Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online)
Volume 30, Issue 3, 2019
DOI: http://doi.org/10.23851/mjs.v30i3.675

Research Article Open Access

Analytical Approximations for Nonlinear Integral and Integro-Differential Equations

Nabaa N. Hasan, Majid R. Nasif
Department of Mathematics, College of Science, Mustansiriyah University, IRAQ
*Correspondent author email: alzaer1972@uomustansiriyah.edu.iq

Article Info

Received
06/06/2019

Accepted
10/06/2019

Published
15/10/2019

Abstract
In this work, we are concerned with how to find a solution for the nonlinear integral and integro-differential equations using two methods Laplace transform series decomposition method (LTSDM) and Sumudu transform series decomposition method (STSDM). In these methods, the nonlinear part of the equation described in Adomian decomposition series. The Laplace and Sumudu methods are found to be reliable and accurate. Four examples are discussed to check the applicability and the simplicity of these methods. Finally, the results are tabulated and displayed graphically to make comparisons between the approximate and exact solutions.

Keywords: Nonlinear Intgro-Differential Equation of Second Kind, Adomian Decomposition Method, Laplace Transform, Sumudu Transform.

الخلاصة
في هذا العمل, نعني كيفية حل المعادلات التكاملية غير خطية والمعادلات التكاملية-التفاضلية غير خطية باستخدام طريقتين (LTSDM) و (STSDM). في هذه الطرق، الجزء غير الخطي من المعادلة الموصوفة في المتسلسلة (Adomian decomposition seriesi). طريقتي لابلاس و سيمودو وجدت لتكون دقيقة و فعالة. نوقشت أربعة امثلة لتدقيق قابلية تطبيق و بساطة هذه الطرق, اخيراً النتائج عرضت بشكل مخططات للمقارنة بين الحلول التقريبية و المضبوطة.

Introduction

For solving nonlinear integral equations, Adomian decomposition method was introduced by George Adomian in 1280, Basically, the technique provides an infinite series solution of the equation and the nonlinear term is decomposed into an infinite series of Adomian polynomials .Several linear and nonlinear ordinary, partial, deterministic and stochastic differential equations are solved by Adomian decomposition method. A comparison was made between Laplace decomposition method homotopy perturbation method and wavelet Galerkin method for solving nonlinear Volterra integro-differential equations; we show that (LTSDM) and (STSDM) are used for finding the better approximate solution of real valued function [7]. In [2], the modified form of Laplace decomposition method has been introduced by Manafianheris, to solve a nonlinear Volterra-Fredholm integral and integro-differential equation using operational matrix with block-ipulse function. Watugalas [4] showed that the Sumudu transform can be effectively used to solve ordinary differential equations and engineering control problems. For more details about Adomian decomposition method see [8]. In this work, we focus to solve nonlinear integral and intgro-differential equation of the secondi kind by (LTSDM) and (STSDM). In section 2, the relation between Laplace and Sumudu transforms given in the Theorem 2.1 [5].

Approximate methods

The nonlinear integral and intgro-differential equation of the second kind is written as follows [2]:

$$u^{(i)}(x) = g(x) + \int_{a}^{b(x)} K(x,t)\, G\big(u(t)\big)dt,\ i \in N \qquad (1)$$

where $u^{(i)}(x) = \frac{d^{\,i}u}{dx^i}$, $i \geq 1$, and $G\big(u(x)\big)$ is a nonlinear function of $u\,(x)$, consider the kernel $K(x,t)$ of equation (1) as difference kernel $K(x,t) = K(x-t)$ such as e^{x-t}, $\sin(x-t)$ and so on [9]. So the equation (1) becomes:

$$u^{(i)}(x) = g(x) + \int_{a}^{b(x)} K(x-t)\, G\big(u(t)\big)dt \qquad (2)$$

Now, in [5] consider the following set of functions:
$A = \{\, f(t) |\, \exists\, M, \tau_1\, , and\ /or\ \tau_2 > 0\, , |f(t)| < Me^{|t|/\tau_j}\, , if\ t \in (-1)^j \times [0\, ,\infty), j = 1, 2\}$
such that the constant M must be finite, while τ_1 and τ_2 need not simultaneously exist. The next theorem shows, the Sumudu transform is closely connected with the Laplace transform.

Theorem 2.1 [5]: Let $f(t) \in A$ with Laplace transform F(s). Then the Sumudu transform R(r) of $f(t)$ is given by:

$$R(r) = \frac{F(\frac{1}{r})}{r} \qquad (3)$$

where R(r) the Sumudu transform of $f(t)$, for example, if $f(t) = \cos(t)$, then:

$$F(s) = L\{f(t)\} = \frac{s}{s^2+1}\,,$$

$$R(r) = \delta\{\, f(t)\, \} = \frac{F\left(\frac{1}{r}\right)}{r} = \frac{1}{r^2+1}$$

Laplace Adomian Decomposition Method (LADM) [3][9][10]
A combined form is proposed for solving the nonlinear integral and intgro-differentiali equations of the second kind based on Laplace Adomian decomposition method. Laplace of the i[th] derivatives of the continuous function $u\,(x)$, is given by:

$$L[u^{(i)}(x)] = S^i\, L[u(x)] - S^{i-1}\, u(0) - S^{i-2}u'(0) - S^{i-3}\, u''(0) \ldots u^{(i-1)}\,(0) \qquad (4)$$

Let us take a simple example that illustrates the equation (4):

$$L[u''\,(x)] = S^2\, L[u(x)] - S\, u(0) - u'(0) \qquad (5)$$

Now, in equation (2) there are two cases:

***Case (1)*:** If a = 0, b(x) = x (Volterra integral equation), then applying the Laplace transform for equation (2) gives:

$$\begin{aligned} &S^i\, L[u(x)] - S^{i-1}\, u(0) - S^{i-2}u'^{(0)} - \\ &S^{i-3}\, u''(0)\ \cdots\ u^{(i-1)}\,(0)\, , i \geq 1 \\ &= L(g(x)) + L(K(x-t)) * L(G(u(x))) \end{aligned} \qquad (6)$$

The important part of equation (6) " $L(K(x-t)) * L(G(u(x)))$" was obtained by The Convolution Theorem for Laplace Transform [9], which states the following:
Consider two functions $f1(x)$ and $f2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f1(x)$ and $f2(x)$ be given by:

$$L\{f1(x)\} = F1(s) \qquad (7)$$

$$L\{f2(x)\} = F2(s) \qquad (8)$$

The Laplace convolution product of these two functions is defined by:

$$(f1\, *\, f2)(x) = \int_0^x f1(x\, -\, t)\, f2(t)\, dt \qquad (9)$$

or;

$$(f2\, *\, f1)(x) = \int_0^x f2(x\, -\, t)\, f1(t)\, dt \qquad (10)$$

Recall that:

$$(f1\, *\, f2)(x) = (f2\, *\, f1)(x) \qquad (11)$$

To show that the Laplace transform of the convolution product $(f1\, *\, f2)(x)$ is given by:

$$\begin{aligned} L\{(f1\, *\, f2)(x)\} &= L\{ \int_0^x \mathrm{f1(x\, -\, t)\, f2(t)dt} \} \\ &= \mathrm{F1(s)F2(s)} \end{aligned} \qquad (12)$$

***Case (2)*:** If a and b(x) are constants (Fredholm integral equation), then applying the Laplace inverse for equation (2) after calculating the integral, equation (6) is equivalent to the following:

$$L[u\,(x)] = \frac{1}{S}\,u(0) + \frac{1}{S^2}u'(0) + \cdots + \frac{1}{S^i}u^{(i-1)}\,(0) + \frac{1}{S^i}\,L\big(g(x)\big) + \frac{1}{S^i}L\big(K(x-t)\big) * L\Big(G\big(u(x)\big)\Big) \quad (13)$$

by putting the linear term in an infinite series of the following form:

$$u(x) = \sum_{n=0}^{\infty} u_n\,(x) \quad (14)$$

and Adomian decomposition series of the nonlinear function $G(u(x))$ is given in the form:

$$G\big(u(x)\big) = \textstyle\sum_{n=0}^{\infty} A_n\,(x),$$

where

$$A_n = \frac{1}{n!}\,\frac{d^n}{d\,\beta^n}\left[G\left(\sum_{i=0}^{\infty}\beta^{\,i}\,u_{\,i}\right)\right]\beta = 0, \quad n = 0\,,1\,,2\,,\ldots \quad (15)$$

substituting equations (14) and (15) in equation (13) leads to:

$$L[\sum_{n=0}^{\infty} u_n\,(x)\,] = \frac{1}{S}u(0) + \frac{1}{S^2}u'(0) + \cdots + \frac{1}{S^{\,i}}u^{(i-1)}\,(0) + \frac{1}{S^{\,i}}\,L\big(g(x)\big) + \frac{1}{S^{\,i}}L\big(K(x-t)\big) * L\left(\sum_{n=0}^{\infty} A_n\,(x)\right) \quad (16)$$

the Adomian decomposition method admits the use of the following recursive relation:

$$L[u_0(x)] = \frac{1}{S}u(0) + \frac{1}{S^2}u'(0) + \cdots + \frac{1}{S^{\,i}}u^{(i-1)}\,(0) + \frac{1}{S^i}\,L\big(g(x)\big)\,,$$
$$L[u_{k+1}(x)] = \frac{1}{S^i}L\big(K(x-t)\big) * L\big(\,A_k\,(x)\big), \quad for\ k \geq 0 \quad (17)$$

Now, applying the Laplace inverse for equation (17), we get:

$$u_{k+1}(x) = L^{-1}\left\{\tfrac{1}{S^{\,i}}L\big(K(x-t)\big) * L\big(\,A_k\,(x)\,\big)\right\},\ k \geq 0$$

thus;

$$u_{approximate}(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

Sumudu Adomian Decomposition Method (SADM): [1][4][10]

A combined form is proposed for solving the nonlinear integral and intgro-differentiali equations of the second kind based on Sumudu Adomian decomposition method. Sumudu of the i[th] derivatives of the continuous function $u\,(x)$, is given by:

$$\delta[u^{(i)}(x)] = \frac{\delta\big(u(x)\big)}{r^i} - \frac{u(0)}{r^i} - \frac{u'(0)}{r^{i-1}} - \frac{u''(0)}{r^{\,i-2}} - \cdots - \frac{u^{(i-1)}(0)}{r},\ i \geq 0 \quad (18)$$

where δ is represent Sumudu transform. Let us take a simple example that illustrates the equation (12):

$$\delta[u''\,(x)] = \frac{\delta\big(u(x)\big)}{r^2} - \frac{u(0)}{r^2} - \frac{u'(0)}{r} \quad (19)$$

Now, in equation (2) there are two cases:

Case (1): If a = 0, b(x) = x (Volterra integral equation). Then applying the Sumudu transform for equation (2), we obtain:

$$\frac{\delta\big(u(x)\big)}{r^{\,i}} - \frac{u(0)}{r^{\,i}} - \frac{u'(0)}{r^{\,i-1}} - \frac{u''(0)}{r^{\,i-2}} - \cdots - \frac{u^{(i-1)}(0)}{r} = \delta\big(g(x)\big) + \delta\big(K(x-t)\big) * \delta\Big(G\big(u(x)\big)\Big),\ i \geq 1 \quad (20)$$

The important part of equation (20) "$\delta(K(x-t)) * \delta(G(u(x)))$" was obtained by The Convolution Theorem for Sumudu Transform [4].

***Case (2)*:** If a and b(x) are constants (Fredholm integral equation), then applying the Sumudu inverse for equation (2) after calculating the integral, equation (20) is equivalent to the following:

$$\delta\big(u(x)\big) = u(0) + r\,u'^{(0)} + r^2\,u''(0) + \cdots + r^{\,i}\delta\big(g(x)\big) + r^{\,i}\left[\delta\big(K(x-t)\big) * \delta\Big(G\big(u(x)\big)\Big)\right],\ i \geq 1 \quad (21)$$

by putting the linear term in an infinite series of the following form:

$$u(x) = \sum_{n=0}^{\infty} u_n\,(x) \quad (22)$$

and Adomian decomposition series of the nonlinear function $G(u(x))$ is given by equation (15). Substituting equations (22) and (15) in equation (21) leads to:

$$\delta\left[\sum_{n=0}^{\infty} u_n(x)\right] = u(0) + r\,u'(0) + r^2\,u''(0) + \cdots + r^i\delta(g(x)) + r^i\left[\delta(K(x-t)) * \delta\left(\sum_{n=0}^{\infty} A_n(x)\right)\right], i \geq 0 \quad (23)$$

the Adomiani decomposition method admits the use of the following recursive relation

$$\delta[u_0(x)] = u(0) + r\,u'(0) + r^2\,u''(0) + \cdots + r^i\delta(g(x))\delta[u_{k+1}(x)] = r^i\,[\delta(K(x-t)) * \delta(A_k(x))\,], k \geq 0 \quad (24)$$

Now, by applying the Sumudu inverse for equation (18), we have:

$u_{k+1}(x) = \delta^{-1}\left\{\frac{1}{s^i}\delta(K(x-t)) * \delta(A_k(x))\right\}, k \geq 0$

thus;

$u_{approximate}(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$

the Laplace transform series decomposition method (LTSDM) and the Sumudu transform series decomposition method (STSDM) are used to solve nonlinear Integral and intgro-differential equations of the secondi kind and we will illustrate this method by using following illustrative examples.

Illustrative examples

To show the presented method is efficiency and convenience, four illustrated examples are given. Notice that in all the examples below we will calculate which is defined as:

$$\text{L.S.E.} = \sum_{i=0}^{10}(E_i - A_i)^2,$$

where E_i, A_i are exact solution and approximate solutions respectively.

Example (1) [9]: Consider the following non-linear Fredholm integral equation of the second kind:

$$u(x) = x + \int_0^1 x\,t\,u^2(t)\,dt \quad (25)$$

The exact solution$u(x) = 2\,x$, to get rapidly convergent to the exact solution, we can modify (19) as follows:

$$u(x) = 1.9999\,x - 0.9999\,x + \int_0^1 x\,t\,u^2(t)\,dt \quad ,n = 2 \quad (26)$$

where n is the number of Adomian polynomials. In this example, using the methods LTSDM and STSDM, we get the same approximate solution:

$$u_{approx.}(x) = 1.99990000749962500 78\,x \quad (27)$$

We found the value of equation (27) after we applied both LTSDM and STSDM 2 times ($n = 2$).

Table 1: shows the exact and approximate Results and the least square error (L.S.E.) foroExample 1.

x	Exact solution	Approximate solution by LTSDM	Approximate solution by STSDM	Absolute error of LTSDM	Absolute error of STSDM
0	0	0	0	0	0
0.1	0.2	0.19999000075	0.19999000075	0.00000999925	0.00000999925
0.2	0.4	0.3999800015	0.3999800015	0.00001999850	0.00001999850
0.3	0.6	0.59997000225	0.59997000225	0.00002999775	0.00002999775
0.4	0.8	0.799960003	0.799960003	0.00003999700	0.00003999700
0.5	1.0	0.99995000375	0.99995000375	0.00004999625	0.00004999625
0.6	1.2	1.1999400045	1.1999400045	0.00005999550	0.00005999550
0.7	1.4	1.39993000525	1.39993000525	0.00006999475	0.00006999475
0.8	1.6	1.599920006	1.599920006	0.00007999400	0.00007999400
0.9	1.8	1.79991000675	1.79991000675	0.00008999325	0.00008999325
1.0	2.0	1.9999000075	1.9999000075	0.00009999250	0.00009999250
L.S.E.		3.84942252e-8	3.84942252e-8		

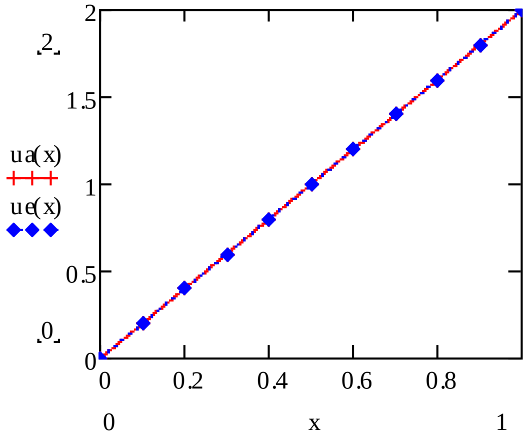

Figure1: Comparison of Exact Solution and Approximate Solution of example (1).

Example (2) [9]: Consider the following non-linear Volterra integral equation of second kind:

$$u(x)=\sin(x)+\frac{2}{3}\cos(x)-\frac{1}{3}(\cos(x))^2-\frac{1}{3}+\int_0^x \sin(x-t)\,u^2(t)\;dt \qquad (28)$$

The exact solution is $u(x)=\sin(x)$. In this example, using the methods LTSDM and STSDM, we get the same approximate solution:

$$u_{approx.}(x)=\frac{x^9}{362880}-\frac{x^7}{5040}+\frac{x^5}{120}-\frac{x^3}{6}+x \quad ,n=4 \qquad (29)$$

where n is the number of Adomian polynomials.

Table 2: Shows the exact and approximate Results and the least square error (L.S.E.) for Example 2.

x	Exact solution	Approximate solution by LTSDM	Approximate solution by STSDM	Absolute error of LTSDM	Absolute error of STSDM
0	0.0	0.0	0.0	0.0	0.0
0.1	0.0998334166468	0.0998334166468	0.0998334166468	0.0	0.0
0.2	0.198669330795	0.198669330795	0.198669330795	0.0	0.0
0.3	0.295520206661	0.295520206661	0.295520206661	0.0	0.0
0.4	0.389418342309	0.38941834231	0.38941834231	1.0496847e-12	1.0496847e-12
0.5	0.479425538604	0.479425538616	0.479425538616	1.221289479e-11	1.221289479e-11
0.6	0.564642473395	0.564642473486	0.564642473486	9.067892851e-11	9.067892851e-11
0.7	0.644217687238	0.644217687732	0.644217687732	4.9380971793e-10	4.9380971793e-10
0.8	0.7173560909	0.717356093043	0.717356093043	2.14315801438e-9	2.14315801438e-9
0.9	0.783326909627	0.783326917448	0.783326917448	7.82095411154e-9	7.82095411154e-9
1.0	0.841470984808	0.8414710097	0.8414710097	2.48922798602e-8	2.48922798602e-8
L.S.E.		6.85624616e-16	6.85624616e-16		

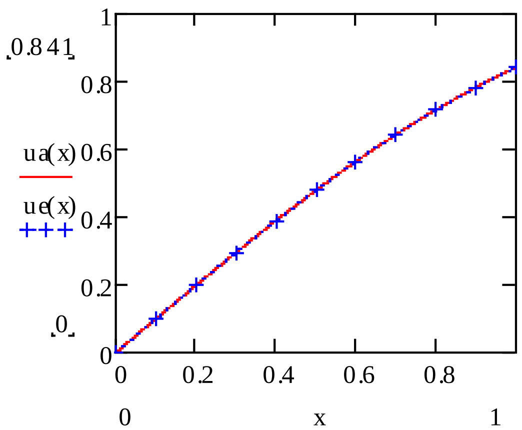

Figure2: Comparison of Exact Solution and Approximate Solution of example (2).

Example (3) [9]: Consider the following non-linear Volterra integro-differential equation of secondi kind:

$$u''(x)=-1-\frac{1}{3}(\sin(x)+\sin(2x))+2\cos(x)+\int_0^x \sin(x-t)\,u^2(t)\;dt \qquad (30)$$

where $u(0)=-1,\ u'(0)=1$.

The exact solution is $u(x)=\sin(x)-\cos(x)$.

By using LTSDM, we get the following approximate solution:

$$u_{approx.}(x)=\frac{x^9}{362880}-\frac{x^8}{40320}-\frac{x^7}{5040}+\frac{x^6}{720}+\frac{x^5}{120}-\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}+x-1\,,\quad n=3 \qquad (31)$$

and by using STSDM, we get the following approximate solution:

$$u_{approx.}(x) = \frac{\sin(2x)}{12} + \frac{\sin(x)}{3} - 2\cos(x) + \frac{x}{2} - \frac{x^2}{2} + 1, \quad n = 2 \qquad (32)$$

where n is the number of Adomiani polynomials.

Table 3: shows the exact and approximate Results and the least square error (L.S.E.) for Example 3.

x	Exact solution	Approximate solution by LTSDM	Approximate solution by STSDM	Absolute error of LTSDM	Absolute error of STSDM
0	-1.0	-1.0	-1.0	0.0	0.0
0.1	-0.895170748631	-0.895170748631	-0.895174747441	0.0	0.00000399881
0.2	-0.781397247046	-0.781397247046	-0.781458516892	0.0	0.00006126985
0.3	-0.659816282464	-0.659816282466	-0.660112703248	1.58176975e-12	0.00029642078
0.4	-0.531642651694	-0.531642651722	-0.532536199661	2.781126425e-11	0.00089354797
0.5	-0.398157023286	-0.398157023543	-0.400234028845	2.563925637e-10	0.00207700556
0.6	-0.260693141515	-0.260693143086	-0.26478714819	1.57107134567e-9	0.00409400668
0.7	-0.120624500047	-0.120624507308	-0.127824334657	7.26163241137e-9	0.00719983461
0.8	0.0206493815524	0.020649354249	0.009003078526	2.73033273583e-8	0.01164630303
0.9	0.161716941357	0.161716853678	0.144043002574	8.76784707177e-8	0.01767393878
1.0	0.30116867894	0.301168430335	0.275660502102	2.48604659787e-7	0.02550817684
L.S.E.		7.029278571656e-14	0.001172474096409		

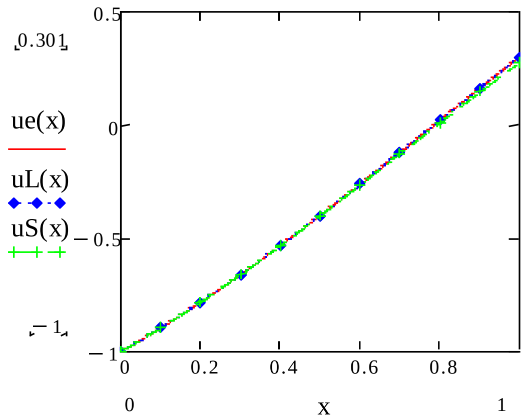

Figure3: Comparison of Exact Solution and [Approximate Solution by using LTSDM uL(x) and by using STSDM uS(x)] of example (3).

Example (4) [9]: Consider the following non-linear Fredholm integro-differential equation of second kind:

$$u''(x) = 2 + \frac{11}{15}x + \frac{19}{35}x^2 + \frac{1}{2}\int_{-1}^{1}(x\,t + x^2t^2)\,(u(t) - u^2(t))\,dt \qquad (33)$$

where $u(0) = 1,\ u'(0) = 1$.

The exact solution is $u(x) = 1 + x + x^2$.

By using LTSDM, we get the following approximate solution:

$$u_{approx.}(x) = \frac{19x^4}{420} + \frac{11x^3}{90} + \frac{1882452191x^2}{2043241200} + x + 1, \quad n=3 \qquad (34)$$

and by using STSDM, we get the following approximate solution:

$$u_{approx.}(x) = \frac{25574111x^4}{269438400} - \frac{87539x^3}{7144200} + x^2 + x + 1, \quad n=3 \qquad (35)$$

where n is the number of Adomiani polynomials.

Table 4: shows the exact and approximate Results and the least square error (L.S.E.) for Example 4.

x	Exact solution	Approximate solution by LTSDM	Approximate solution by STSDM	Absolute error of LTSDM	Absolute error of STSDM
0	1.0	1.0	1.0	0.0	0.0
0.1	1.11	1.1093398149	1.1099972385	0.000660185090	0.000002761521
0.2	1.24	1.2379024342	1.2400538409	0.002097565758	0.000053840918
0.3	1.39	1.3865840485	1.3904379873	0.003415951526	0.000437987258
0.4	1.56	1.5563894195	1.5616456567	0.003610580491	0.001645656695
0.5	1.75	1.7484318807	1.7544006277	0.001568119319	0.004400627677
0.6	1.96	1.9639333368	1.9696544779	0.003933336751	0.009654477908
0.7	2.19	2.2042242639	2.2085865843	0.014224263912	0.018586584344
0.8	2.44	2.4707437098	2.4726041232	0.030743709780	0.032604123194
0.9	2.71	2.7650392934	2.7633420699	0.055039293405	0.053342069921
1.0	3.0	3.0887672053	3.0826631992	0.088767205262	0.082663199243
L.S.E.		0.0121039167024	0.0112025482923		

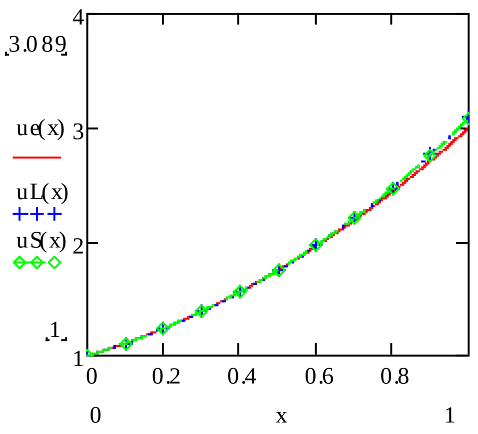

Figure 4: Comparison of Exact Solution and [Approximate Solution by using LTSDM uL(x) and by using STSDM uS(x)] of example (4).

Remark**:** In the above examples, the interval [-1, 1] can be chosen because the error value is very small and the error is too small in all partial intervals of [-1, 1], for example [0, 1] was too small as shown in the previous graphs.

Conclusion

ADM has main advantages such as simplicity, high accuracy and the solution when it exists is found in a rapidly convergent series form. A combined from of the Adomian decomposition method with the Laplace transform and with the Sumudu transform are effectively used to solve nonlinear integral and integro-differential equations.

References

[1] E.I. Akinola and I.A. Olopade , "Sumudu Transform Series Decomposition Method for Solving Nonlinear Volterra Integro-Differential Equations", International Journal of Innovation and Applied Studies ,Vol. 18 , No. 1 , , p.p. 262-267 , October 2016.

[2] Ahmed A Hamoud and Dr.Kirtiwant P Ghadle , "The combined Modified Laplace with Adomian decomposition method for Solving the nonlinear Volterra-Fredholm Integro Differential Equations", J. KSIAM, Vol.21, No.1, p.p. 17–28, March 2017.

[3] Ilias S. Kotsireas , "A Survey on Solution Methods for Integral Equations", CiteSeerx, Vol.11,No.3, p.p. 1–47, June , 2008.

[4] Fethi B. Muhammad and Ahmed A. Karaballi , "Sumudu transform fundamental properties investigations and applications", Journal of Applied Mathematics and Stochastic Analysis, Vol.6, No.91083, p.p.1–23, May, 2006.

[5] Fethi B. Muhammad and Ahmed A. Karaballi, "Analytical Investigations of the Sumudu transform and Applications to integral Production Equayions ", Hindawi Publishing Corporation,Vol.3,No.410, p.p. 103–118 , Oct 2002.

[6] M. Rahman, "Integral Equations and their Applications", printed in Great Britain by Athenaeum Press Ltd.,WIT Press 2007.

[7] Dimple Rani and Vinod Mishra , "Modification of Laplace Adomian Decomposition Method for Solving Nonlinear Volterra Integral and Integro-differential Equations based on Newton Raphson Formula", European Journal of Pure and Applied Mathematics , Vol. 11, No. 1 , p.p202-214 , January 2018.

[8] S. Z. Rida and A. S. Abedl-Rady "A Domian Decomposition Sumudu Transform Method for Solving Fractional Nonlinear Equations" Mathematical Sciences Letters ,Vol. 5, No. 1, p.p 39 – 48 , Jan. 2016.

[9] Abdul-Majid Wazwaz, "Linear and Nonlinear Integral Equation", Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg 2011, p.p 404,431, 472,520.

[10] Abdul-Majid Wazwaz , "A First Course in integral equations", p.p212, 2015 by World Scientific Publishing Co. Pte. Ltd.