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Harmonic Functions of the Class of Barzilai'c Type Related to New Derivative Operator

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Abstract

In this article, we define and investigate the class of Bazilevi´c type harmonic univalent functions $\mathcal{F}_{\lambda,\beta}^{m}(\eta,\sigma,\gamma)$, which related with a new linear operator. We have also obtained the harmonic structures in terms of its coefficient bounds, extreme points, distortion bound, convolution and we proved the function belongs to this class be closed under linear combination.

Keywords: Harmonic, Univalent functions, Bazilevi´c type, Derivative operator..

الخلاصة

في هذه المقالة, نعرف صنف من دوال بازا ليفج التوافقية احاديه التكافؤ والتحقق منها التي ترتبط بمؤثر خطي جديد. لقد حصلنا أيضًا على التراكيب التوافقية من حيث حدود المعامل، والنقاط المتطرفة، والتشويه، والالتفاف. وأثبتنا أن الدالة التي تنتمي إلى هذا الصنف تكون مغلقة تحت التركيب الخطي.

Introduction

Let A refer to the class of functions has been the expressed as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in C : |z| < 1\}$ and normalized under two conditions with $f(0) = 0$ and $f'(0) = 1$.

In addition to that, let P refer to the class of functions $h(z)$, with positive real part in $\mathcal{U}$ as follows:

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

A function $f(z)$ in the form (1) is called starlike functions, if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad (z \in \mathcal{U})$$

and denoted by S^*(see [7]). From (1), we write that:

$$f(z)^{\alpha} = \left(z + \sum_{n=2}^{\infty} a_n z^n\right)^{\alpha} \qquad (2)$$

By applying binomial expansion on (2), we obtain:

$$f(z)^{\alpha} = z^{\alpha} + \alpha a_2 z^{\alpha+1} + \left[\alpha a_3 + \frac{\alpha(\alpha-1)}{2!} a_2^2\right] z^{\alpha+2} + \left[\alpha a_4 + \frac{\alpha(\alpha-1)}{2!} 2a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} a_2^3\right] z^{\alpha+3} + \cdots.$$

Then we define the class of analytic functions of fractional power A_{α} as follows:

$$f(z)^{\alpha} = z^{\alpha} + \sum_{n=2}^{\infty} a_n(\alpha) z^{\alpha+n-1} \qquad (3)$$

Thus, we shall define the differential operator

$\mathcal{L}_{\lambda,\beta}^{m}: A_\alpha \longrightarrow A_\alpha$ as follows:

$$\mathcal{L}_{\lambda,\beta}^{1} f(z)^\alpha = (\lambda-\beta+1)\left(\mathcal{L}_{\lambda,\beta}^{0} f(z)^\alpha\right) + (\beta-\lambda) z \left(\mathcal{L}_{\lambda,\beta}^{0} f(z)^\alpha\right)',$$
$$= [1+(1-\alpha)(\lambda-\beta)] z^\alpha + \sum_{n=2}^{\infty} [1+(\beta-\lambda)(\alpha+n-2)] a_n(\alpha) z^{\alpha+n-1},$$
$$\mathcal{L}_{\lambda,\beta}^{2} f(z)^\alpha = (\lambda-\beta+1)\left(\mathcal{L}_{\lambda,\beta}^{1} f(z)^\alpha\right) + (\beta-\lambda) z \left(\mathcal{L}_{\lambda,\beta}^{1} f(z)^\alpha\right)',$$
$$= [1+(1-\alpha)(\lambda-\beta)]^2 z^\alpha + \sum_{n=2}^{\infty} [1+(\beta-\lambda)(\alpha+n-2)]^2 a_n(\alpha) z^{\alpha+n-1}.$$

For general

$$\mathcal{L}_{\lambda,\beta}^{m} f(z)^\alpha = \mathcal{L}(\mathcal{L}_{\lambda,\beta}^{n-1} f(z)^\alpha) = [1+(1-\alpha)(\lambda-\beta)]^m z^\alpha + \sum_{n=2}^{\infty} [1+(\beta-\lambda)(\alpha+n-2)]^m a_n(\alpha) z^{\alpha+n-1}. \quad (4)$$

Has been remarked that the linear operator which is defined in (4) is generalized many operators by giving specific values to the parameters which studied by several earlier authors as follows:

(i) If $\lambda = 0$ and $\beta = 1$, then operator (4) reduces A. T. Oladipo and D. Breaz operator [9].

(ii) If $\lambda = 0, \beta = 1$ and $\alpha = 1$, then operator (4) reduces to the Salagean derivative operator [11].

(iii) If $\lambda = 0$ and $\alpha = 1$, then operator (4) generates the operator which presented by Al-Oboudi[1].

A continuous function $f(z) = u + iv$ which defined in a simply connected domain $\mathcal{U}$ is a complex- valued harmonic function in $\mathcal{U}$ if both u and v are real harmonic. We write:

$$f = h + \bar{g}, \quad (5)$$

where h and g are two analytic functions in $\mathcal{U}$, where h and g are the analytic and co-analytic part of the function f respectively. A sufficient and necessary condition for a function:

$$f(z) = h(z) + \overline{g(z)},$$

to be sense preserving and locally univalent in unit disk $\mathcal{U}$ is:

$$|h'(z)| >$$

Let A_H denoted the family of all functions f in the form (5) which
are univalent, sense preserving function and harmonic in $\mathcal{U} = \{z : |z| < 1\}$.
In the present work, we will express the functions h^α and g^α as follows:

$$h(z)^\alpha = z^\alpha + \sum_{n=2}^{\infty} a_n(\alpha) z^{\alpha+n-1},$$
$$g(z)^\alpha = \sum_{n=1}^{\infty} b_n(\alpha) z^{\alpha+n-1}, (z \in \mathcal{U}, 0 \le |b_1(\alpha)| < 1) \quad (6)$$

Hence

$$f(z)^\alpha = h(z)^\alpha + \overline{g(z)^\alpha}. \quad (7)$$

Note that if g is identically zero; that is $g = 0$, and $\alpha = 1$, then A_H will generates a known class A.
We define our linear operator as given in (4) such that

$$\mathcal{L}_{\lambda,\beta}^{m} f(z)^\alpha = \mathcal{L}_{\lambda,\beta}^{m} h(z)^\alpha + (-1)^m \overline{\mathcal{L}_{\lambda,\beta}^{m} g(z)^\alpha}, \quad (8)$$

Where

$$\mathcal{L}_{\lambda,\beta}^{m} h(z)^\alpha = [1+(1-\alpha)(\lambda-\beta)]^m z^\alpha + \sum_{n=2}^{\infty} [1+(\beta-\lambda)(\alpha+n-2)]^m a_n(\alpha) z^{\alpha+n-1},$$
$$\mathcal{L}_{\lambda,\beta}^{m} g(z)^\alpha = \sum_{n=1}^{\infty} [1+(\beta-\lambda)(\alpha+n-2)]^m \times b_n(\alpha) z^{\alpha+n-1}$$

Now, we shall define generalization class of Bazilevi´c type harmonic univalent functions involving new general linear operator.

Definition 1.1 Let $f(z)$ in A_H , belongs to the class $\mathcal{F}_{\lambda,\beta}^{m}(\eta,\sigma,\gamma)$ if it satisfies the following condition:

$$Re\left\{\sigma e^{i\eta} - (\sigma e^{i\eta} - 1) \frac{\mathcal{L}_{\lambda,\beta}^{m} f(z)^\alpha}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha}\right\} \ge \gamma, \quad (9)$$

where $0 \le \gamma < 1, \eta \in \mathbb{R}, \sigma \ge 0, \beta \ge 0, \lambda \ge 0, \alpha > 0, n, m \in \mathbb{N}, z \in \mathcal{U}$ and $\mathcal{L}_{\lambda,\beta}^{m} f(z)^{\alpha}$ is earlier defined in (8).

Furthermore, let $\mathcal{NF}_{\lambda,\beta}^{m}(\eta,\sigma,\gamma)$ be subclass of $\mathcal{F}_{\lambda,\beta}^{m}(\eta,\sigma,\gamma)$ consist of harmonic functions

$$f_m^{\alpha} = h^{\alpha} + \overline{g_m^{\alpha}} \tag{10}$$

where h^{α} and $\overline{g_m^{\alpha}}$, which has the following representation:

$$h(z)^{\alpha} = z^{\alpha} + \sum_{n=2}^{\infty} |a_n(\alpha)| z^{\alpha+n-1},$$

$$g(z)^{\alpha} = -(-1)^m \sum_{n=1}^{\infty} |b_n(\alpha)| z^{\alpha+n-1},$$
$$(z \in \mathcal{U}, 0 \le |b_1(\alpha)| < 1)$$

Starting point in the study of functions characterized in (9) was discovered in 1955 by Bazilevi´c [2], when the Bazilevi´c function defined in $\mathcal{U}$ by the form:

$$f(z) = \left\{ \frac{\xi}{1+\eta^2} \int_0^z (h(t) - i\eta) t^{-(1+\frac{i\xi\eta}{1+\eta^2})} g(t)^{\frac{\xi}{1+\eta^2}} dt \right\}^{\frac{1+i\eta}{\xi}} \tag{11}$$

where the function $h(t)$ belongs to P and $g(z) \in S^*, \xi, \eta \in \mathbb{R}$ with $\xi > 0$.
The class of harmonic functions have been studied by many authors for variant properties. By the earlier papers for contributors such as [[3],[4], [5], [6],[8], [9]] and [10] regarding of the theory of analytic functions which have a wide application in many physical problem: as electrostatic potential in heat conduction, fluid flows, and theory of fractals constitute practical examples.

The aim of this paper requests to generate class of Bazilevi´c type harmonic univalent function related to new derivative operator. Also, we obtain coefficient bounds for functions f^{α} which is define in (6) belongs in the class $\mathcal{F}_{\lambda,\beta}^{n}(\eta,\sigma,\gamma)$.
As well as, the distortion bounds, inclusion results and extreme points for functions in this class are also obtained.

1. Main Results

In this result, we present a sufficient condition for coefficient of functions in the class $\mathcal{F}_{\lambda,\beta}^{m}(\eta,\sigma,\gamma)$.

Theorem 2.1. Let $f^{\alpha} = h^{\alpha} + \overline{g^{\alpha}}$ be given by (7). If:

$$\sum_{n=2}^{\infty} \frac{\sigma-1}{1-\gamma} \left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)} \right)^n |a_n(\alpha)| + \sum_{n=1}^{\infty} \frac{\sigma-1}{1-\gamma} \left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)} \right)^n |b_n(\alpha)| \le 1 \tag{12}$$

where $0 \le \gamma < 1, \eta \in \mathbb{R}, \sigma \ge 0, \beta \ge 0, \lambda \ge 0, \alpha > 0, n, m \in \mathbb{N}, z \in \mathcal{U}$, then f^{α} be harmonic univalent and sense-preserving in $\mathcal{U}$ and $f^{\alpha} \in \mathcal{F}_{\lambda,\beta}^{m}(\eta,\sigma,\gamma)$.

Proof. First suppose that the inequality (12) holds.

If $z_1 \ne z_2$

$$\left| \frac{f^{\alpha}(z_1) - f^{\alpha}(z_2)}{h^{\alpha}(z_1) - h^{\alpha}(z_2)} \right| \ge 1 - \left| \frac{g^{\alpha}(z_1) - g^{\alpha}(z_2)}{h^{\alpha}(z_1) - h^{\alpha}(z_2)} \right|$$
$$= 1 - \left| \frac{\sum_{n=1}^{\infty} b_n(\alpha)({z_1}^{\alpha+n-1} - {z_2}^{\alpha+n-1})}{(z_1^{\alpha} - z_2^{\alpha}) + \sum_{n=2}^{\infty} a_n(\alpha)({z_1}^{\alpha+n-1} - {z_2}^{\alpha+n-1})} \right|$$

$$> 1 - \frac{\sum_{n=1}^{\infty} (\alpha+n-1) b_n(\alpha)}{\alpha + \sum_{n=2}^{\infty} (\alpha+n-1) a_n(\alpha)}$$

$$> -1 \frac{\sum_{n=1}^{\infty} \big(1+(\beta-\lambda)(\alpha+n-1)\big) b_n(\alpha)}{\big(1+(1-\alpha)(\lambda-\beta)\big) + \sum_{n=2}^{\infty} \big(1+(\beta-\lambda)(\alpha+n-1)\big) a_n(\alpha)}$$

$$\geq 1 - \frac{\sum_{n=1}^{\infty} \frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^n |b_n(\alpha)|}{1+\sum_{n=2}^{\infty} \frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^n |a_n(\alpha)|} \geq 0$$

Hence proved the univalent, and also note that f^α be sense- preserving in $\mathcal{U}$ since:

$$|h'(z)^\alpha| \geq \alpha|z|^{\alpha-1} - \sum_{n=2}^{\infty} (\alpha+n-1)|a_n(\alpha)|\,|z|^{\alpha+n-2}$$

$$> \alpha - \sum_{n=2}^{\infty} (\alpha+n-1)|a_n(\alpha)|$$

$$\geq \big(1+(1-\alpha)(\lambda-\beta)\big) - \sum_{n=2}^{\infty} \frac{\sigma-1}{1-\gamma}(1+(\beta-\lambda)(\alpha+n-1))|a_n(\alpha)|$$

$$\geq \sum_{n=2}^{\infty} \frac{\sigma-1}{1-\gamma}(1+(\beta-\lambda)(\alpha+n-1))|b_n(\alpha)|$$

$$\geq \sum_{n=2}^{\infty} (\alpha+n-1))|b_n(\alpha)|$$

$$> \sum_{n=2}^{\infty} (\alpha+n-1)|b_n(\alpha)|\,|z|^{\alpha+n-2}$$

$$\geq |g'(z)^\alpha|$$

The remaining condition needs to investigate the function $f^\alpha(z)$ which belongs to the class $\mathcal{F}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$. By (9) and (10), we have

$$Re\left\{\sigma e^{i\eta} - (\sigma e^{i\eta}-1)\frac{\mathcal{L}_{\lambda,\beta}^m f\,(z)^\alpha}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha}\right\} = Re\left\{\sigma e^{i\eta} - (\sigma e^{i\eta}-1)\frac{\mathcal{L}_{\lambda,\beta}^m h\,(z)^\alpha + (-1)^m\overline{\mathcal{L}_{\lambda,\beta}^m g\,(z)^\alpha}}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha}\right\} \geq \gamma$$

By applying the fact that $Re\{w\} \geq \gamma$, if and only if $|1-\gamma+w| \geq |1+\gamma-w|$ for $\gamma(0 \leq \gamma < 1)$, it suffices to show that

$$\left|(1-\gamma) + \sigma e^{i\eta} - (\sigma e^{i\eta} - 1)\frac{\mathcal{L}_{\lambda,\beta}^m f\,(z)^\alpha}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha}\right| - \left|(1+\gamma) - \sigma e^{i\eta} + (\sigma e^{i\eta} - 1)\frac{\mathcal{L}_{\lambda,\beta}^m f\,(z)^\alpha}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha}\right| \geq 0.$$

That is

$$\left|(1-\gamma+\sigma e^{i\eta})[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha - (\sigma e^{i\eta}-1)\mathcal{L}_{\lambda,\beta}^m f\,(z)^\alpha\right| - \left|(1+\gamma-\sigma e^{i\eta})[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha + (\sigma e^{i\eta}-1)\mathcal{L}_{\lambda,\beta}^m f\,(z)^\alpha\right| \geq 0,$$

$$\left|\begin{array}{l}(2-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha - \sum_{n=2}^{\infty}(\sigma e^{i\eta}-1)\,[1+(\beta-\lambda)(\alpha+n-2)]^m a_n(\alpha) z^{\alpha+n-1} \\ -(-1)^m \sum_{n=1}^{\infty}(\sigma e^{i\eta}-1)\,[1+(\beta-\lambda)(\alpha+n-2)]^m b_n(\alpha)\overline{z^{\alpha+n-1}}\end{array}\right|$$

$$- \left|\begin{array}{l}\gamma[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha + \sum_{n=2}^{\infty}(\sigma e^{i\eta}-1)\,[1+(\beta-\lambda)(\alpha+n-2)]^m a_n(\alpha) z^{\alpha+n-1} \\ +(-1)^m \sum_{n=2}^{\infty}(\sigma e^{i\eta}-1)\,[1+(\beta-\lambda)(\alpha+n-2)]^m b_n(\alpha)\overline{z^{\alpha+n-1}}\end{array}\right| \geq 0.$$

$$\geq 2(1-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m \left[|z|^\alpha - \sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |a_n(\alpha)||z|^{\alpha+n-1} - (-1)^m \sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_n(\alpha)||\overline{z}|^{\alpha+n-1}\right],$$

$$\geq 2(1-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m \left[1 - \left(\sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |a_n(\alpha)| + (-1)^m \sum_{n=1}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_n(\alpha)|\right)\right] \geq 0 \tag{13}$$

The expression (13) is non-negative by (12), and furthermore $f(z)^\alpha \in \mathcal{F}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$.
The harmonic functions of the form:

$$f^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty}\frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m x_k z^{\alpha+n-1} + \sum_{n=1}^{\infty}\frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^n y_n \overline{z^{\alpha+n-1}}, \tag{14}$$

where $0 \leq \gamma < 1,\ \eta \in \mathbb{R},\ \sigma \geq 0,\ \beta \geq 0,\ \lambda \geq 0,\ \alpha > 0, n, m \in \mathbb{N}, z \in \mathcal{U}$, and

$$\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$$

Note that the coefficient bound s which given in (12) is a sharp. Therefore, functions in the form (14) belong to the class $\mathcal{F}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$, since

$$\sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |a_n(\alpha)| + \sum_{n=1}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^n |b_n(\alpha)| = \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$$

Next theorem investigates that a condition in (12) as a necessary condition for the function f_m^α which given by (10).

Theorem 2.2. Let $f_m^\alpha = h^\alpha + \overline{g_m^\alpha}$ be given by (10) belongs to class 〖NF〗 _(λ,β)^m (η,σ,γ) if and only if:

$$\sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |a_n(\alpha)| + \sum_{n=1}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_n(\alpha)| \leq 1 \tag{15}$$

where $0 \leq \gamma < 1, \eta \in \mathbb{R},\ \sigma \geq 0,\ \beta \geq 0,\ \lambda \geq 0,\ \alpha > 0, n, m \in \mathbb{N}, z \in \mathcal{U}$.

Proof. Since $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma) \subset \mathcal{F}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$, we just have to state the only part of theorem. To this end, suppose that $f_m^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$ and by virtue of (9), we get:

$$Re\left\{(\sigma e^{i\eta}-\gamma) - (\sigma e^{i\eta}-1)\frac{\mathcal{L}_{\lambda,\beta}^m f\,(z)^\alpha}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha}\right\} \geq 0 \tag{16}$$

This is equivalent to:

$$Re\left\{\frac{(\sigma e^{i\eta}-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha - (\sigma e^{i\eta}-1)\mathcal{L}_{\lambda,\beta}^m f\,(z)^\alpha}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha}\right\}$$

$$Re\left\{\frac{(\sigma e^{i\eta}-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha - (\sigma e^{i\eta}-1)\Big[[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha - \sum_{n=2}^{\infty}[1+(\beta-\lambda)(\alpha+n-2)]^m|a_n(\alpha)|z^{\alpha+n-1} + (-1)^m\sum_{n=1}^{\infty}[1+(\beta-\lambda)(\alpha+n-2)]^m|b_n(\alpha)|\overline{z^{\alpha+n-1}}\Big]}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha}\right\} \geq 0$$

$$= Re\left\{\frac{(1-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha - \sum_{n=2}^{\infty}(\sigma e^{i\eta}-1)[1+(\beta-\lambda)(\alpha+n-2)]^m|a_n(\alpha)|z^{\alpha+n-1} + (-1)^m\sum_{n=1}^{\infty}(\sigma e^{i\eta}-1)[1+(\beta-\lambda)(\alpha+n-2)]^m|b_n(\alpha)|\overline{z^{\alpha+n-1}}}{[1+(1-\alpha)(\lambda-\beta)]^n z^\alpha}\right\} \geq 0$$

$$= Re\left\{\frac{(1-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m - \sum_{n=2}^{\infty}(\sigma e^{i\eta}-1)[1+(\beta-\lambda)(\alpha+n-2)]^m|a_n(\alpha)|z^{n-1} - \left(\frac{\overline{z}^\alpha}{z^\alpha}\right)(-1)^m\sum_{n=1}^{\infty}(\sigma e^{i\eta}-1)[1+(\beta-\lambda)(\alpha+n-2)]^m|b_n(\alpha)|\overline{z^{n-1}}}{[1+(1-\alpha)(\lambda-\beta)]^m}\right\} \geq 0.$$

This condition must be true $\forall z \in \mathcal{U}$ and for real η. Therefore, choose $0 \leq |z| = r < 1$ and $\eta = 0$, so that the above inequality reduces to:

$$\frac{(1-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m - \sum_{n=2}^{\infty}(\sigma-1)[1+(\beta-\lambda)(\alpha+n-2)]^m|a_n(\alpha)|z^{n-1} - (-1)^m\sum_{n=1}^{\infty}(\sigma-1)[1+(\beta-\lambda)(\alpha+n-2)]^m|b_n(\alpha)|\overline{z^{n-1}}}{[1+(1-\alpha)(\lambda-\beta)]^m} \geq 0$$

In the following theorem, we will determine the extreme points of closed convex hulls for functions belong to the class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$, and we refer to it with the symbol $clco\ \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$.

Theorem 2.3. Let $f_m^\alpha(z)$ be given by (10) belongs to class $clco\ \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$ if and only if

$$f_m^\alpha(z) = \sum_{n=1}^{\infty}\left(x_n h_n(z) + y_n g_n(z)\right),$$

where

$h_1(z)^\alpha = z^\alpha$,

$$h_n^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty}\frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m z^{\alpha+n-1}, (n = 2,3,\dots)$$

$$g_{nm}^\alpha(z) = z^\alpha - (-1)^m\sum_{n=1}^{\infty}\frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m\overline{z^{\alpha+n-1}},$$

$(n = 2,3,\dots)$,

$\sum_{n=2}^{\infty}x_n + \sum_{n=1}^{\infty}y_n = 1,\ x_n \geq 0, y_n \geq 0$ and $x_1 = 1 - \left(\sum_{n=2}^{\infty}x_n + \sum_{n=1}^{\infty}y_n\right) \geq 0$.

In particular case, the extreme points of $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$ are $\{h_n\}$ and $\{g_{nm}\}$.

Proof. First, we have

$$f_m^\alpha(z) = \sum_{n=1}^{\infty}(x_n h_n(z) + y_n g_{nm}(z)),$$

$$= \sum_{n=1}^{\infty}(x_n + y_n)z^\alpha + \sum_{n=2}^{\infty}\frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m x_n z^{\alpha+n-1}$$

$$- (-1)^m\sum_{n=2}^{\infty}\frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^n y_n\overline{z^{\alpha+n-1}}.$$

Then

$$\sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^n|a_n(\alpha)| + \sum_{n=1}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m|b_n(\alpha)|$$

$$= \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1 - x_1 \leq 1.$$

This means that $f_m^\alpha \in clco\ \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$.

Conversely, assume that $f_m^\alpha \in clco\ \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$. Putting

$$x_n = \frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m|a_n(\alpha)|,\quad (0 \leq x_n \leq 1, n \geq 2)$$

$$y_n = \frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m|b_n(\alpha)|,\quad (0 \leq y_n \leq 1, n \geq 1)$$

and $x_1 = 1 - \left(\sum_{n=2}^{\infty}x_n + \sum_{n=1}^{\infty}y_n\right)$. Therefore, $f_m^\alpha(z)$ can be written as

$$f_m^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty} |a_n(\alpha)|\, z^{\alpha+n-1} - (-1)^m \sum_{n=1}^{\infty} |b_n(\alpha)|\, \overline{z^{\alpha+n-1}}$$
$$= z^\alpha + \sum_{n=2}^{\infty} \frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta))}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m x_n z^{\alpha+n-1} - (-1)^m \sum_{n=1}^{\infty} \frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta))}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m y_k \overline{z^{\alpha+n-1}}$$
$$= z^\alpha + \sum_{n=2}^{\infty} (h_n(z) - z^\alpha) x_n + \sum_{n=1}^{\infty} (g_{nm}(z) - z^\alpha) y_n$$
$$= z^\alpha\{1 - (\sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n)\} + \sum_{n=2}^{\infty} h_n(z) x_n + \sum_{n=1}^{\infty} g_{nm}(z) y_n$$
$$= \sum_{n=2}^{\infty} x_n h_n(z) \sum_{n=1}^{\infty} y_n g_{nm}(z).$$

We give distortion bounds for functions belongs to the class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$.

Theorem 2.4. Let $f_m^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$. Then for $|z| = r < 1$, we have

$$|f_m^\alpha(z)| \le (1 - |b_1(\alpha)|) r^\alpha + \left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha)}\right)^m \left[\frac{1-\gamma}{\sigma-1} - \frac{1+\gamma}{\sigma-1}|b_1(\alpha)|\right] r^{\alpha+1},$$

$$|f_m^\alpha(z)| \ge (1 + |b_1(\alpha)|) r^\alpha - \left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha)}\right)^m \left[\frac{1-\gamma}{\sigma-1} - \frac{1+\gamma}{\sigma-1}|b_1(\alpha)|\right] r^{\alpha+1},$$

Proof. This proof for only a right part inequality because the left part is similar to the right part. Let $f_m^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$.

By taking the absolute value of f_m^α, we get

$$|f_m^\alpha(z)| = \left| z^\alpha + \sum_{n=2}^{\infty} a_n(\alpha) z^{\alpha+n-1} - (-1)^m \sum_{n=1}^{\infty} b_n(\alpha)\, \overline{z^{\alpha+n-1}} \right|$$
$$\le (1 - |b_1(\alpha)|) r^\alpha + \sum_{n=2}^{\infty} (|a_n(\alpha)| - |b_n(\alpha)|)\ r^{\alpha+n-1}$$
$$\le (1 - |b_1(\alpha)|) r^\alpha + r^{\alpha+1} \sum_{n=2}^{\infty} (|a_n(\alpha)| - |b_n(\alpha)|)$$
$$\le (1 - |b_1(\alpha)|) r^\alpha + \frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha)}\right)^m \times \left[\sum_{n=2}^{\infty} \frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |a_n(\alpha)| - \sum_{n=1}^{\infty} \frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_n(\alpha)|\right]$$
$$\le (1 - |b_1(\alpha)|) r^\alpha + \frac{1-\gamma}{\sigma-1}\left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha)}\right)^m \times \left[\sum_{n=2}^{\infty} \frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |a_k(\alpha)| - \sum_{n=1}^{\infty} \frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_n(\alpha)|\right]$$
$$\le (1 - |b_1(\alpha)|) r^\alpha + \left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha)}\right)^m \left[\frac{1-\gamma}{\sigma-1} - \frac{1+\gamma}{\sigma-1}|b_1(\alpha)|\right] r^{\alpha+1},$$

for $|b_1(\alpha)| < 1$. This shown that the bound which given in theorem 2.4, be sharp for harmonic functions

$$f_m^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty} |a_n(\alpha)| z^{\alpha+n-1} - (-1)^m \sum_{n=1}^{\infty} |b_n(\alpha)|\, \overline{z^{\alpha+n-1}},$$
$$F_m^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty} |A_n(\alpha)| z^{\alpha+n-1} - (-1)^m \sum_{n=1}^{\infty} |B_n(\alpha)|\, \overline{z^{\alpha+n-1}}.$$

The convolution of f_n^α and F_n^α is given by

$$(f_m^\alpha * F_m^\alpha)(z) = f_m^\alpha(z) * F_m^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty} |a_n(\alpha)||A_n(\alpha)| z^{\alpha+n-1} - (-1)^m \sum_{n=1}^{\infty} |b_n(\alpha)||B_n(\alpha)|\, \overline{z^{\alpha+n-1}} \quad (17)$$

In this theorem, using the definition in (17) to show that the class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$ is closed under convolution.

Theorem 2.5 For $0 \le \mu \le \gamma < 1$, let $f_m^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$ and $F_n^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\mu)$. Then $f_m^\alpha * F_m^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma) \subset \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\mu)$.

Proof. We wish to show that the coefficients of $f_m^\alpha * F_m^\alpha$ satisfy the required condition given in Theorem 2.2. For the function $F_m^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\mu)$, we note that $|A_n(\alpha)| \le 1$ and $|B_n(\alpha)| \le 1$.

Now, for the convolution function $(f_m^\alpha * F_m^\alpha)(z)$, we obtain

$$\sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |a_n(\alpha)||A_n(\alpha)| + \sum_{n=1}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_n(\alpha)||B_n(\alpha)|$$

$$\sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^n |a_n(\alpha)| + \sum_{n=1}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_n(\alpha)| \le 1.$$

Since $0 \le \mu \le \gamma < 1$ and $f_m^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$. Therefore $f_m^\alpha * F_m^\alpha \in \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma) \subset \mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\mu)$.
Here, let $f_{mi}^\alpha(z)$ be defined as

$$f_{mi}^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty}|a_{n,i}(\alpha)|z^{\alpha+n-1} - (-1)^m\sum_{n=1}^{\infty}|b_{n,i}(\alpha)|\,\overline{z^{\alpha+n-1}} \quad (18)$$

where $i = 1,2,\dots,k$.

Theorem 2.6 Let $f_{mi}^\alpha(z)$ which defined by (18) belongs to the class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$ for every $i = 1,2,\dots,k$. Then the function

$$t_i^\alpha(z) = \sum_{i=1}^{k} v_i\, f_{mi}^\alpha(z), \qquad (0 \le v_i \le 1)$$

are also in the class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$, where $\sum_{i=1}^k v_i = 1$.

Proof. According to a definition of t_i^α, can be written as

$$t_i^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty}\left(\sum_{i=1}^k v_i\, a_{n,i}(\alpha)\right) z^{\alpha+n-1} - (-1)^m\sum_{n=1}^{\infty}\left(\sum_{i=1}^k v_i b_{n,i}(\alpha)\right)\,\overline{z^{\alpha+n-1}}.$$

Furthermore, since $f_{mi}^\alpha(z)$ belongs to the class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$ for every $i = 1,2,\dots,k$, then by (12), we have

$$\sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m\left(\sum_{i=1}^k v_i\,|a_{n,i}(\alpha)|\right) + \sum_{n=1}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m\left(\sum_{i=1}^k v_i\,|b_{n,i}(\alpha)|\right).$$

$$\sum_{i=1}^{k} v_i\left(\sum_{n=2}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^n |a_{n,i}(\alpha)| + \sum_{n=1}^{\infty}\frac{\sigma-1}{1-\gamma}\left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_{n,i}(\alpha)|\right) \le \sum_{i=1}^{k} v_i \le 1.$$

Corollary 2.7 The class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$ be closed under convex linear combination.

Proof. Let $f_{mi}^\alpha(z)$, $(i = 1,2)$ defined by (18) belongs to the class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$. Then the function $\Phi^\alpha(z)$ defined by

$\Phi^\alpha(z) = \mu\, f_{m1}^\alpha(z) + (1-\mu)\, f_{m2}^\alpha(z), \quad 0 \le \mu \le 1)$
is in the class $\mathcal{NF}_{\lambda,\beta}^m(\eta,\sigma,\gamma)$. By choosing $k = 2$, $v_1 = \mu$ and $v_2 = 1-\mu$ in Theorem 2.6, we obtain the above corollary.

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