_{n_{q-1}}\|} \mu(t) dt \\ & \leq \int_0^\varepsilon \mu(t) dt \end{aligned} \quad (8)$$

Using (4), (6) and (8) we get

$$\begin{aligned} & \int_0^\varepsilon \mu(t) dt \leq \int_0^{\|w_1, \dots, w_{n-1}, u_{m_q} - u_{n_q}\|} \mu(t) dt \\ & \leq h \int_0^{\|w_1, \dots, w_{n-1}, u_{m_{q-1}} - u_{n_{q-1}}\|} \mu(t) dt \\ & \leq h \int_0^\varepsilon \mu(t) dt \end{aligned}$$

Which is a contradiction, since  $h \in [0,1)$ . Therefore,  $\{u_n\}$  is Cauchy sequence, and convergent. The Limit is called  $s$ . By (2) we have

$$\begin{aligned} & \int_0^{\|w_1, \dots, w_{n-1}, Ts - u_{n+1}\|} \mu(t) dt = \\ & \int_0^{\|w_1, \dots, w_{n-1}, Ts - Tu_n\|} \mu(t) dt \end{aligned}$$

$$\leq \alpha \left( \begin{aligned} & \int_0^{\|w_1, \dots, w_{n-1}, S - u_n\|} \mu(t) dt, \\ & \int_0^{\|w_1, \dots, w_{n-1}, S - Ts\|} \mu(t) dt, \\ & \int_0^{\|w_1, \dots, w_{n-1}, u_n - u_{n+1}\|} \mu(t) dt \end{aligned} \right)$$

Taking limit as  $n \rightarrow \infty$ , to get

$$\begin{aligned} & \int_0^{\|w_1, \dots, w_{n-1}, Ts - s\|} \mu(t) dt = \\ & \propto \left( 0, \int_0^{\|w_1, \dots, w_{n-1}, S - Ts\|} \mu(t) dt, 0 \right) \end{aligned}$$

By condition  $N_2$  of function  $\alpha$ ,

$$\int_0^{\|w_1, \dots, w_{n-1}, Ts - s\|} \mu(t) dt = h \cdot 0 = 0$$

Which by (3), implies that  $\|w_1, \dots, w_{n-1}, Ts - s\| = 0$  or,  $Ts = s$ .

Suppose that  $r$  be another fixed point,  $r \neq s$ . So, by (2) getting

$$\begin{aligned} & \int_0^{\|w_1, \dots, w_{n-1}, S - r\|} \mu(t) dt = \\ & \int_0^{\|w_1, \dots, w_{n-1}, Ts - Tr\|} \mu(t) dt \\ & \leq \alpha \left( \begin{aligned} & \int_0^{\|w_1, \dots, w_{n-1}, S - r\|} \mu(t) dt, \\ & \int_0^{\|w_1, \dots, w_{n-1}, S - Ts\|} \mu(t) dt, \\ & \int_0^{\|w_1, \dots, w_{n-1}, r - Tr\|} \mu(t) dt \end{aligned} \right) \\ & = \alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, S - r\|} \mu(t) dt, \int_0^{\|w_1, \dots, w_{n-1}, S - s\|} \mu(t) dt, \int_0^{\|w_1, \dots, w_{n-1}, r - r\|} \mu(t) dt \right) \\ & = \alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, S - r\|} \mu(t) dt, 0, 0 \right) \end{aligned}$$

By condition  $N_2$  of function  $\alpha$ ,  
 $\int_0^{\|w_1, \dots, w_{n-1}, S - r\|} \mu(t) dt = 0$

which, from (3), implies that  $\|w_1, \dots, w_{n-1}, S - r\| = 0$  or  $s = r$ , i.e., fixed point is unique.

We will extend the idea of the previous theorem for a pair of mappings (F,T) to be a Picard

operators where H has two related  $n$ -norms in integral setting.

**Theorem (2.5):** Let H be a linear space with two  $n$ -norms  $\|\cdot, \dots, \cdot\|_p$  and  $\|\cdot, \dots, \cdot\|_q$  and

- i.  $\int_0^{\|w_1, \dots, w_{n-1}, u-v\|_p} \mu(t) dt \leq \int_0^{\|w_1, \dots, w_{n-1}, u-v\|_q} \mu(t) dt$ , for all  $u, v \in H$
- ii. H is complete depending on  $\|\cdot, \dots, \cdot\|_p$
- iii. F, T are self-mappings on H such that T is continuous depending on  $\|\cdot, \dots, \cdot\|_p$  and

$$\int_0^{\|w_1, \dots, w_{n-1}, Tu-Fv\|_q} \mu(t) dt \leq \alpha \left( \begin{matrix} \int_0^{\|w_1, \dots, w_{n-1}, u-v\|_q} \mu(t) dt, \\ \int_0^{\|w_1, \dots, w_{n-1}, u-Tu\|_q} \mu(t) dt, \\ \int_0^{\|w_1, \dots, w_{n-1}, v-Fv\|_q} \mu(t) dt \end{matrix} \right) \quad (9)$$

$\forall u, v \in H$  with some  $\alpha \in A$ , where  $\mu : [0, +\infty) \rightarrow [0, +\infty)$  is a summable such that for all  $\varepsilon > 0, \int_0^\varepsilon \mu(t) dt > 0$ . Then F and T are Picard operators.

**Proof:** For each integer  $m \geq 0$ , we define  $u_{2m+1} = Tu_{2m}, u_{2m+2} = Fu_{2m+1}$

Then by (9) we get,  $\int_0^{\|w_1, \dots, w_{n-1}, u_1-u_2\|_q} \mu(t) dt = \int_0^{\|w_1, \dots, w_{n-1}, Tu_0-Fu_1\|_q} \mu(t) dt$

$$\leq \alpha \left( \begin{matrix} \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt, \\ \int_0^{\|w_1, \dots, w_{n-1}, u_0-Tu_0\|_q} \mu(t) dt, \\ \int_0^{\|w_1, \dots, w_{n-1}, u_1-Fu_1\|_q} \mu(t) dt \end{matrix} \right) \leq \alpha \left( \begin{matrix} \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt, \\ \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt, \\ \int_0^{\|w_1, \dots, w_{n-1}, u_1-u_2\|_q} \mu(t) dt \end{matrix} \right)$$

By condition  $N_2$  of function  $\alpha$  then

$$\int_0^{\|w_1, \dots, w_{n-1}, u_1-u_2\|_q} \mu(t) dt \leq h \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt, \quad (10)$$

for some  $h \in [0,1)$

Similarly, one can show that  $\int_0^{\|w_1, \dots, w_{n-1}, u_2-u_3\|_q} \mu(t) dt \leq h \int_0^{\|w_1, \dots, w_{n-1}, u_1-u_2\|_q} \mu(t) dt$  (11)

In general, for any  $i \in N$  odd or even,

$$\int_0^{\|w_1, \dots, w_{n-1}, u_i-u_{i+1}\|_q} \mu(t) dt \leq h \int_0^{\|w_1, \dots, w_{n-1}, u_{i-1}-u_i\|_q} \mu(t) dt \quad (12)$$

for some  $h \in [0,1)$ . So, for any  $m \in N$  odd or even, obtaining that

$$\int_0^{\|w_1, \dots, w_{n-1}, u_m-u_{m+1}\|_q} \mu(t) dt \leq h^m \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt \quad (13)$$

By Theorem (2.5-i) getting

$$\int_0^{\|w_1, \dots, w_{n-1}, u_m-u_{m+1}\|_p} \mu(t) dt \leq \int_0^{\|w_1, \dots, w_{n-1}, u_m-u_{m+1}\|_q} \mu(t) dt \leq h^n \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt$$

As  $m \rightarrow \infty \Rightarrow \lim_{m \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, u_m-u_{m+1}\|_p} \mu(t) dt = 0, h \in [0,1)$

From (3) implies that  $\lim_{m \rightarrow \infty} \|w_1, \dots, w_{n-1}, u_m - u_{m+1}\|_p = 0$  (14)

To prove that  $\{u_m\}$  is a Cauchy sequence with respect to  $(H, \|\cdot, \dots, \cdot\|)$ . For any integer  $a > 0$

$$\begin{aligned} & \int_0^{\|w_1, \dots, w_{n-1}, u_m-u_{m+a}\|_p} \mu(t) dt \leq \int_0^{\|w_1, \dots, w_{n-1}, u_m-u_{m+a}\|_q} \mu(t) dt \\ & \leq \int_0^{\|w_1, \dots, w_{n-1}, u_m-u_{m+1}\|_q} \mu(t) dt + \int_0^{\|w_1, \dots, w_{n-1}, u_{m+1}-u_{m+2}\|_q} \mu(t) dt + \dots + \\ & + \int_0^{\|w_1, \dots, w_{n-1}, u_{m+a-1}-u_{m+a}\|_q} \mu(t) dt \\ & \leq h^m \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt + h^{m+1} \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt + \dots + \\ & + h^{m+a-1} \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt \\ & \leq \frac{h^m}{1-h} \int_0^{\|w_1, \dots, w_{n-1}, u_0-u_1\|_q} \mu(t) dt \rightarrow 0, \text{ as } m \rightarrow \infty. \end{aligned}$$

since  $h \in [0,1)$ . Therefore,  $\{u_m\}$  is Cauchy. Hence by completeness of  $H$ ,  $\{u_m\}$  converges to some  $s \in H$ , i.e.  $\|w_1, \dots, w_{n-1}, u_m - s\|_p \rightarrow 0$  as for some  $s \in H$ . Since  $T$  is continuous with respect to  $\|\cdot, \dots, \cdot\|_p$  we have

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, u_{2m+1} - s\|_p} \mu(t) dt = \\ \lim_{m \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, u_{2m} - s\|_p} \mu(t) dt \\ &= \lim_{m \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, Ts - s\|_p} \mu(t) dt \end{aligned}$$

So, by (3)  $\|w_1, \dots, w_{n-1}, u_{2m} - s\|_p = 0$  i.e.  $Ts = s$ . Now by (9)

$$\begin{aligned} &\int_0^{\|w_1, \dots, w_{n-1}, s - Fs\|_q} \mu(t) dt = \\ &\int_0^{\|w_1, \dots, w_{n-1}, Ts - Fs\|_q} \mu(t) dt \\ &\leq \alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, s - s\|_q} \mu(t) dt, \right. \\ &\quad \left. \int_0^{\|w_1, \dots, w_{n-1}, s - Ts\|_q} \mu(t) dt, \right. \\ &\quad \left. \int_0^{\|w_1, \dots, w_{n-1}, s - Fs\|_q} \mu(t) dt \right) \end{aligned}$$

$$\leq \alpha \left( 0, 0, \int_0^{\|w_1, \dots, w_{n-1}, s - Fs\|_q} \mu(t) dt \right)$$

Then by condition  $N_2$  of function  $\alpha$ ,  $\int_0^{\|w_1, \dots, w_{n-1}, s - Fs\|_q} \mu(t) dt \leq h \cdot 0 = 0$  (15)

and by (3)  $Fs = s$ , thus  $s$  is a fixed point of  $F$  and  $T$ . To prove the uniqueness, assume that  $r \neq s$  be fixed point of  $F$  and  $T$ . By (9), having

$$\begin{aligned} &\int_0^{\|w_1, \dots, w_{n-1}, s - r\|_q} \mu(t) dt \\ &= \int_0^{\|w_1, \dots, w_{n-1}, Ts - Fr\|_q} \mu(t) dt \\ &\leq \alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, s - r\|_q} \mu(t) dt, \right. \\ &\quad \left. \int_0^{\|w_1, \dots, w_{n-1}, s - Ts\|_q} \mu(t) dt, \right. \\ &\quad \left. \int_0^{\|w_1, \dots, w_{n-1}, r - Fr\|_q} \mu(t) dt \right) \end{aligned}$$

$$\leq \alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, s - r\|_q} \mu(t) dt, 0, 0 \right)$$

$$\leq h \cdot 0 = 0 \text{ as } \alpha \in A.$$

Then by (3) we get  $\|w_1, \dots, w_{n-1}, s - r\|_q = 0$  and so  $s = r$ .

If  $T = F$ , then the Theorem (2.5) gives as follows.

**Remark (2.6):** If replace the mapping  $F$  by identity mapping in Theorem (2.5) then  $T$  is Picard operator.

Another similar result is achieved if the continuity of  $T$  with respect to  $\|\cdot, \dots, \cdot\|_p$  is replaced by assuming the continuity at a point. Then the same conclusion under much less restricted condition is obtained.

**Theorem (2.7):** Let  $H$  be a linear space with two norms  $\|\cdot, \dots, \cdot\|_p$  and  $\|\cdot, \dots, \cdot\|_q$  and

- i.  $\int_0^{\|w_1, \dots, w_{n-1}, u - v\|_p} \mu(t) dt \leq \int_0^{\|w_1, \dots, w_{n-1}, u - v\|_q} \mu(t) dt$ , for all  $u, v \in H$
- ii. The mapping  $T : H \rightarrow H$  is continuous at  $s \in H$  with respect to  $\|\cdot, \dots, \cdot\|_p$  and

$$\begin{aligned} &\int_0^{\|w_1, \dots, w_{n-1}, Tu - Tv\|_q} \mu(t) dt \leq \\ &\alpha \left( \int_0^{\|w_1, \dots, w_{n-1}, u - v\|_q} \mu(t) dt, \right. \\ &\quad \left. \int_0^{\|w_1, \dots, w_{n-1}, u - Tu\|_q} \mu(t) dt, \right. \\ &\quad \left. \int_0^{\|w_1, \dots, w_{n-1}, v - Tv\|_q} \mu(t) dt \right) \end{aligned} \quad (16)$$

For all  $u, v \in H$ , some  $\alpha \in A$ , where  $\mu : [0, +\infty) \rightarrow [0, +\infty)$  is a summable such that for all  $\varepsilon > 0$ ,  $\int_0^\varepsilon \mu(t) dt > 0$

- iii. There exists a point  $u_0 \in H$  such that  $\{T^m u_0\}$  has a subsequence  $\{T^{m_i} u_0\}$  converging to  $s$  in  $(H, \|\cdot, \dots, \cdot\|_p)$ .

Then  $T$  is Picard operator.

**Proof:** Construct a sequence  $\{u_m\}$  by  $u_{m+1} = Tu_m$  for  $m \geq 0$  i.e.  $u_1 = Tu_0, u_2 = Tu_1 = T^2 u_0, \dots, u_m = T^m u_0$ . By similar argument in the proof of Theorem (2.5), the sequence  $\{u_m\}$  is Cauchy with respect to  $\|\cdot, \dots, \cdot\|_p$ . Hypothesis(iii) guarantee the convergence subsequence of a subsequence  $\{u_{m_i}\}$  of  $\{u_m\}$  to  $s$ , therefore  $\{u_m\}$  converges to  $s$  in  $H$  with respect to  $\|\cdot, \dots, \cdot\|_p$  i.e.  $\lim_{m \rightarrow \infty} u_m = s$ . And the continuity of  $T$  at  $s$  with respect to  $\|\cdot, \dots, \cdot\|_p$  implies that

$$\begin{aligned}
0 &= \lim_{m \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, u_{m+1} - s\|_p} \mu(t) dt = \\
&\lim_{m \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, T u_m - s\|_p} \mu(t) dt \\
&= \lim_{m \rightarrow \infty} \int_0^{\|w_1, \dots, w_{n-1}, T s - s\|_p} \mu(t) dt .
\end{aligned}$$

So, by (3)  $\|w_1, \dots, w_{n-1}, T s - s\|_p = 0$  i.e.  $T s = s$ . Thus T is Picard operator.

**Remark (2.8):** If  $\mu(t) = 1$  over  $[0, +\infty)$  in each above results then the contractive condition of integral type convert into a contractive condition not involving integrals.

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