Research Article Open Access

Some Properties of Magic Squares of Distinct Squares and Cubes

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Article Info

Received
27/05/2019

Accepted
10/06/2019

Published
15/10/2019

Abstract

Magic squares is n×n matrix with positive integer entries as well as the sum of rows, columns and mains diagonal have the same magic constant, one of the most oldest magic square was discovered in china. In this paper, the history of magic square is displayed and some definitions of its kind is also given by prove of two theorems about properties of magic square is introduced.

Keywords: Magic square, modular, Liner system.

الخلاصة

المربعات السحرية هي مصفوفات n×n ذات مدخلات من اعداد صحيحة موجبة بالاضافة الى مجموع الصفوف والاعمدة والاقطار الرئيسية لها نفس الثابت السحري, قد تم اكتشاف احد اقدم المربعات السحرية في الصين. في هذا البحث عرض تاريخ المربعات السحرية وتعريف بعض انواع المربعات السحرية. ولقد تم تقديم اثبات مبرهنتين حول خصائص المربعات السحرية.

Introduction

The magic squares have long fascinated human beings even in ancient Babylonian civilization. People believed that these squares possessed magic powers, which humans used as religious symbols, and were considered a magic shield. People dug them into stones and carried them to the belief that they protected their bearer from dangers and evils. In the eighth century A.D. the importance of magic squares was considered as being useful in turning the ordinary metal into gold. The magic squares were used as magic powers in many civilizations. In India, for example, they were used to search for and retrieve missing persons. The Arabs believed that magic square had an effect on all aspects of life and were used to protect and heal children with lame [13].

The magic squares history goes back to China [4], after which it moved to the Islam and India[17], and through the Arabs moved to Europe in the middle Ages [17].

There is an ancient and strange legend in China for the discovery of the first magic square, which speaks about the discovery of the magic square with matrix of order 3×3 through the pattern that appeare on the shell of the tortoise. It was found that the sum of rows, sum of columns as well as the sum of the main diagonals are equal to the number 15 [18], the magic square matrix of order four Was discovered by the German artist Albrecht Dürer's, where he dug in the upper right part of his etching Melencolia I, which included a collection of small details and scientists believe that the explanation is the lack of knowledge possessed by human access to heavenly wisdom and knowledge of the secrets of nature [13].

Although the magic squares over time have lost the belief that they have magical powers, but it continued to be a favorite subject for mathematicians and also used in entertainment games and puzzles for the possession of the mystery. Magic squares are also an important subject in art see [12].

The question that was put forward more than 160 years ago is which integer can be written in the form of sum of three integer cubes:

$$x^3 + y^3 + z^3 = n$$

The first reference to point out this problem was S. Ryley, [16]. In 1908 A.S. Werebrusov found the parametric for n = 2, [9] and Later in 1936 Mahler discovered a first parametric for n = 1, [8]. In 1954 year – Miller and Woollett discovered explicit representations for 69 values of n between 1 and 100, their search exhausted the region$|x|, |y|, |z| \leq 3164$, [10]. In 1992, Brown et al. found the first solution for n = 39, [6]. 1994 Koyama used modern computers and successfully found first solutions for 16 integers between 100 and 1000 after expand the search region to$|x|, |y|, |z| \leq 2^{21}$, [6]. 1997 Koyama, Tsuruoka, and Sekigawa used a new algorithm to find first solutions for five more values between 100 and 1000, [7]. In 2007, it's also found that there are no solution for n=$\mp 4 \; mod$ 9.

Basic Definitions

Definition (1) [1]

A magic square is a square matrix with positive integer entries from 1, . . . ,n^2such that sum in every row , sum in every column, and the sum of the two main diagonal are the same number it's called the magic constant as shown in Figure 1.

The magic constant calculated by the following formula:

$$N = \frac{n(n^2+1)}{2}$$

4	9	2
3	5	7
8	1	6

Figure 1: Magic square.

Definition (2) [2]

A semi magic square is n × n matrix whose entries are positive integer with row sum and column sum equal to the magic constant and the sum of one of the main diagonal or both failed to be equal to the magic constant as shown in Figure 2.

3	0	0
0	1	2
0	2	1

Figure 2: Semi magic square.

Definition (3) [15]

A magic square whose entries are square integers or cubes integers is called magic square of square or cubes as shown in Figure 3.

68^2	29^2	41^2	37^2
17^2	31^2	79^2	32^2
59^2	28^2	23^2	61^2
11^2	77^2	8^2	49^2

Figure 3: Magic square of square.

Example (1) [3]: The way to construct a magic square of square of order 3 is not found, that shown in Figure 4.

373^2	289^2	565^2
360721	425^2	23^2
205^2	527^2	222121

Figure 4: Magic square of order 3 with seven distinct square.

Theorem (1) [11]: (Gauss) Appositive integer N can be represented as a sum of three squares if and only if N is not of the form:

$$\mathrm{N} = 4^s(8k+7) \qquad (1)$$

where s, k are integers.

Theorem (2) [3]: The construction of a 3×3 magic square using seven (or eight or nine) distinct square integers different is impossible

Proof:

A. If the entries of magic square is 1,…,n^2 then,

I. According to Gauss theorem [11]

The magic constant for odd magic square is $\frac{n(n^2+1)}{2}$ is odd, that is when n = 3 the magic constant is 15, therefore the value of s in equation 1 must be zero to match the two sides for the equation. That is when n = 15, then:

$$n = 4^s(8k+7),$$

$\Longrightarrow 15 = (8k+7) \Longrightarrow k = 1.$

This means that the magic constant 15 can be written as the form $4^s(8k+7)$, when s=0 and k = 1, thus it is impossible to write it as sum of three squares.

II. In general, if the magic constant is $\frac{n(n^2+1)}{2}$ then:

$$\frac{n(n^2+1)}{2} = 8k+7,$$

$$\Longrightarrow k = \frac{n^3+n-14}{16}$$

$$\Longrightarrow n^3 + n - (14 + 16k) = 0,$$

when k = 46, then by using the following program in MATLAB:

```
clc
syms n
x = solve (n^3+n-750)
```

we get:

x_1 = 9.0489

x_2 = -4.5245 - 7.9001i

x_3 = -4.5245 + 7.9001i

This solution is approximate solution. n cannot be written as a sum of three squares because it is not integer.

B- If all the entries of magic square are integers which satisfying the following condition of magic square:

$$\begin{bmatrix} a^2 & b^2 & c^2 \\ d^2 & e^2 & f^2 \\ g^2 & h^2 & i^2 \end{bmatrix}$$

The condition of magic square are:

$a^2+b^2+c^2$ = s

$d^2+e^2+f^2$ = s

$g^2+h^2+i^2$ = s

$a^2+d^2+g^2$ = s

$b^2+e^2+h^2$ = s

$c^2+f^2+i^2$ = s

$a^2+e^2+i^2$ = s

$c^2+e^2+g^2$ = s

The equations can be arranged as a matrix in the form:

$$\begin{array}{c} \begin{matrix} a^2 & b^2 & c^2 & d^2 & e^2 & f^2 & g^2 & h^2 & i^2 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Then by using elementary matrix operation on rows we get the coefficient matrix for the above matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix is:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & s \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rank of coefficient matrix is equal to the rank of augmented matrix which is 7, then the system must have at least one solution [19]. In [14], the author illustrates some properties for magic square that satisfy the magic square of square.

Theorem (3) [11]: If n is a positive integer such that n ≡1, 3 or 5 (mod8), then n can be represented as the sum of three squares.

Example (2) [3]: The construction of 3×3 semi magic square with cubes entries is impossible.

51^3	619^3	165^3
618^3	162^3	115^3
178^3	72^3	235788435

Figure 5: 3×3 semi magic square with eight distinct cubes.

Theorem (4) [3]: The constructing of 3×3 semi magic square using a positive distinct cubed integer is impossible**.**

Proof:

Let n $=$ 4 mod 9 from the magic constant of odd magic square of order n the magic constant is $\frac{n(n^2+1)}{2}$, therefore:

$\frac{n(n^2+1)}{2}$ mod 9 = 4, implies that:

$$n^3 + n - 72 = 0.$$

Using the following program in MATLAB:

```
clc
syms n
x = solve (n^3+n-72)
```

we get:

x_1 = 4.0801

x_2 = -2.0400 - 3.6722i

x_3 = -2.0400 + 3.6722i

This solution is approximate solution. Since n = 4.0801, therefore it cannot be written as a sum of three cubes, where the other are complex.

Conclusion

Magic squares have been inspiring to many artists and mathematicians. Therefore, there are types of magic squares that have specific properties. Many open problems have emerged one of them if can be constructing 3×3 magic square of squares and the other constructed 3×3 magic square of cubes, which is taken in this paper.

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