Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 31, Issue 4, 2020 DOI: http://doi.org/10.23851/mjs.v31i4.863

Research Article **Open Access**

Certain Subclasses of Meromorphic Univalent Function Involving Differential Operator

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Article Info

Received
19/03/2020

Accepted
24/08/2020

Published
20/12/2020

ABSTRACT

The main object of the present paper is to introduce the class of meromorphic univalent function K* (σ,τ,S) defined by differential operator with study some geometric properties like coefficient inequality , growth theorem and distortion theorem, radii of starlikeness and convexity of f(z) in the class K* (σ,τ,S) .Also the concept of convolution (Hadamard product) investigate and Neighborhoods of the elements of class K* (σ,τ,S) are obtained.

KEYWORDS: Meromorphic univalent function, Differential operator, Hadamard product, Starlike function, Convex function, Neighborhood.

الخلاصة

الهدف الرئيسي من هذا البحث هو تقديم صنف من الدوال الميرومورفيه الاحاديه التكافؤ (σ,τ,S) K* المعرفه بواسطه المؤثر التفاضلي مع دراسة بعض الخصائص الهندسية مثل اللامساوات في المعامل ، نظرية النمو ونظرية التشويه ، نصف قطر النجم والتحدب من f(z) في الفئة (σ,τ,S) K*. كما يتم التحقق من مفهوم الالتفاف (ضرب هادمرد) ويتم الحصول على جوارات عناصر الصنف (σ,τ,S) K*

INTRODUCTION

Denote by $\sum$ the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{l=0}^{\infty}a_l z^l \qquad (1)$$

which are analytic and univalent in the punctured open unit disk
$U^*=\{z\in C: 0<|z|<1\}$

A function $f\in\sum$ is said to be meromorphic starlike if

$$R\left(\frac{zf'(z)}{f(z)}\right)<0 \ , z\in U^*$$

We denote by $\sum^*$ the class of all meromorphic starlike function.

A function $f\in\sum$ is said to be meromorphic convex if

$$R\left(1+\frac{zf''(z)}{f'(z)}\right)<0 \ , z\in U^*$$

The class of all meromorphic convex functions will be denoted by $\sum^c$..

Let $f\in\sum$ be of the form (1) and γ,δ be real numbers with $\gamma\ge\delta\ge 0$, then the analogue of the differential operator given in [6] is defined as follows:

$$D^0_{\gamma,\delta}f(z)=f(z)$$

$$D^1_{\gamma,\delta}f(z)=D_{\gamma,\delta}f(z)$$

$$=\gamma\delta(z^2f(z))''+(\gamma-\delta)\frac{(z^2f(z))'}{z}+(1-\gamma+\delta)f(z)$$

$$D^S_{\gamma,\delta}f(z)=D_{\gamma,\delta}\left(D^{S-1}_{\gamma,\delta}f(z)\right),$$

$z\in U^*$, $s\in N=\{1,2,3\cdots\cdots\}$.

If $f\in\sum$ is given by (1), then we have

$$D^S_{\gamma,\delta}f(z)=\frac{1}{z}+\sum_{l=0}^{\infty}B(\gamma,\delta,l)^S a_l z^l ,$$

$$z\in U^* \qquad (2)$$

where

$$B(\gamma,\delta,l)=[(l+2)\gamma\delta+\gamma-\delta](l+1)+1.$$

Note that for δ=0 and γ=1, we obtain the differential operator introduced by [1].

In this paper, we shall try to obtain the coefficient estimates of the class $K^*(\sigma,\tau,S)$, growth and distortion theorems, radii of starlikeness and convexity, convolution and neighborhoods of the elements for the class $K^*(\sigma,\tau,S)$.

A SET OF MAIN RESULTS

In this section, we define the following subclasses of meromorphic function by utilizing the operator $D^S_{\gamma,\delta}$.

Definition 1:

A function $f(z) \in \sum$ of the form (1) is in the class $K^*(\sigma,\tau,S)$ if it satisfies the following inequality

$$\left|\frac{z(D^S_{\gamma,\delta} f(z))'' + 2(D^S_{\gamma,\delta} f(z))'}{z(D^S_{\gamma,\delta} f(z))'' + 2\tau(D^S_{\gamma,\delta} f(z))'}\right| < \sigma \quad (3)$$

$(0 \le \tau < 1 \ ,0 < \sigma \le 1 \ ,S = 1,2,3 \dots)$

This class was studied by many researchers by (for example Juma [3] and Mahmoud [5]).

1. Coefficient Inequality:

We derive the coefficient inequality for the class $K^*(\sigma,\tau,S)$ in the next theorem.

Theorem 1:

The function $f(z)$ given by (1) is in the class$K^*(\sigma,\tau,s)$, if and only if

$$\sum_{l=1}^{\infty} Bl[l(1+\sigma) + (1 + \sigma(2\tau-1))]\, a_l \le 2\sigma(1-\tau) \quad (4)$$

The result is sharp for the function $f(z)$ is given by:

$$f(z) = \frac{1}{z} + \frac{2\sigma(1-\tau)}{Bl[l(1+\sigma) + (1+\sigma(2\tau-1)]} z^l \,, l \ge 1 \ (5)$$

Proof: Suppose (4) holds, and if $|z| = 1$. Then by (3) , we have

$$\left|\frac{z(D^S_{\gamma,\delta} f(z))'' + 2(D^S_{\gamma,\delta} f(z))'}{z(D^S_{\gamma,\delta} f(z))'' + 2\tau(D^S_{\gamma,\delta} f(z))'}\right| < \sigma$$

$$\left|z(D^S_{\gamma,\delta} f(z))'' + 2(D^S_{\gamma,\delta} f(z))'\right| - \sigma\left|z(D^S_{\gamma,\delta} f(z))'' + 2\tau(D^S_{\gamma,\delta} f(z))'\right| \le 0$$

$$\left|z[z^{-1} + \sum_{l=1}^{\infty} B(\gamma,\delta,l)^S a_l z^l]'' + 2[z^{-1} + \sum_{l=1}^{\infty} B(\gamma,\delta,l)^S a_l z^l]'\right| - \sigma\left|z[z^{-1} + \sum_{l=1}^{\infty} B(\gamma,\delta,l)^S a_l z^l]'' + 2\tau[z^{-1} + \sum_{l=1}^{\infty} B(\gamma,\delta,l)^S a_l z^l]'\right| \le 0$$

Let $B = B(\gamma,\delta,l)^S$, Then we get

$$\left|z[2z^{-3} + \sum_{l=1}^{\infty} B(l(l-1))a_l z^{l-2}] + 2[-z^{-2} + \sum_{l=1}^{\infty} B(l)a_l z^{l-1}]\right| - \sigma\left|z[2z^{-3} + \sum_{l=1}^{\infty} B(l(l-1))a_l z^{l-2}] + 2\tau[-z^{-2} + \sum_{l=1}^{\infty} B(l)a_l z^{l-1}]\right| \le 0$$

$$\left|2z^{-2} + \sum_{l=1}^{\infty} B(l(l-1))a_l z^{l-1} - 2z^{-2} + \sum_{l=1}^{\infty} B(l)a_l z^{l-1}\right| - \sigma\left|2z^{-2} + \sum_{l=1}^{\infty} B(l(l-1))a_l z^{l-1}] - 2\tau z^{-2} + 2\tau\sum_{l=1}^{\infty} B(l)a_l z^{l-1}]\right| \le 0$$

$$\left|\sum_{l=1}^{\infty} B(l(l-1))a_l z^{l-1} + 2\sum_{l=1}^{\infty} B(l)a_l z^{l-1}\right| - \sigma\left|(2-2\tau)z^{-2} + \sum_{l=1}^{\infty} B(l(l-1))a_l z^{l-1} + 2\tau\sum_{l=1}^{\infty} B(l)a_l z^{l-1}\right| \le 0$$

$$\left|\sum_{l=1}^{\infty} Bl(l-1+2)a_l z^{l-1})\right| - \sigma\left|\frac{2-2\tau}{z^2} + \sum_{l=1}^{\infty} Bl(l-1+2\tau)a_l z^{l-1}\right| \le 0$$

$$\le \sum_{l=1}^{\infty} Bl(l+1)a_l|z|^{l-1} + \sigma\sum_{l=1}^{\infty} Bl(l-1+2\tau)a_l|z|^{l-1} - 2\sigma(1-\tau)|z|^{-2} \le 0.$$

ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 31, Issue 4, 2020 DOI: http://doi.org/10.23851/mjs.v31i4.863

Since $|z| = 1$, we have

$$\sum_{l=1}^{\infty} Bl[(l+1)+\sigma(l-1+2\tau)]a_l \leq 2\sigma(1-\tau),$$

then

$$\sum_{l=1}^{\infty} Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]a_l \leq 2\sigma(1-\tau).$$

Thus, by the maximum modulus theorem,
We get $\in K^*(\sigma,\tau,S)$.
Conversely, if $f(z)$ of the form (1) is in the class $K^*(\sigma,\tau,S)$, then by (3) we get

$$\left|\frac{z(D_{\gamma,\delta}^S f(z))'' + 2(D_{\gamma,\delta}^S f(z))'}{z(D_{\gamma,\delta}^S f(z))'' + 2\tau(D_{\gamma,\delta}^S f(z))'}\right| < \sigma$$

Thus,

$$\left|\frac{\sum_{l=1}^{\infty} Bl(l+1)a_l z^{l-1}}{\frac{2(1-\tau)}{z^2} + \sum_{l=1}^{\infty} Bl(l-1+2\tau)\, a_l z^{l-1}}\right| < \sigma$$

Since $|R(z)| \leq |z|$ for all z, we have

$$R\left\{\frac{\sum_{l=1}^{\infty} Bl(l+1)a_l z^{l-1}}{\frac{2(1-\tau)}{z^2} + \sum_{l=1}^{\infty} Bl(l-1+2\tau)a_l z^{l-1}}\right\} < \sigma \quad (6)$$

Now, we taking the worth of z on the real axis so that the worth

$$\frac{z(D_{\gamma,\delta}^S f(z))''}{(D_{\gamma,\delta}^S f(z))'}$$

is real, then the denominate of (6) and $z \to 1^-$ through positive real value, we have the inequality (4) .
The result is sharp for $f(z)$, defined by

$$f(z) = \frac{1}{z} + \frac{2\sigma(1-\tau)}{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]} z^l . l \geq 1 \quad (7)$$

Corollary 1: Let $f \in K^*(\sigma,\tau,S)$. T

$$a_l \leq \frac{2\sigma(1-\tau)}{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]},$$

where

$$0 \leq \tau \leq 1$$

and

$$0 < \sigma \leq 1 \quad , s = 1,2,3,\ldots$$

2. Growth and Distortion Theorems:

We derive some properties distortion and growth of $f \in K^*(\sigma,\tau,s)$ in the next theorems.

Theorem 2:
If $f(z) \in K^*(\sigma,\tau,S)$ of the form (1), then for $0 < |z| = r$,we get

$$\frac{1}{r} + \frac{\sigma(1-\tau)}{B(1+\tau\sigma)} r \leq |f(z)| \leq \frac{1}{r} + \frac{\sigma(1-\tau)}{B(1+\tau\sigma)} r ,$$

with equality for

$$f(z) = \frac{1}{z} + \frac{\sigma(1-\tau)}{B(1+\tau\sigma)} z .$$

Proof: By hypothesis $f \in K^*(\sigma,\tau,s)$, we get from Theorem 1

$$\sum_{l=1}^{\infty} Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]\, a_l \leq 2\sigma(1-\tau),$$

that is,

$$\sum_{l=1}^{\infty} a_l \leq \frac{2\sigma(1-\tau)}{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]},$$

then

$$a_l \leq \frac{\sigma(1-\tau)}{B(1+\sigma\tau)} .$$

Thus, for $0 < |z| = r < 1$, we have

$$|f(z)| \leq \frac{1}{|z|} + \sum_{l=1}^{\infty} a_l |z|^l,$$

$$|f(z)| \leq \frac{1}{r} + \sum_{l=1}^{\infty} a_l\, r ,$$

$$\leq \frac{1}{r} + \frac{\sigma(1-\tau)}{B(1+\sigma\tau)}, \qquad |z| = r$$

So,

$$|f(z)| \geq \frac{1}{|z|} - \sum_{l=1}^{\infty} a_l |z|^l \geq \frac{1}{r} - \sum_{l=1}^{\infty} a_l\, r$$

$$\geq \frac{1}{r} - \frac{\sigma(1-\tau)}{B(1+\sigma\tau)}$$

Thus, the proof is completed.

Theorem 3:
If $f(z) \in K^*(\sigma,\tau,S)$ of the form (1), then for $0 < |z| = r < 1$, we get

$$\frac{1}{r^2} - \frac{\sigma(1-\tau)}{B(1+\tau\sigma)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{\sigma(1-\tau)}{B(1+\tau\sigma)},$$

with equailt for

$$f(z)=\frac{1}{z}+\frac{\sigma(1-\tau)}{B(1+\tau\sigma)}z.$$

Proof: Utilizing Theorem 1, we get

$$\sum_{l=1}^{\infty}Bl\big[l(1+\sigma)+\big(1+\sigma(2\tau-1)\big)\big]a_l \le 2\sigma(1-\tau),$$

then

$$a_l\le\frac{\sigma(1-\tau)}{B(1+\sigma\tau)}$$

Thus, for $0<|z|=r<1$

$$|f'(z)|\le\left|\frac{-1}{z^2}\right|+\sum_{l=1}^{\infty} l\,a_l|z|^{l-1},$$

$$\le\frac{1}{r^2}+\frac{\sigma(1-\tau)}{B(1+\sigma\tau)}$$

Also,

$$|f'(z)|\ge\left|\frac{-1}{z^2}\right|-\sum_{l=1}^{\infty} l\,a_l|z|^{l-1},$$

$$\ge\frac{1}{r^2}-\frac{\sigma(1-\tau)}{B(1+\sigma\tau)}$$

3. Radii of Starlikness and Convexity:

The radius of starlikeness and convexity for$K^*(\sigma,\tau,s)$ is given by the following theorems:

Theorem 4:

If $f(z)\in\ K^*(\sigma,\tau,S)$ of the form (1), then f is meromorphically starlike of order λ $(0\le\lambda<1)$ in the disk $|z|<r_1(\sigma,\tau,S,\lambda)$, wehere

$$r_1(\sigma,\tau,S,\lambda)=inf_{l\ge1}\left\{\frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1)](1-\lambda)}{2\sigma(l+2-\lambda)(1-\tau)}\right\}^{\frac{1}{l+1}}$$

The result is sharp for $f(z)$ is given by the following

$$f(z)=\frac{1}{z}+\frac{2\sigma(1-\tau)}{Bl\big[l(1+\sigma)+\big(1+\sigma(2\tau-1)\big)\big]}z^l,\quad l\ge1$$

Proof: It is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\le1-\lambda,$$

$$\left|\frac{zf'(z)}{f(z)}+1\right|=\left|\frac{\sum_{l=1}^{\infty}(l+1)a_lz^l}{z^{-1}+\sum_{l=1}^{\infty}a_lz^l}\right|\le\frac{\sum_{l=1}^{\infty}(l+1)\,a_l|z|^{l+1}}{1-\sum_{l=1}^{\infty}a_l\,|z|^{l+1}}$$

Thus,

$$\frac{\sum_{l=1}^{\infty}(l+1)\ a_l|z|^{l+1}}{1-\sum_{l=1}^{\infty}a_l\,|z|^{l+1}}\le1-\lambda,$$

$$\sum_{l=1}^{\infty}(l+2-\lambda)\ a_l|z|^{l+1}\ \le1-\lambda, \tag{8}$$

by Theorem 1, we have

$$\sum_{l=1}^{\infty}Bl[l(1+\sigma)+(1+\sigma(2\tau-1)]\,a_l \le 2\sigma(1-\tau) \tag{9}$$

Therefore,

$$|z|^{l+1}\le\frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))](1-\lambda)}{2\sigma(1-\tau)(l+2-\lambda)},$$

$$|z|\le\left\{\frac{Bl\big[l(1+\sigma)+\big(1+\sigma(2\tau-1)\big)\big](1-\lambda)}{2\sigma(1-\tau)(l+2-\lambda)}\right\}^{\frac{1}{l+1}}$$

Theorem 5:

If $f(z)\in\ K^*(\sigma,\tau,S)$ of the form (1), then f is meromorphically convex of order ϕ $(0\le\phi<1)$ in $|z|<r_2(\sigma,\tau,S,\phi)$ where

$$r_2(\sigma,\tau,S,\phi)=inf_{l\ge1}\left\{\frac{B[l(1+\sigma)+(1+\sigma(2\tau-1)](1-\phi)}{2\sigma(1-\tau)(l+2-\phi)}\right\}^{\frac{1}{l+1}}.$$

The result is sharp for $f(z)$ is given by the following:

$$f(z)=\frac{1}{z}+\frac{2\sigma(1-\tau)}{Bl\big[l(1+\sigma)+\big(1+\sigma(2\tau-1)\big)\big]}z^l,\quad l\ge1.$$

Proof: It is sufficient to prove that

$$\left|\frac{z\big(f(z)\big)''}{(f(z))'}+2\right|\le1-\phi,$$

$$\left|\frac{z\big(f(z)\big)''}{(f(z))'}+2\right|=\left|\frac{z[z^{-1}+\sum_{l=1}^{\infty}a_lz^l]''+2\,[z^{-1}+\sum_{l=1}^{\infty}a_lz^l]'}{[z^{-1}+\sum_{l=1}^{\infty}a_lz^l]'}\right|=\left|\frac{\sum_{l=1}^{\infty}l(l+1)a_lz^{l-1}}{z^{-2}+\sum_{l=1}^{\infty}(l)a_lz^{l-1}}\right|\le\frac{\sum_{l=1}^{\infty}l(l+1)\ a_l|z|^{l+1}}{1-\sum_{l=1}^{\infty}l\,a_l|z|^{l+1}}\ \le1-\phi,$$

if

ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 31, Issue 4, 2020 DOI: http://doi.org/10.23851/mjs.v31i4.863

$$\sum_{l=1}^{\infty} l(l+1)\, a_l |z|^{l+1}$$
$$\leq (1-\phi)\left(1-\sum_{l=1}^{\infty} l a_l\, |z|^{l+1}\right),$$

that is, if

$$\sum_{l=1}^{\infty}[l(l+1)+l-l\phi]\, a_l|z|^{l+1} \leq 1-\phi,$$
$$\sum_{l=1}^{\infty} l[l+2-\phi]\, a_l|z|^{l+1} \leq 1-\phi \quad (10)$$

From (10) and Theorem 1, we obtain

$$|z|^{l+1} \leq \frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))](1-\phi)}{2\sigma(1-\tau)l[l+2-\phi]},$$
$$|z| \leq \left[\frac{B\big[l(1+\sigma)+\big(1+\sigma(2\tau-1)\big)\big](1-\phi)}{2\sigma(1-\tau)(l+2-\phi)}\right]^{\frac{1}{l+1}}$$

4. Convolution

Theorem 6:
If $f(z)$ and $g(z) \in K^*(\sigma,\tau,S)$,then $(f*g)(z) \in K^*(\rho,\tau,S)$ for

$$f(z)=\frac{1}{z}+\sum_{l=0}^{\infty} a_l z^l \;\; ,g(z)=\frac{1}{z}+\sum_{l=0}^{\infty} b_l z^l,$$

where

$$(f*g)(z)=\frac{1}{z}+\sum_{l=0}^{\infty} a_l b_l z^l\,, \quad \rho \leq \frac{2\sigma^2(1-\tau)(l+1)}{2\sigma^2(1-\tau)(l+2\tau-1)-Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]^2}$$

Proof: Hence $f,g \in K^*(\sigma,\tau,S)$, then by using Theorem 1, we get

$$\sum_{l=1}^{\infty}\frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]}{2\sigma(1-\tau)}\, a_l \leq 1$$

and,

$$\sum_{l=1}^{\infty}\frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]}{2\sigma(1-\tau)}\, b_l \leq 1$$

We need to find the largest ρ such that

$$\sum_{l=1}^{\infty}\frac{Bl[l(1+\rho)+(1+\rho(2\tau-1))]}{2\rho(1-\tau)}\, a_l b_l \leq 1.$$

By Cauchy – Schwarz inequality, we have

$$\sum_{l=1}^{\infty}\frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]}{2\sigma(1-\tau)}\sqrt{a_l b_l} \leq 1 \quad (11)$$

To proof Theorem 6, it is sufficient to prove that

$$\frac{Bl[l(1+\rho)+(1+\rho(2\tau-1))]}{2\rho(1-\tau)}\, a_l b_l \leq \frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]}{2\sigma(1-\tau)}\sqrt{a_l b_l}\,,$$

which is equivalent to

$$\sqrt{a_l b_l} \leq \frac{\rho[l(1+\sigma)+(1+\sigma(2\tau-1))]}{\sigma[l(1+\rho)+(1+\rho(2\tau-1))]}$$

From (11), we get

$$\sqrt{a_l b_l} \leq \frac{2\sigma(1-\tau)}{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]}$$

We must prove that

$$\frac{2\sigma(1-\tau)}{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]} \leq \frac{\rho[l(1+\sigma)+(1+\sigma(2\tau-1))]}{\sigma[l(1+\rho)+(1+\rho(2\tau-1))]}$$

such that

$$2\sigma^2(1-\tau)\big[l(1+\rho)+\big(1+\rho(2\tau-1)\big)\big] \leq Bl\rho\big[l(1+\sigma)+\big(1+\sigma(2\tau-1)\big)\big]^2,$$
$$2\sigma^2(1-\tau)[(l+1)+\rho(l+2\tau-1)] \leq Bl\rho[l(1+\sigma)+(1+\sigma(2\tau-1))]^2,$$
$$2\sigma^2(1-\tau)(l+1) \leq Bl\rho[l(1+\sigma)+(1+\sigma(2\tau-1))]^2 - 2\sigma^2\rho(1-\tau)(l+2\tau-1),$$
$$2\sigma^2(1-\tau)(l+1) \geq \rho[2\sigma^2(1-\tau)(l+2\tau-1) - Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]^2],$$

which gives

$$\rho \leq \frac{2\sigma^2(1-\tau)(l+1)}{2\sigma^2(1-\tau)(l+2\tau-1)-Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]^2}$$

Theorem 7:
If the functions $f_j(z)(j=1,2)$ denoted by

$$f_j(z)=\frac{1}{z}+\sum_{l=1}^{\infty} a_{l,j} z^l\;, \qquad (a_{l,j}\geq 0\,, j=1,2)$$

be in $K^*(\sigma,\tau,S)$,then the function $g(z)$ defined by

$$g(z) = \frac{1}{z} + \sum_{l=1}^{\infty}\left(a_{l,1}^2 + a_{l,2}^2\right)z^l,$$

is in the class $K^*(\varphi, \tau, S)$,where

$$\varphi \le \frac{4\sigma^2(\tau-1)(l+1)}{4\sigma^2(1-\tau)(l+2\tau-1) - Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]^2}$$

Proof: Hence $f_j \in K^*(\sigma, \tau, S), (j = 1,2)$.
Then utilizing Theorem 1,we get

$$\sum_{l=1}^{\infty}\frac{Bl\left[l(1+\sigma)+\left(1+\sigma(2\tau-1)\right)\right]}{2\sigma(1-\tau)}a_{l,j} \le 1 \ , j = 1,2$$

since

$$\sum_{l=1}^{\infty}\left(\frac{Bl\left[l(1+\sigma)+\left(1+\sigma(2\tau-1)\right)\right]}{2\sigma(1-\tau)}\right)^2 a_{l,j}^2 \le \left(\sum_{l=1}^{\infty}\frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1)]}{2\sigma(1-\tau)}a_{l,j}\right)^2 \le 1, (j = 1,2)$$

and

$$\sum_{l=1}^{\infty}\frac{1}{2}\left(\frac{Bl(l(1+\sigma)+(1+\sigma(2\tau-1))}{2\sigma(1-\tau)}\right)^2\left(a_{l,1}^2 + a_{l,2}^2\right) \le -1$$

Therefore, we need to find the largest φ such that

$$\frac{l(1+\varphi)+\left(1+\varphi(2\tau-1)\right)}{\varphi} \le \frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]^2}{4\sigma^2(1-\tau)}, \ \ l \ge 1.$$

Hence

$$\varphi \le \frac{4\sigma^2(\tau-1)(l+1)}{4\sigma^2(1-\tau)(l+2\tau-1) \ - Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]^2}$$

Theorem 8:
Let

$$f(z) = \frac{1}{z} + \sum_{l=1}^{\infty}a_l z^l \in K^*(\sigma, \tau, S)$$

,
and

$$g(z) = \frac{1}{z} + \sum_{l=1}^{\infty}b_l z^l$$

with $|b_l| \le 1$,is in the class $K^*(\sigma, \tau, S)$. Then $f(z) * g(z) \in K^*(\sigma, \tau, S)$.
Proof : From Theorem 1, we get

$$\sum_{l=1}^{\infty}Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]a_l \le 2\sigma(1-\tau)$$

Hence

$$\sum_{l=1}^{\infty}\frac{Bl\left[l(1+\sigma)+\left(1+\sigma(2\tau-1)\right)\right]}{2\sigma(1-\tau)}|a_l b_l| = \sum_{l=1}^{\infty}\frac{Bl\left[l(1+\sigma)+\left(1+\sigma(2\tau-1)\right)\right]}{2\sigma(1-\tau)}a_l|b_l \le \frac{Bl[l(1+\sigma)+(1+\sigma(2\tau-1))]}{2\sigma(1-\tau)}$$

Thus,
$f(z) * g(z) \in K^*(\sigma, \tau, k)$.

Corollary 2:
Let
$(z) = \frac{1}{z} + \sum_{l=1}^{\infty}a_l z^l \in K^*(\sigma, \tau, S)$,
and

$$g(z) = \frac{1}{z} + \sum_{l=1}^{\infty}b_l z^l$$

for $0 \le b \le 1$
is in the class $K^*(\sigma, \tau, S)$.Then
$f(z) * g(z) \in K^*(\sigma, \tau, S)$.

5. Neighborhood on $K^*(\sigma, \tau, S)$:

In the section, the concept of neighborhood of analytic function was first introduced by Goodman [2] and then generalized by Ruscheweyh [7]. Lin and Srivastava [4], investigated this concept for the elements of several famous subclass of analytic function.
We define the (l, ξ)Neighborhood of a function $f(z) \in \sum$
by

$$N_{l,\xi}(f) = \left\{g \in \sum: g(z) = z^{-1} + \sum_{l=1}^{\infty}b_l z^l \ \ and \ \sum_{l=1}^{\infty}l|a_l - b_l| \le \xi\right\}$$

For the identity function $e(z) = z$, we get

$$N_{l,\xi}(e) = \left\{g \in \sum : g(z) = z^{-1} + \sum_{l=1}^{\infty}b_l z^l \ and \ \sum_{l=1}^{\infty}l|b_l| \le \xi\right\}$$

Lemma:
The function $f(z) \in \sum$ is said to be in the class $K^*(\sigma, \tau, S, \xi)$ if there exists a function $g(z) \in K^*(\sigma, \tau, S)$, such that

ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 31, Issue 4, 2020 DOI: http://doi.org/10.23851/mjs.v31i4.863

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\vartheta \ , \qquad (13)$$

$(z \in U, 0 \le \vartheta < 1)$

Theorem 9:
If $g(z) \in K^*(\sigma,\tau,s)$ and

$$\vartheta = 1-\frac{\xi\left[B((1+\sigma)+\left(1+\sigma(2\tau-1)\right))\right]}{B((1+\sigma)+(1+\sigma(2\tau-1))-2\sigma(1-\tau)}, (14)$$

then $N_{l,\xi}(g) \subset K^*(\sigma,\tau,S,\xi)$.

Proof: Assume that $f \in N_{l,\xi}(g)$. Then we have from (12) that

$$\sum_{l=1}^{\infty} l|a_l-b_l| \le \xi,$$

which suggests the coefficient inequality

$$\sum_{l=1}^{\infty} |a_l-b_l| \le \xi \ , \quad (l \in N)$$

Hence $g \in K^*(\sigma,\tau,S)$,we get from corollary 1

$$\sum_{l=1}^{\infty} b_l \le \frac{2\sigma(1-\tau)}{Bl((1+\sigma)+\left(1+\sigma(2\tau-1)\right))} \ ,$$

From (13),we get

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{l=1}^{\infty}|a_l-b_l|}{1-\sum_{l=1}^{\infty} b_l}$$

$$\le \frac{\xi\left[B((1+\sigma)+\left(1+\sigma(2\tau-1)\right))\right]}{B((1+\sigma)+\left(1+\sigma(2\tau-1)\right))-2\sigma(1-\tau)}$$

$= 1-\vartheta.$

Since, by lemma $f \in K^*(\sigma,\tau,S,\xi)$ for ϑ given by (14).

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