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On $m\widehat{\omega}$-light functions

Nadia Kadum Humadi, Haider Jebur Ali
Department of Mathematics, College of Science, Mustansiriyah University, IRAQ.
*Correspondent author email: m.a.nadia313@gmail.com

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Abstract

In this work we introduced new types of m-light functions namely $m\widehat{\omega}$-light functions, also we introduced $m\widehat{\omega}$-disconnected, and $m\widehat{\omega}$-totally disconnected spaces. Some examples, facts, and theorems have been given to support our work.

Keywords: $m_X\widehat{\omega}$-open sets, $m\widehat{\omega}$-disconnected space, $m\widehat{\omega}$-totally disconnected space, $m\widehat{\omega}T_1$-space, $m\widehat{\omega}T_2$-space, and $m\widehat{\omega}$-light functions.

الخلاصة

في عملنا هذا قدمنا أنواع جديدة من الدوال m- واهنة والتي تدعى الدوال $m\widehat{\omega}$- واهنة كذلك قدمنا الفضاءات-$m\widehat{\omega}$غيرالمترابطة، والفضاءات $m\widehat{\omega}$- غير المترابطة كليا". بعض الامثلة, حقائق ومبرهنات أُعطيت لتدعم عملنا.

Introduction

The concept of functions is one of the important topics in the study of subject of topology. In our work we dealt with one of functions which is light function, it has been studied by the researcher Gorgees Shaheed Mohammad in (2001) [3]. We used m-structure spaces to define another types namely $m\widehat{\omega}$-light functions which are weaker than m-light function which studied by H. F. Abass and H. J. Ali in (2016) [1]. Whene m-light function (a function f from m-space X into m-space Y is m-light, if $f^{-1}(\{y\})$ is m-totally disconnected subspace of X for each $y \in Y$). Also they introduced m-disconnected and m-totally disconnected spaces, and in (2018) [4], also they introduced θ, θ^*, and θ^{**}-totally disconnected mapping. By the same context and using $m_X\ \widehat{\omega}$-open sets we define $m\widehat{\omega}$-disconnected, $m\widehat{\omega}$-totally disconnected spaces.
The relationship between these types has been studied and there are some examples, facts and results have been given.

1- On $m_X\ \widehat{\omega}$-totally disconnected, spaces.

In this section, by using $m_X\ \widehat{\omega}$-open sets we provided $m_X\ \widehat{\omega}$-disconnected and $m_X\ \widehat{\omega}$-totally disconnected spaces, and illustrate the relation between them.

Definition (1.1) [5]: Let X be a non-empty set, a sub collection m_X of the power set $p(X)$ is called m-structure if it contains $\emptyset$ and X, the pair (X, m_X) is called m-structure space (briefly m-space). Any elements $\mathcal{W}$ in m_X is said to be m_X-open set, $\mathcal{W}^C$ is said to be m_X-closed.

Remark (1.2) [1]: Every topology is m-structure, since every topology has $\emptyset$ and X.

Example (1.3): Consider X= {h, i, g}, m_X= {$\emptyset$, X, {h}, {i}, {h, g}, {i, g}}, then m_X is an m-structure on X, but not a topology.

Definition (1.4) [1]: Consider (X, m_X) is an m-space and $\mathcal{W}, \mathcal{J}$ are two non-empty m_X-open sets. $\mathcal{W} \cup \mathcal{J}$ is called m-disconnection to X iff $X = \mathcal{W} \cup \mathcal{B}, \mathcal{W} \cap \mathcal{B} = \emptyset$.

Definition (1.5) [1]: An m-space (X, m_X) is called m-disconnected if there is m-disconnection to X. If X is not m-disconnected then it is m-connected.

Example (1.6): Let X= {1, 2, 3} and m_X= {$\emptyset$, X, {1}, {2}, {1, 3}, {2, 3}}. (X, m_X) is m-disconnected space.

Definition (1.7): Suppose $\mathcal{W}$ is a subset of an m-space (X, m_X), it is said to be $m_X\widehat{\omega}$ -open set if for any element $x \in \mathcal{W}$ there exists an m_X-open set $\mathcal{B}$ containing x and $\mathcal{B}$-$\mathcal{W}$ is countable. The complement of $m_X\widehat{\omega}$-open set is $m_X\ \widehat{\omega}$-closed set. The family of all $m_X\ \widehat{\omega}$-open sets is denoted by $m_{\widehat{\omega}}$.

Remark (1.8):

1- Every m_X–open set is $m_X\widehat{\omega}$ open but not conversely, since if (X, m_X) is an m-space and $\mathcal{W}$ is an m_X-open set, let $x \in \mathcal{W}$ and suppose $\mathcal{B}$ is an m_X-open set, take $\mathcal{B}$=$\mathcal{W}$, hence $\mathcal{W}$ contains x and $\mathcal{W}$-$\mathcal{W}$ is countable, so $\mathcal{W}$ is $m_X\ \widehat{\omega}$-open set. Example (1.9) demonstrates that the converse is false.

Example (1.9): Consider X= $\{d, e, f\}$ and m_X= $\{\emptyset, X\}$, so $\{d\}$ is $m_X\ \widehat{\omega}$-open set, while it is not m_X-open set**.**

Definition (1.10):

1- $m_X\ \widehat{\omega}$ -closure of a set $\mathcal{W}$ in m-space X, is the intersection of all $m_X\ \widehat{\omega}$-closed sets which contain $\mathcal{W}$ and we symbolize it by $m_X\ \widehat{\omega} - cl(\mathcal{W})$.

2- $m_X\widehat{\omega}$-interior of a set $\mathcal{W}$ in m-space X, is the union of every $m_X\widehat{\omega}$ -open sets which contained in $\mathcal{W}$, we symbolize it $m_X\widehat{\omega}$-$Int(\mathcal{W})$).

Example (1.11): Let $\mathcal{W}$ = $\mathcal{Z}$- {1} be a subset of $(\mathcal{Z}, m_{ind})$, then $m_{\mathcal{Z}}\ \widehat{\omega}$ -$cl(\mathcal{W})$ = $\mathcal{Z}$-{1}, and $m_{\mathcal{Z}}\widehat{\omega}$ -$Int(\mathcal{W})$ = $\mathcal{Z}$-{1}.

Definition (1.12): Suppose $\mathcal{W}$ and $\mathcal{B}$ are non-empty $m_X\ \widehat{\omega}$-open sets, then $\mathcal{W} \cup \mathcal{B}$ is said to be $m_X\ \widehat{\omega}$-disconnection to X if $\mathcal{W} \cup \mathcal{B} = X$ and $\mathcal{W} \cap \mathcal{B} = \emptyset$.

Definition (1.13): An m-space (X, m_X) is called $m\widehat{\omega}$-disconnected if there is $m\widehat{\omega}$-disconnection to X. X is $m\widehat{\omega}$ –connected space if it is not $m\widehat{\omega}$-disconnected.

Example (1.14): Suppose X= $\{d, e, f\}$ and m_X= $\{\emptyset, X, \{f\}, \{e\}\}$, so (X, m_X) is $m\widehat{\omega}$-disconnected, but not $m\widehat{\omega}$-connected space**.**

Remark (1.15):

1- Every $m\widehat{\omega}$-connected space is m-connected but not conversely, since if (X, m_X) is an $m\widehat{\omega}$-connected space, and if it is not m-connected space that means if it is m-disconnected space, so that there exist non-empty disjoint m_X-open sets $\mathcal{W}$ and $\mathcal{B}$ where $\mathcal{W} \cup \mathcal{B} = X$, by Remark (1-8) X is $m\widehat{\omega}$**-**disconnected, this contradicts the assumption, so (X, m_X) is m-connected**..**

2- Every m-disconnected space is $m\widehat{\omega}$–disconnected, since if (X, m_X) is m-disconnected space, so there exist non-empty disjoint m_X-open sets $\mathcal{W}$ and $\mathcal{B}$ where $\mathcal{W} \cup \mathcal{B} = X$, by Remark (1-8), X is $m\widehat{\omega}$-disconnected. Example (1.16) demonstrates that the converse of (1) (respectively (2)) is false.

Example (1.16): 1- Let X= $\{e, f, g\}$ and m_X= $\{\emptyset, X, \{e\}, \{g\}\}$, then (X, m_X) is m-connected and then it is not m-disconnected, also not $m\widehat{\omega}$-connected space since {1}, {2, 3} are two non-empty disjoint $m_X\widehat{\omega}$–open sets and X= {1} $\cup$ {2, 3}. So X is $m\widehat{\omega}$-disconnected space.

Example (1.17): Let $(\mathcal{R}, m_u)$ be the usual m-space and $p \in \mathcal{R}$, then ($\mathcal{R}$-$\{p\}$,m_u) is m-disconnected space, since $(-\infty, p)$ and (p, ∞) are two non-empty $m_{\mathcal{R}}$-open which separate $\mathcal{R}$-$\{p\}$. Also it is $m\widehat{\omega}$-disconnected**.**

Proposition (1.18): 1- Let (X, m_X) be an m-space and $\mathcal{L}$ be a non-empty subset of X, then $m_X\widehat{\omega}$-$cl(\mathcal{L}) = \mathcal{L} \cup$ ($m_X\ \widehat{\omega}$-$d(\mathcal{L})$).

Proof:

1- To prove $\mathcal{L} \cup (m_X\ \widehat{\omega}$-$d(\mathcal{L})) \subseteq m_X\ \widehat{\omega}$-$cl(\mathcal{L})$. If $y \in \mathcal{L} \cup (m_X\widehat{\omega}$-$d(\mathcal{L}))$ and suppose $y \notin m_X\widehat{\omega}$-$cl(\mathcal{L})$, hence there is $m_X\widehat{\omega}$-closed set $\mathcal{F}$ with $\mathcal{L} \subseteq \mathcal{F}$ and $y \notin \mathcal{F}$, so $\mathcal{W} = X$-$\mathcal{F}$ is $m_X\ \widehat{\omega}$-open set contains y and $\mathcal{W} \cap \mathcal{L} = \emptyset$, so $y \notin m_X\widehat{\omega}$-$d(\mathcal{L})$ and since $y \notin \mathcal{L}$, so $y \notin \mathcal{L} \cup (m_X\widehat{\omega}$-$d(\mathcal{L}))$ C!, so $y \in m_X\widehat{\omega}$-$cl(\mathcal{L})$, therefore $\mathcal{L} \cup (m_X\widehat{\omega}$-$d(\mathcal{L})) \subseteq m_X\ \widehat{\omega}$-$cl(\mathcal{L})$.

Now to prove $m_X\widehat{\omega}$-$cl(\mathcal{L}) \subseteq \mathcal{L} \cup (m_X\widehat{\omega}$-$d(\mathcal{L}))$. Let $y \in m_X\ \widehat{\omega}$-$cl(\mathcal{L})$ and suppose $y \notin \mathcal{L} \cup (m_X\widehat{\omega}$-$d(\mathcal{L}))$, so there is an $m_X\widehat{\omega}$-open set $\mathcal{B}$ with $y \in \mathcal{B}$ and $\mathcal{B} \cap \mathcal{L} = \emptyset$. Also $H = X$-$\mathcal{B}$ is $m_X\widehat{\omega}$-closed set containing $\mathcal{L}$ with $y \notin H$, so $y \notin m_X\widehat{\omega}$-$cl(\mathcal{L})$ which is a contradiction. Implies that $y \in \mathcal{L} \cup (m_X\widehat{\omega}$-$d(\mathcal{L}))$, then $m_X\widehat{\omega}$-$cl(\mathcal{L}) \subseteq \mathcal{L} \cup (m_X\widehat{\omega}$-$d(\mathcal{L}))$, thus $m_X\widehat{\omega}$-$cl(\mathcal{L})$ = $\mathcal{L} \cup (m_X\widehat{\omega}$-$d(\mathcal{L}))$.

Lemma (1.19): The infinite union of $m_X\widehat{\omega}$-open sets is $m_X\widehat{\omega}$-open set too**.**

Proof: Take (X, m_X) as an m-space and $\{\mathcal{L}_i \setminus i \in I\}$ as a collection of $m_X\widehat{\omega}$-open sets, let $y \in \bigcup_{i \in I} \mathcal{L}_i$ then $y \in \mathcal{L}_j$ for some $j \in I$, this implies there is m_X-open set $\mathcal{W}_j$ with $y \in \mathcal{W}_j$ and $\mathcal{W}_j$-$\mathcal{L}_j$ is countable set, since $\mathcal{W}_j$-$\bigcup_{i\in I}\mathcal{L}_i \subseteq \mathcal{W}_j$-$\mathcal{L}_j$, then $\mathcal{W}_j$-$\bigcup_{i\in I}\mathcal{L}_i$ is countable set. Thus $\bigcup_{(i\in I)}\mathcal{L}_i$ is $m_X\widehat{\omega}$-open set**.**

Proposition (1.20): Assume $\mathcal{W}$ is a subset of an m-space X. $\mathcal{W}$ is $m_X\widehat{\omega}$-open set iff every point in $\mathcal{W}$ is an $m_X\widehat{\omega}$-interior point to $\mathcal{W}$**.**

Proof: Suppose $\mathcal{W}$ is an $m_X\widehat{\omega}$-open set and $x \in \mathcal{W}$, then x is an $m_X\widehat{\omega}$-interior point to $\mathcal{W}$ since $\mathcal{W}$ is an $m_X\widehat{\omega}$-open set which is containing every point in it and contained in itself. Conversely, since $\mathcal{W} = \bigcup_{x\in\mathcal{W}}\{x\}$ and every point in $\mathcal{W}$ is $m_X\widehat{\omega}$-interior point, so for each $x \in \mathcal{W}$ there exists $m_X\widehat{\omega}$-open set $\mathcal{B}$ with $x \in \mathcal{B} \subseteq \mathcal{W}$, then $\mathcal{W} = \bigcup_{x_\alpha\in\mathcal{W}}\mathcal{B}_{x_\alpha}$, for each $\alpha \in \wedge$, therefore $\mathcal{W}$ is $m_X\widehat{\omega}$-open set (by lemma (1-19)).

Proposition (1.21): 1- Consider $\mathcal{F}$ is a subset of m-space X. $\mathcal{F}$ is $m_X\widehat{\omega}$-closed iff $m_X\widehat{\omega}$-$d(\mathcal{F}) \subseteq \mathcal{F}$.

Proof: Let $\mathcal{F}$ be an $m_X\widehat{\omega}$-closed set, and let $x \in m_X\widehat{\omega}$-$d(\mathcal{F})$ and suppose $x \notin \mathcal{F}$, so $x \in \mathcal{F}^c$, where $\mathcal{F}^C$ is $m_X\widehat{\omega}$-open set, since $\mathcal{F}^c \cap \mathcal{F} = \emptyset$, so x is not $m_X\widehat{\omega}$-limit point to $\mathcal{F}$, that means $x \notin m_X\ \widehat{\omega}$-$d(\mathcal{F})$, which is a contradiction. Thus $x \in \mathcal{F}$, and therefore $m_X\widehat{\omega}$-$d(\mathcal{F}) \subseteq \mathcal{F}$. Conversely, let $m_X\widehat{\omega}$-$d(\mathcal{F}) \subseteq \mathcal{F}$. Let $x \in X$ with $x \notin \mathcal{F}$, so $x \in \mathcal{F}^c$, then x is not $m_X\widehat{\omega}$-limit point to $\mathcal{F}$, that means there is an $m_X\widehat{\omega}$-open set $\mathcal{W}$ containing x, and $\mathcal{W} \cap \mathcal{F} = \emptyset$, then $\mathcal{W} \subseteq \mathcal{F}^c$, which implies x is $m_X\widehat{\omega}$-interior point to $\mathcal{F}^c$, so $\mathcal{F}^c$is $m_X\widehat{\omega}$-open set, thus $\mathcal{F}$ is $m_X\widehat{\omega}$-closed set**.**

Proposition (1.22): An m-space (X, m_X) is $m\widehat{\omega}$-connected iff the only $m_X\widehat{\omega}$-clopen sets in X are $\emptyset$ and X**.**

Proof: Take X as an $m\widehat{\omega}$-connected space and $\mathcal{K}$ as an $m_X\widehat{\omega}$-clopen and non-empty proper subset of X, so $\mathcal{K}^c$ is also $m_X\widehat{\omega}$-clopen set and $\mathcal{K} \cup \mathcal{K}^c = X$, since $\mathcal{K}^c \neq \emptyset$. So X is $m\widehat{\omega}$-disconnected which is a contradiction, so there is no non-empty proper subset of X which is $m_X\widehat{\omega}$-clopen set. Conversely, suppose X is $m\widehat{\omega}$-disconnected then there exist non-empty disjoint $m_X\widehat{\omega}$-open sets $\mathcal{W}$ and $\mathcal{B}$ in which $\mathcal{W} \cup \mathcal{B}$=$X$, also $\mathcal{W}$ is $m_X\widehat{\omega}$-closed set because $\mathcal{W} = \mathcal{B}^c$, that is a contradiction, therefore X is $m\widehat{\omega}$-connected**.**

Proposition (1.23): An m-space X is $m\widehat{\omega}$-disconnected iff there is $m_X\ \widehat{\omega}$ -clopen set $\mathcal{W}$ in which $\emptyset \neq \mathcal{W} \neq X$.

Proof: Let $\mathcal{W}$ be a non-empty proper subset of X which is both $m_X\widehat{\omega}$-open and $m_X\widehat{\omega}$-closed. Put $\mathcal{L} = \mathcal{W}^C$. Then $\mathcal{L}$ is non-empty subset of X(because$\mathcal{W} \neq X$). Moreover, $\mathcal{W} \cup \mathcal{L} = X$ and $\cap \mathcal{L} = \emptyset$. , $\mathcal{L}$ is $m_X\ \widehat{\omega}$ -clopen (since $\mathcal{W}$ is both $m_X\widehat{\omega}$-closed and $m_X\widehat{\omega}$-open). So X is $m\widehat{\omega}$-disconnected space. Now Take X as $m\widehat{\omega}$-disconnected, hence there exist non-empty $m_X\widehat{\omega}$-open sets $\mathcal{W}, \mathcal{L}$ with $\mathcal{W} \cap \mathcal{L} = \emptyset$ and $\mathcal{W} \cup \mathcal{L} = X$, then $\mathcal{W} = \mathcal{L}^C$ so $\mathcal{W}$ is $m_X\widehat{\omega}$-closed set, which implies $\mathcal{W}$ is $m_X\widehat{\omega}$-clopen set and since $\mathcal{L}$ is non-empty set, then $\mathcal{W}$ is proper subset of X, therefore $\mathcal{W}$ is $m_X\widehat{\omega}$-clopen set in which $\emptyset \neq \mathcal{W} \neq X$.

Proposition (1.24): A subset $\mathcal{E}$ of m-space X is $m_X\widehat{\omega}$-disconnected if and only if there exist $m_X\widehat{\omega}$-open sets $\mathcal{W}$ and $\mathcal{L}$ in which $\mathcal{E} \subseteq \mathcal{W} \cup \mathcal{L}$, $\mathcal{W} \cap \mathcal{E} \neq \emptyset$, $\mathcal{L} \cap \mathcal{E} \neq \emptyset$, and $\mathcal{W} \cap \mathcal{L} \cap \mathcal{E} = \emptyset$.

Proof: Suppose that $\mathcal{E}$ is $m_X\widehat{\omega}$-disconnected set, then there exist two non-empty disjoint $m_{\mathcal{E}}\widehat{\omega}$-open subsets $\mathcal{L}$ and $\mathcal{H}$ and $\mathcal{L} \cup \mathcal{H} = \mathcal{E}$. So there are $m_X\widehat{\omega}$-open sets $\mathcal{W}$ and $\mathcal{L}$ in X, where $\mathcal{L} = \mathcal{W} \cap \mathcal{E}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{E}$, so $\mathcal{E} \subseteq \mathcal{W} \cup \mathcal{L}$, $\mathcal{W} \cap \mathcal{E} \neq \emptyset$, $\mathcal{L} \cap \mathcal{E} \neq \emptyset$ and $\mathcal{W} \cap \mathcal{L} \cap \mathcal{E} = \emptyset$. Conversely, since $\mathcal{W} \cap \mathcal{E}$ and $\mathcal{L} \cap \mathcal{E}$ are separate $\mathcal{E}$, so $\mathcal{E}$ is $m_X\widehat{\omega}$-disconnected set**.**

Definition (1.25) [2]: An m-function $h: (X, m_X) \longrightarrow (Y, m_Y)$ is called m-continuous iff $h^{-1}(\mathcal{W})$ is m_X-open set for any m_Y-open set $\mathcal{W}$**.**

Proposition (1.26): If X is m-connected space and $h: (X, m_x) \longrightarrow (Y, m_Y)$ is m-continuous function, then the image of X is m_Y-connected set**.**

Proof: Suppose $h(X)$ is not m_Y-connected set, so there exist non-empty m_Y-open sets $\mathcal{W}$ and $\mathcal{L}$ such that $h(X)$ =$\mathcal{W} \cup \mathcal{L}$ and $\mathcal{W} \cap \mathcal{L}$=$\emptyset$, but h is m-continuous, then $h^{-1}(\mathcal{W})$ and $h^{-1}(\mathcal{L})$ are two non-empty m_X-open sets, also $h^{-1}(\mathcal{W}) \cup h^{-1}(\mathcal{L}) = X$, so $X \subseteq h^{-1}(\mathcal{W}) \cup h^{-1}(\mathcal{L})$, also

$h^{-1}(\mathcal{W}) \cap h^{-1}(\mathcal{L}) \cap X = \emptyset$. Hence, $h^{-1}(\mathcal{W})$ and $h^{-1}(\mathcal{L})$ are disjoint and separation of X, which is a contradiction, so $h(X)$ is m_Y-connected set.

Definition (1.27): An m-space X is called $m\widehat{\omega}$-totally disconnected if every distinct points x and y in X has an $m_X\widehat{\omega}$-disconnection $\mathcal{W} \cup \mathcal{B}$ to X such that $x \in \mathcal{W}$ and $y \in \mathcal{B}$.

Example (1.28):

1- Let $(\mathcal{R}, m_{\mathcal{R}})$ be an m-space where $m_{\mathcal{R}}$ is the usual m-structure on $\mathcal{R}$, $(\mathcal{Q}, m_{\mathcal{Q}})$ where $m_{\mathcal{Q}}$ is the relative usual m-structure on $\mathcal{Q}$, is $m\widehat{\omega}$-disconnected subspace of $(\mathcal{R}, m_{\mathcal{R}})$, since for any q_1 and $q_2 \in \mathcal{Q}$ with $q_1 \neq q_2$, there is $e \in \mathcal{Q}^c$ such that $q_1 < e < q_2$, then $(-\infty, e)$ and (e, ∞) are disjoint $m_{\mathcal{R}}$-open sets, but $(-\infty, e) \cap \mathcal{Q} = \mathcal{W}_1$ and $\mathcal{W}_2 = (e, \infty) \cap \mathcal{Q}$ are $m_{\mathcal{Q}}$-open sets, so they are $m_{\mathcal{Q}}\ \widehat{\omega}$-open sets (by Remark (1-8)) and $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{Q}$, $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, and $q_1 \in \mathcal{W}_1$, $q_2 \in \mathcal{W}_2$, therefore $(\mathcal{Q}, m_{\mathcal{Q}})$ is $m\widehat{\omega}$-totally disconnected.

2- Let $X = \{e, f, g\}$ and $m_X = \{\emptyset, X, \{e\}, \{f, g\}, \{f\}, \{e, g\}\}$, so (X, m_X) is $m\widehat{\omega}$-totally disconnected.

Remark (1.29): Every m-totally disconnected space is $m\widehat{\omega}$-totally disconnected, since if (X, m_X) is m-totally disconnected space, then for each distinct points x, y in X there are two non-empty disjoint m_X-open sets $\mathcal{W}$ and $\mathcal{L}$ containing x, y respectively where $\mathcal{W} \cup \mathcal{L} = X$, and y Remark (1-8) (X, m_X) is $m\widehat{\omega}$-totally disconnected space.

Example (1.30): Let $X = \{e, f\}$ and $m_x = \{\emptyset, X, \{e\}\}$, so (X, m_X) is $m\widehat{\omega}$-totally disconnected space, but not m-totally disconnected.

Remark (1.31): Every m-totally disconnected space is $m\widehat{\omega}$-disconnected, but not conversely, since if (X, m_X) is m-totally disconnected space, then for any distinct points x, y in X there are two non-empty disjoint m_X-open sets $\mathcal{W}$ and $\mathcal{L}$ containing x, y respectively where $\mathcal{W} \cup \mathcal{L} = X$, so X is m-disconnected and then $m\widehat{\omega}$-disconnected (by Remark (1-15)). Example (1.32) demonstrates that the converse is false.

Example (1.32): Consider $X = \{1, 2, 3\}$ and $m_X = \{\emptyset, X, \{2, 3\}\}$, then (X, m_X) is $m\widehat{\omega}$-disconnected space but not m-totally disconnected since $2 \neq 3$, but there is no two m_X-open sets containing 2 and 3 respectively such that they are m_X-disconnection to X.

Remark (1.33): The m-continuous image of $m_X\widehat{\omega}$-totally disconnected set is not $m_Y\widehat{\omega}$-totally disconnected.

Example (1.34): Consider the m-function $I_{\mathcal{R}}: (\mathcal{R}, m_D) \longrightarrow (\mathcal{R}, m_{ind})$, $(\mathcal{R}, m_D)$ is $m\widehat{\omega}$-totally disconnected space but $(\mathcal{R}, m_{ind})$ is not $m\widehat{\omega}$-totally disconnected space.

Definition (1.35): An m-function $f: (X, m_X) \longrightarrow (Y, m_Y)$ is called $m\widehat{\omega}$-totally disconnected function if the image of any $m_X\widehat{\omega}$-totally disconnected set is $m_Y\widehat{\omega}$-totally disconnected set.

Example (1.36):

1- The identity function $I_{\mathcal{Z}}: (\mathcal{Z}, m_D) \longrightarrow (\mathcal{Z}, m_{ind})$ is $m\widehat{\omega}$-totally disconnected function.

2- 2- Let $X = Y = \{e, f, g\}$, $m_X = \{\emptyset, X, \{e, f\}\}$, $m_Y = \{\emptyset, Y, \{e\}, \{f\}\}$ and $h: X \longrightarrow Y$ defined as $h(e) = f$, $h(f) = e$ and $h(g) = g$. h *is* $m\widehat{\omega}$-totally disconnected function since the image of each $m_X\widehat{\omega}$ -totally disconnected set is $m_Y\widehat{\omega}$-totally disconnected set.

2- Some $m\widehat{\omega}$-separation axioms and $m\widehat{\omega}$ -light functions.

We introduced in this part the definitions of $m\widehat{\omega}T_1$-space and $m\widehat{\omega}T_2$-space, and we illustrate the relation between them, also we introduced $m\widehat{\omega}$-light and inversely $m\widehat{\omega}$-light function, and we connected them with the previous concepts.

Definition (2.1): An m-space (X, m_X) is called $m\widehat{\omega}T_1$-space when for every distinct points x, y in X there exist non-empty $m_X\widehat{\omega}$-open sets $\mathcal{W}$ and $\mathcal{B}$ in X in which $x \in \mathcal{W}$, $y \notin \mathcal{W}$ and $y \in \mathcal{B}$, $x \notin \mathcal{B}$.

Example (2.2):

1- Let $X = \{e, f\}$ and $m_x = \{\emptyset, X, \{e\}, \{f\}\}$, then (X, m_X) is $m\ \widehat{\omega}T_1$-space.

2- The indiscrete m-space $(\mathcal{Z}, m_{ind})$ is $m\widehat{\omega}T_1$-space.

Definition (2.3): An m-space (X, m_X) is called $m\widehat{\omega}$-Hausdorff when for every distinct points a and b in X there exist non-empty disjoint $m_X\widehat{\omega}$-open sets $\mathcal{W}$ and $\mathcal{L}$ containing a and b respectively.

Example (2.4): Let X= $\{e, f\}$ and m_x= $\{\emptyset, X, \{e\}\}$ is $m\widehat{\omega}T_2$-space.

Remark (2.5): Every $m\widehat{\omega}T_2$-space is $m\widehat{\omega}T_1$-space, since if (X, m_X) is $m\widehat{\omega}T_2$-space, then for each distinct points x, y in X there exist non-empty disjoint $m_X\widehat{\omega}$-open sets $\mathcal{W}$ and $\mathcal{B}$ containing x and y respectively, since $\mathcal{W} \cap \mathcal{B} = \emptyset$ so $\mathcal{W}$ does not contain y and $\mathcal{B}$ does not contain x, so (X, m_X) is $m\widehat{\omega}T_1$-space.

Example (2.6): $(\mathcal{R}, m_{cof})$ is $m\widehat{\omega}T_1$-space while it is not $m\widehat{\omega}T_2$-space**.**

Remark (2.7): Every $m\widehat{\omega}$-totally discon-nected space is $m\widehat{\omega}$-Hausdorff, but not conversely. Since if (X, m_X) is $m\widehat{\omega}$-totally disconnected space and x, y be different points in X, so there are exist non-empty disjoint $m_X\widehat{\omega}$-open sets $\mathcal{W}, \mathcal{B}$ containing x, y respectively with $\mathcal{W} \cup \mathcal{B} = X$, so (X, m_X) is $m\widehat{\omega}T_2$-space. Example (2.8) demonstrates that the converse is false.

Example (2.8): The usual $m_{\mathcal{R}}$-structure on $\mathcal{R}$ is $m\widehat{\omega}$-Hausdorff space, but not $m\widehat{\omega}$-totally disconnected space**.**

Remark (2.9): Every m-Hausdorff space is $m\widehat{\omega}$-Hausdorff, but not conversely. Since if (X, m_X) is m-Hausdorff space, so for any distinct points a, b in X there exist non-empty disjoint m_X-open sets $\mathcal{W}$ and $\mathcal{B}$ containing a, b respectively, by remark (1-8), (X, m_X) is $m\widehat{\omega}$-Hausdorff. Example (2.10) demonstrates that the converse is false.

Example (2.10): Let X= {1, 2} and m_x= $\{\emptyset, X, \{1\}\}$, then (X, m_X) is $m\widehat{\omega}$-Hausdorff space but not m-Hausdorff.

Definition (2.11): A surjective m-function h from m-space (X, m_X) onto m-space (Y, m_Y) is called $m\widehat{\omega}$-light function if the $h^{-1}(y)$ is $m_X\widehat{\omega}$-totally disconnected set for any $y \in Y$**.**

Example (2.12): Let $h: (X, m_X) \longrightarrow (Y, m_Y)$ be an m-function where X={1, 2, 3, 4, 5, 6}, m_X={$\emptyset$, X, {1}, {2}, {1, 2}, {3}, {4}, {3, 4}, {5}, {6}, {5, 6}} and Y={7, 8, 9} in which h(1)=h(2)=7, h(3)=h(4)=8 and h(5)=h(6)=9, then h is $m\widehat{\omega}$-light function**.**

Remark (2.13): Every m-light function is $m\widehat{\omega}$-light function, but not conversely. Since if $h: (X, m_X) \longrightarrow (Y, m_Y)$ is m-light function, so the inverse image of each $y \in Y$ is m_X-totally disconnected set, then it is $m_X\widehat{\omega}$-totally disconnected set (by Remark (1.29)), thus h is $m\widehat{\omega}$-light function. . Example (2.14) demonstrates that the converse is false.

Example (2.14): Let $h: (\mathcal{Z}, m_{ind}) \longrightarrow (\mathcal{Z}, m_{cof})$ be an m-function such that $h(a)$ =c for any $a \in \mathcal{Z}$, then h is $m\widehat{\omega}$-light function but not m-light function**.**

We found the following Remark in [1] without proof. We provided its proof here for completeness**.**

Remark (2.15): Every m-homeomorphism function is m-light function, but not conversely. Since if $h: (X, m_X) \longrightarrow (Y, m_Y)$ is m-homeomorphism function then for every $y \in Y$ there exists a unique $x \in X$ such that $h(x) = y$ (because h is bijective function), and since $\{x\}$ is m_X-totally disconnected set, thus h is m-light function**.** Example (2.16) demonstrates that the converse is false.

Example (2.16): The m-function $h: (X, m_X) \longrightarrow (Y, m_Y)$, where $X = \{e, f, g\}$, $m_X = \{\emptyset, X, \{e\}, \{g\}, \{f, g\}, \{e, g\}\}$, and Y= $\{k\}$ where $h(x) = k$ for each $x \in X$, then h is m-light function but not m-homeomorphism**.**

Remark (2.17): Every m-homeomorphism function is $m\widehat{\omega}$-light function, but not conversely. Since h is bijective then for any $y \in Y$ there is a unique $\ell \in X$ such that $h(\ell) = y$, and $\{\ell\}$ is m_X-totally disconnected and then $m_X\widehat{\omega}$-totally disconnected (by Remark (1.29)), so h is $m\widehat{\omega}$-light function. Example (2.18) demonstrates that the converse is false.

Example (2.18): In (2.16), h is $m\widehat{\omega}$-light function but not m-homeomorphism**.**

Proposition (2.19): Whenever (Y, m_Y) is an m-subspace of an m-space (X, m_X), and if $\mathcal{L}$ is $m_X\widehat{\omega}$-open set, hence it is $m_Y\widehat{\omega}$-open set in which $\mathcal{L} \subseteq Y$**.**

Proof: Suppose (X, m_X) is an m-space and $\mathcal{L}$ is $m_X\widehat{\omega}$-open set, then for each $f \in \mathcal{L}$ there is an m_X-open set $\mathcal{W}$ contains f and $\mathcal{W}$-$\mathcal{L}$ is countable, and since (Y, m_Y) is m-subspace of (X, m_X) then $\mathcal{W} \cap Y$ is m_Y-open set in Y containing f (since $f \in \mathcal{W}$ and $f \in \mathcal{L} \subseteq Y$), hence $(\mathcal{W}\cap Y)$-$\mathcal{L}$ is countable (since $(\mathcal{W}\cap Y)$-$\mathcal{L} \subseteq \mathcal{W}$-$\mathcal{L}$], then $\mathcal{L}$ is $m_Y\widehat{\omega}$-open set

Example (2.20): Let (Q, m_{ind}) be the indiscrete m-space and $(\mathcal{Z}, m_{ind})$ be an m-subspace of (Q, m_{ind}), then $\{0\}$ is $m_{\mathcal{Z}}\widehat{\omega}$-open set in $(\mathcal{Z}, m_{ind})$, since $\{0\}$ is a subset of $(\mathcal{Z}, m_{ind})$ and it is $m_Q\widehat{\omega}$-open set.

Proposition (2.21): Let $h: (X, m_X) \longrightarrow (Y, m_Y)$ be $m\widehat{\omega}$-light function, then $h|_U: U \longrightarrow f(U)$ is also $m\widehat{\omega}$-light function where $U \subseteq X$.

Proof: Let $\mathscr{b} \in h(U) \subseteq Y$, then $\mathscr{b} \in Y$ but h is $m\widehat{\omega}$-light function so $h^{-1}(\mathscr{b})$ is $m_X\widehat{\omega}$-totally disconnected set. We will prove that $h^{-1}(\mathscr{b})\cap U$ is $m_U\widehat{\omega}$-totally disconnected set for any $\mathscr{b} \in h(U)$. Let $x, y \in h^{-1}(\mathscr{b})\cap U$, so $x, y \in h^{-1}(\mathscr{b})$ and since $h^{-1}(\mathscr{b})$ is $m_X\widehat{\omega}$-totally disconnected set, so there is an $m_X\widehat{\omega}$-disconnection $H \cup V$ for $h^{-1}(\mathscr{b})$ such that $\left(H \cap h^{-1}(\mathscr{b})\right) \cup \left(V \cap h^{-1}(\mathscr{b})\right) = h^{-1}(\mathscr{b})$ and $(H \cap h^{-1}(\mathscr{b})) \cap (V \cap h^{-1}(\mathscr{b})) = \emptyset$, in which H and V are $m_X\widehat{\omega}$-open sets, and $x \in H, y \in V$. Now to prove that $h^{-1}(\mathscr{b})\cap U$ is $m_U\widehat{\omega}$-totally disconnected set. Since $((U\cap h^{-1}(\mathscr{b})) \cap H) \cup ((U\cap h^{-1}(\mathscr{b}))\cap V) = (U \cap (h^{-1}(\mathscr{b})\cap H)) \cup (U\cap(h^{-1}(\mathscr{b})\cap V)) = U \cap ((h^{-1}(\mathscr{b})\cap H) \cup (h^{-1}(\mathscr{b})\cap V)) = U\cap h^{-1}(\mathscr{b})$, and $((U \cap h^{-1}(\mathscr{b}))\cap H) \cap ((U \cap h^{-1}(\mathscr{b}))\cap V) = (U \cap (h^{-1}(\mathscr{b})\cap H)) \cap (U \cap (h^{-1}(\mathscr{b})\cap V) = U \cap ((h^{-1}(\mathscr{b})\cap H) \cap (h^{-1}(\mathscr{b})\cap V)) = U\cap\emptyset = \emptyset$, where $x \in (U \cap h^{-1}(\mathscr{b})) \cap H$ and $y \in (U \cap h^{-1}(\mathscr{b})) \cap V$, then $(U \cap h^{-1}(\mathscr{b}))\cap V$ and $(U \cap h^{-1}(\mathscr{b}))\cap H$ are disjoint $m_U\widehat{\omega}$-open sets and their union is equal to $h^{-1}(\mathscr{b})\cap U$, therefore $h^{-1}(\mathscr{b})\cap U$ is $m_U\widehat{\omega}$-totally disconnected set, which means $h|_U$ is $m\widehat{\omega}$-light function.

Definition (2.22): A surjective m-function $h: (X, m_X) \longrightarrow (Y, m_Y)$ is called inversely $m\widehat{\omega}$-totally disconnected function if $h^{-1}(\mathcal{W})$ is $m_X\widehat{\omega}$-totally disconnected set for any $m_Y\,\widehat{\omega}$-totally disconnected set.

Example (2.23): The identity function $I_{\mathcal{Z}}: (\mathcal{Z}, m_{ind}) \longrightarrow (\mathcal{Z}, m_{\mathcal{Z}})$ is inversely $m\widehat{\omega}$-totally disconnected.

Proposition (2.24): inversely $m\widehat{\omega}$-totally disconnected function, is $m\widehat{\omega}$-light function.

Proof: Let $y \in Y$, to prove $h^{-1}(y)$ is $m_X\widehat{\omega}$-totally disconnected, since h is inversely $m\widehat{\omega}$-totally disconnected function then it is surjective m-function, and since $\{y\}$ is $m_Y\widehat{\omega}$-totally disconnected set, then $h^{-1}(\{y\})$ is $m_X\widehat{\omega}$-totally disconnected set, which implies h is $m\widehat{\omega}$-light function.

Proposition (2.25): Let $\mathfrak{F}: (X, m_X) \longrightarrow (\mathcal{Z}, m_{\mathcal{Z}})$ and $\mathcal{G}: (\mathcal{Z}, m_{\mathcal{Z}}) \longrightarrow (Y, m_Y)$ be surjective m-functions, then the m-function $h: (X, m_X) \longrightarrow (Y, m_Y)$ where $h = \mathcal{G} \circ \mathfrak{F}$ is $m\widehat{\omega}$-light function if $\mathfrak{F}$ is inversely $m\widehat{\omega}$-totally disconnected function and $\mathcal{G}$ is $m\widehat{\omega}$-light function.

Proof: We will prove that for any $y \in Y$, $h^{-1}(y)$ is $m_X\widehat{\omega}$-totally disconnected set. Let $y \in Y$, then $h^{-1}(y) = (\mathcal{G} \circ \mathfrak{F})^{-1}(y) = \mathfrak{F}^{-1}(\mathcal{G}^{-1}(y))$, and since $\mathcal{G}$ is $m\widehat{\omega}$-light function, so $\mathcal{G}^{-1}(y)$ is $m_{\mathcal{Z}}\widehat{\omega}$-totally disconnected set, and since $\mathfrak{F}$ is inversely $m\widehat{\omega}$-totally disconnected function, hence $\mathfrak{F}^{-1}(\mathcal{G}^{-1}(y))$ is $m_X\,\widehat{\omega}$-totally disconnected set, but $h^{-1}(y) = \mathfrak{F}^{-1}(\mathcal{G}^{-1}(y))$, therefore $h^{-1}(y)$ is $m_X\widehat{\omega}$-totally disconnected set, and then h is $m\widehat{\omega}$-light function.

Proposition (2.26): Let $h: (X, m_X) \longrightarrow (Y, m_Y)$ be a surjective m-function and $h = \mathcal{G} \circ f\mathfrak{F}$ where $\mathfrak{F}: (X, m_X) \longrightarrow (\mathcal{Z}, m_{\mathcal{Z}})$ and $\mathcal{G}: (\mathcal{Z}, m_{\mathcal{Z}}) \longrightarrow (Y, m_Y)$ are m-functions, if $\mathcal{G}$ is bijective m-function and $\mathfrak{F}$ is $m\widehat{\omega}$-light function, then h is $m\widehat{\omega}$-light function too.

Proof: Let $y \in Y$, since $\mathcal{G}$ is bijective m-function, then there exists one point $z \in \mathcal{Z}$ such that $\mathcal{G}(z) = y$ and since $h^{-1}(y) = (\mathcal{G} \circ \mathfrak{F})^{-1}(y) = \mathfrak{F}^{-1}(\mathcal{G}^{-1}(y)) = \mathfrak{F}^{-1}(\mathcal{G}^{-1}(\mathcal{G}(z)) = \mathfrak{F}^{-1}(z)$ and since $\mathfrak{F}$ is $m\widehat{\omega}$-light function, so $\mathfrak{F}^{-1}(z)$ is $m_X\widehat{\omega}$-totall disconnected set, but $h^{-1}(y) = \mathfrak{F}^{-1}(z)$, this implies $h^{-1}(y)$ is $m_X\widehat{\omega}$-totally disconnected set, therefore h is $m\widehat{\omega}$-light function.

Proposition (2.27): Let $h: (X, m_X) \longrightarrow (Y, m_Y)$ be a surjective m-function and $h = \mathcal{G} \circ \mathfrak{F}$ where $\mathfrak{F}: (X, m_X) \longrightarrow (\mathcal{Z}, m_{\mathcal{Z}})$ and $\mathcal{G}: (\mathcal{Z}, m_{\mathcal{Z}}) \longrightarrow (Y, m_Y)$ are m-functions, then if h is $m\widehat{\omega}$-light function and $\mathcal{G}$ is one-to-one m-function, then $\mathfrak{F}$ is $m\widehat{\omega}$-light function.

Proof: Take $z \in \mathcal{Z}$, then $\mathcal{G}(z) \in Y$ and since h is $m\widehat{\omega}$-light function, then $h^{-1}(\mathcal{G}(z))$ is $m_X\widehat{\omega}$-totally disconnected set, but $h^{-1}(\mathcal{G}(z)) = (\mathcal{G} \circ \mathfrak{F})^{-1}(\mathcal{G}(z)) = \mathfrak{F}^{-1}(\mathcal{G}^{-1}(\mathcal{G}(z))) = \mathfrak{F}^{-1}(z)$, that means $\mathfrak{F}^{-1}(z)$ is $m_X\widehat{\omega}$-totally disconnected set, therefore $\mathfrak{F}$ is $m\widehat{\omega}$-light function.

Proposition (2.28): Let $h:(X,m_X)\longrightarrow(Y,m_Y)$ be a surjective m-function and $h=\mathcal{G}\circ\mathfrak{F}$ where $\mathfrak{F}:(X,m_X)\longrightarrow(\mathcal{Z},m_{\mathcal{Z}})$ and $\mathcal{G}:(\mathcal{Z},m_{\mathcal{Z}})\longrightarrow(Y,m_Y)$ are m-functions, then if h is $m\widehat{\omega}$-light function and $\mathfrak{F}$ is $m\widehat{\omega}$-totally disconnected function then $\mathcal{G}$ is $m\widehat{\omega}$-light function.

Proof: Let $y\in Y$, since h is $m\widehat{\omega}$-light function, then $h^{-1}(y)$ is $m_X\widehat{\omega}$-totally disconnected, and since $\mathfrak{F}$ is $m\widehat{\omega}$-totally disconnected function, so $\mathfrak{F}(h^{-1}(y))$ is $m_{\mathcal{Z}}\widehat{\omega}$-totally disconnected set, but $\mathfrak{F}(h^{-1}(y))$ = $\mathfrak{F}((\mathcal{G}\circ\mathfrak{F})^{-1}(y))=\mathfrak{F}(\mathfrak{F}^{-1}(\mathcal{G}^{-1}(y))=\mathcal{G}^{-1}(y)$, therefore $\mathcal{G}^{-1}(y)$ is $m_{\mathcal{Z}}\widehat{\omega}$-totally disconnected set $\mathcal{Z}$,
which implies that $\mathcal{G}$ is $m\widehat{\omega}$-light function.

Definition (2.29): A zero dimension m-space (X,m_X) is an m-space having a base of sets that are $m_X\widehat{\omega}$-clopen.

Lemma (2.30): Every zero dimension metric m-space is $m\widehat{\omega}$-totally disconnected.

Proof: Let (X,m_X) be a zero dimension metric m-space and let x,y be two distinct points in X, since X is metric m-space, so it is m-Hausdorff space, then x has a neighbourhood G which does not contain y, since X is zero dimension m-space, hence there is a basic m_X-open set $\mathcal{B}$ which is also m_X-closed set and then $\mathcal{B}$ is $m_X\widehat{\omega}$-clopen set such that $x\in\mathcal{B}\subseteq G$, and since $\mathcal{B}$ is $m_X\widehat{\omega}$-clopen set then $\mathcal{B}^C$ is also $m_X\widehat{\omega}$-clopen set which contains y, it is clear that $X=\mathcal{B}\cup\mathcal{B}^C$ and $\mathcal{B}\cap\mathcal{B}^C=\emptyset$, so X is $m\widehat{\omega}$-totally disconnected space.

Proposition (2.31): A surjective m-function $h:(X,m_X)\longrightarrow(Y,m_Y)$ where X and Y are metric m-spaces and X is $m\widehat{\omega}$-compact space, is called $m\widehat{\omega}$-light function if the inverse image for any $y\in Y$ is zero dimension.

Proof: Consider $y\in Y$, since $h^{-1}(y)$ is subset of X which is metric m-space then $h^{-1}(y)$ is metric m-subspace of X (metric is a hereditary property), so $h^{-1}(y)$ is zero dimension metric m-subspace of X, so by lemma (2.30) $f^{-1}(y)$ is $m_X\,\widehat{\omega}$-totally disconnected subspace and hence f is $m\widehat{\omega}$-light function.

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