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Best multiplier Approximation in $\mathbf{L_{p,\emptyset_n}(B)}$

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Abstract

The purpose of this paper is to find best multiplier approximation of unbounded functions in $L_{p,\emptyset_n}$ –space by using Trigonometric polynomials and by de *la Vallee-Poussin* operators. Also we will estimate the degree of the best multiplier approximation by Weighted –Ditzian-Totik modulus.

Keywords: multiplier convergence, multiplier Integral.

الخلاصة

الغرض من هذا البحث هو ايجاد افضل تقريب مضاعف للدوال الغير مقيدة في الفضاء $L_{p,\emptyset_n}$ بستخدام الحدوديات المثلثية و بستخدام مؤثر دو لا فاليه - بوسان وكذلك سوف نقدر درجة افضل تقريب مضاعف بواسطة نموذج دتزيان-توتيك الوزني .

Introduction

Main Approximation problems of bounded periodic functions using *de la Vallee-Poussin* have been studied by several authors [1,2], in Morrey spaces .The Approximation of periodic bounded functions in C(I)-spaces, I= $[-\pi,\pi]$ by *de la Vallee –Poussin* sums was obtained by [3] in tow dimension . Also the Approximation of bounded μ −measurable functions using Trigonometric polynomial have been studied by [4]. In the present paper we generalize these results in Multiplier spaces, $L_{p,\emptyset_n}(B)$, $B=[-\pi,\pi]$ using *de la Vallee-Poussin* sums by means of Weighted –Ditzian-Totik modulus.

Let us introduce some definitions and some results that used throughout this paper.

Definition: 1.1 [5]:

A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergence if there is a sequence $\{\emptyset_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} a_n \ \emptyset_n < \infty$, and we will say that $\{\emptyset_n\}$ is a multiplier for the convergence.

Note:

If $\sum a_n$ is convergent series then it is multiplier convergent, this by taken $\{\emptyset_n\}_{n=0}^{\infty} = \{1\}_{n=0}^{\infty}$. But the converse is not true.

Example:

The series $\sum_{n=1}^{\infty}\frac{1}{n}$ divergent series and the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ convergent sequence. Since $\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{n}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ which is convergent series then the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is a multiplier convergent.

Definition 1.2 [5]:

For any real valued function f if there is a sequence $\{\emptyset_n\}_{n=0}^{\infty}$ such that $\int_B f\emptyset_n(x) < \infty$, then we say that $\emptyset_n$ is a multiplier for the Integral.

Definition 1.3:

Let $L_{p,\emptyset_n}(B), 1 \le p < \infty$ be the space of all real valued unbounded functions f such that $\int_B f\emptyset_n(x)\,dx < \infty$, with the following norm:

$\|f\|_{L_{p,\emptyset_n}} = \sup\left\{(\int_B |f\emptyset_n(x)|^p\, dx)^{\frac{1}{p}} : x \in B\right\}$,
where $\emptyset_n$ is the multiplier for the integral, and $B = [-\pi, \pi]$.

Let us define the norm $\|f\|_{L_{p,\emptyset_n}}$ by $\|f\|_{p,\emptyset_n}$.

Definition 1.4 [2]:

For $f \in L_p(B)\,, B = [-\pi, \pi]$. The Fourier series of f is given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n(f)\cos nx + b_n(f)\sin nx), \qquad \ldots\ldots(1)$$

where $a_n(f)$ and $b_n(f)$ are Fourier coefficients of function f, the n −th partial sums of (1) is given by:

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_n(f)\cos kx + b_n(f)\,sin\,kx).$$

The *de la Vallee-Poussin* partial sum of (1) is defined by:

$$V_{n,m}(f,x) = \frac{1}{m+1}\sum_{k=n}^{n+m} S_k\,(f,x)$$

$m = 0,1,2, \ldots, n = 0,1,2, \ldots$.

Definition 1.5:

For $2\pi -$ periodic $f \in L_{P,\emptyset_n}(B)$ and from [4] let us consider the following Trigonometric polynomial, which has the representation:
$S_n^{**}(f,x) = \frac{1}{n}\sum_{k=1}^{2n} f(x_k)\, D_n(x - x_k)$,
where $D_n(t) = 1 + 2\sum_{k=1}^{n} coskx$, and
$x_k = \frac{2\pi k}{2n+1}$.

Definition 1.6:

For $f \in L_{P,\emptyset_n}(B)$, let us define the modify *de la Vallee-Poussin* operator as:
$\mathcal{V}_{3n,2n}(f,x) = \frac{2}{3n+2}\sum_{k=0}^{3n} f\emptyset_n(x_k) q_{2n}(x - x_k)$,
where $q_{2n} = \frac{1}{n+1}\sum_{k=0}^{n} D_{n+k}(t)$, and $D_{n+k}(t)$ is the Dirchlet kernel.

Definition 1.7:

Let $f \in L_{P,\emptyset_n}(B)$ then the degree of best multiplier approximation of a function f with respect to trigonometric polynomial $g_n \in \Pi_n$ is given by:
$E_n(f)_{P,\emptyset_n} = \inf\{\|f - g_n\|_{P,\emptyset_n}, g_n \in \Pi_n\}$,
where Π_n be the set of all trigonometric polynomial.

Definition 1.8:

For $f \in L_{P,\emptyset_n}(B)$ and $\delta > 0$, we will define the following concepts:

1. $\omega^k(f,\delta)_{P,\emptyset_n} = sup_{|h|<\delta}\left\|\Delta_h^k f(\cdot)\right\|_{P,\emptyset_n}$, the multiplier modulus of smoothness of order (k) of function f where $\Delta_h^k(f,x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right)$
$x \mp \frac{kh}{2} \in B$ the k^{th} symmetric difference of the function f.

2. $\omega^{r,\theta}(f,\delta)_{p,\emptyset} = sup_{|h|<\delta}\left\|\Delta_{h,\theta}^r(f,.)\right\|_{p,\emptyset_n}$,
where $\Delta_{h,\theta}^r\,(f,x) = \sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k} f\big(x - \frac{rh\theta}{2} + kh\theta\big)$, $x \pm \frac{rh\theta}{2} \in B$, $\theta(x) = \sqrt{x - x^2}$, $x \in [0,1]$ is the multiplier Ditzian –Totik modulus of smoothness of f**.**

3. $\omega_k^{r,\theta}(f,\delta)_{p,\emptyset} = sup\left\|\theta^k(x)\Delta_{h,\theta}^r(f,.)\right\|_{p,\emptyset_n}$,
the k-th locally weighted Ditzian –Totik modulus of smoothness of f where r, k are nonnegative integers and $r + k > 0$.

Note: For r, k are nonnegative integers and for $f \in L_{P,\emptyset_n}(B), 1 \le p \le \infty$, then:

$$\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} \cong K^{r,k,\theta}(f,\delta)_{p,\emptyset_n} \qquad \cdots\cdots(2)$$

where $K^{r,k,\theta}$ is the multiplier weighted Ditzian – Totik K-Functional which is defined by:

$$K^{r,k,\theta}(f,\delta)_{p,\emptyset_n} = inf\left\{\left\|\theta^k(x)(f - T_n)\right\|_{p,\emptyset_n} + \delta^r\left\|\theta^r T_n^{(r)}\right\|_{p,\emptyset_n}\right\}.$$

Proof:

For $f \in L_P(B)$, we have [4]:

$$\omega_k^{r,\theta}(f,\delta)_P \approx K^{r,k,\theta}(f,\delta)_P,$$

then,

$$Sup\left\|\theta^k(x)\Delta_{h\emptyset}^r(f,x)\right\|_P \approx in\,f\left\{\|\theta^k(x)(f - T_n)\|_P + \delta^r\left\|\theta^r T_n^{(r)}\right\|_P\right\}.$$

Since$(f\emptyset_n) \in L_P(B)$, then:

$Sup\left\|\theta^k(x)\Delta^r_{h\emptyset}(f\emptyset_n,x)\right\|_P$
$\approx in\,f\,\{\|\theta^k(x)(f\emptyset_n - T_n\emptyset_n)\|_P +$
$\delta^r\|\theta^r(T_n)^r\emptyset_n\|_P\} =$
$in\,f\left\{\left\|\theta^k(x)(f - T_n)\emptyset_n\right\|_P + \delta^r\|\theta^r(T_n)^r\emptyset_n\|_P\right\}.$
Then,
$Sup\left\|\theta^k(x)\Delta^r_{h\emptyset}(f,x)\right\|_{P,\emptyset_n}$
$\approx in\,f\,\{\|\theta^k(x)(f - T_n)\|_{P,\emptyset_n}$
$+ \delta^r\|\theta^r(T_n)^r\|_{P,\emptyset_n}\}.$
Thus,
$\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} = Sup\left\|\theta^k(x)\Delta^r_{h\emptyset}(f,x)\right\|_{P,\emptyset_n}$
$\approx in\,f\,\{\|\theta^k(x)(f - T_n)\|_{P,\emptyset_n}$
$+ \delta^r\|\theta^r(T_n)^r\|_{P,\emptyset_n}\}$
$= K^{r,k,\theta}(f,\delta)_{p,\emptyset_n}.$
Hence, $\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} \approx K^{r,k,\theta}(f,\delta)_{p,\emptyset_n}.$

2. Auxiliary Results:

In this section, we mention some basic results, which used to prove the main results.

Lemma 2.1 [6]:

If $f \in L_p(X), 1 \le p < \infty, X = [\,a,b\,]$, then we have:

$$\int_a^b f(x)dx = \frac{b-a}{n}\sum_{k=1}^{n} f(x_k)\,,$$

where $x_k = a + \frac{(b-a)(2k-1)}{2n}\ \ ,1 \le k \le n.$

Lemma 2.2:

For $f \in L_{P,\emptyset_n}(B), 1 \le p < \infty$ and $B = [-\pi,\pi]$, we have:
$\mathcal{V}_{3n,2n}(f,x) = \frac{1}{\pi}\ \int_{-\pi}^{\pi} f\emptyset_n\,(x)q_{2n}(x-t)dt.$

Proof:

Since,
$\mathcal{V}_{3n,2n}(f,x) = \frac{2}{3n+2}\sum_{k=0}^{3n} f\emptyset_n(x_k)q_{2n}(x-x_k)$
then:
$\mathcal{V}_{3n,2n}(f,x)$
$= \frac{2}{3n+2}\sum_{k=0}^{3n}\frac{2\pi}{3n}\,\frac{3n}{2\pi} f\emptyset_n(x_k)q_{2n}(x-x_k)$
$= \frac{2}{3n+2}\,\frac{3n}{2\pi}\,\sum_{k=0}^{3n}\frac{2\pi}{3n}\ f\emptyset_n(x_k)q_{2n}(x-x_k)$
$= \frac{6n}{6n+4}\,\frac{1}{\pi}\sum_{k=0}^{3n}\frac{2\pi}{3n}\ f\emptyset_n(x_k)q_{2n}(x-x_k).$
Using lemma (2.1) above we have:
$\mathcal{V}_{3n,2n}(f,x) = \frac{6n}{6n+4}\,\frac{1}{\pi}\ \int_{-\pi}^{\pi} f\emptyset_n\,(t)q_{2n}(x-t)\,dt,$
and since $\lim_{n\to\infty}\frac{6n}{6n+4} = 1$, then we get:
$\mathcal{V}_{3n,2n}(f,x) = \frac{1}{\pi}\ \int_{-\pi}^{\pi} f\emptyset_n\,(t)q_{2n}(x-t)\,dt.$ ■

Lemma 2.3:

For $f \in L_{P,\emptyset_n}(B)$,$1 \le p < \infty, \delta > 0$
$B = [-\pi,\pi]$, we have:
1. $\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} \le A\delta\omega_k^{r-1,\theta}(f',\delta)_{p,\emptyset_n}$
2. $\omega_k^{1,\theta}(f,\delta)_{p,\emptyset_n}\ \le A\delta\|f'\|_{p,\emptyset_n}$, where A be a positive constant, f' is the first derivative of f.

Proof:

1. $\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}$
$= sup\left\|\theta^k(.)\Delta^r_{h,\theta}(f,.)\ \ \right\|_{p,\emptyset_n}$
$= sup\left\|\theta^k(.)\Delta^{r-1}_{h,\theta}(\Delta^1_{h,\theta}(f,.))\ \ \right\|_{p,\emptyset_n}$
$= sup\{\int_B \left|\theta^k(x)\Delta^{r-1}_{h,\theta}\left(\Delta^1_{h,\theta}(f,x)\right)\emptyset_n\right|^p dx\,\}^{\frac{1}{p}}$
$\le sup\{\int_B \left|\theta^k(x)\Delta^{r-1}_{h,\theta}(f\emptyset_n\left(x-\frac{h\theta}{2}+h\theta\right) - f\emptyset_n(x-\frac{h\theta}{2}))\right|^p dx\,\}^{\frac{1}{p}}$

$$\le sup\left\{\int\limits_B\left|\theta^k(x)\Delta^{r-1}_{h,\theta}\int\limits_{x-\frac{h\theta}{2}}^{x-\frac{h\theta}{2}+h\theta}(f\emptyset_n)'(t)dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\le sup\left\{\int\limits_B\left|\theta^k(x)\Delta^{r-1}_{h,\theta}(f\emptyset_n)'(t)\int\limits_{x-\frac{h\theta}{2}}^{x-\frac{h\theta}{2}+h\theta}dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\le sup\left\{\int\limits_B\left|\theta^k(x)\Delta^{r-1}_{h,\theta}(f\emptyset_n)'(t)[x-\frac{h\theta}{2}+h\theta-x+\frac{h\theta}{2}\right|^p dx\right\}^{\frac{1}{p}}$$

$$\le sup\left\{\int\limits_B\left|\theta^k(x)\Delta^{r-1}_{h,\theta}(f\emptyset_n)'(t)[h\theta]\right|^p dx\right\}^{\frac{1}{p}}$$

$= \ h\theta.\left\|\theta^k(.)\Delta^{r-1}_{h,\theta}f'\right\|_{p,\emptyset_n}$
$\le A\delta\omega_k^{r-1,\theta}(f',\delta)_{p,\emptyset_n}.$
Then
$\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} \le A\delta\omega_k^{r-1,\theta}(f',\delta)_{p,\emptyset_n}\ \ ,|h| < \delta$

2. $\omega_k^{1,\theta}(f,\delta)_{p,\emptyset_n} = sup\left\|\theta^k(.)\Delta_{h,\theta}(f,\ .)\right\|_{p,\emptyset_n}$

$= sup\{\int_B \left|\theta^k(x)\left(\Delta_{h,\theta}(f,x)\right)\right|^p dx\}^{\frac{1}{p}}$

$\leq sup\{\int_B \left|\theta^k(x)(f\emptyset_n(x-\frac{h\theta}{2}+h\theta) - f\emptyset_n(x-\frac{h\theta}{2}))\right|^p dx\}^{\frac{1}{p}}$

$$\leq sup\left\{\int_B \left|\theta^k(x)\int_{x-\frac{h\theta}{2}}^{x-\frac{h\theta}{2}+h\theta}(f\emptyset_n)'(t)dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leq sup\left\{\int_B \left|\theta^k(x)(f\emptyset_n)'(t)\int_{x-\frac{h\theta}{2}}^{x-\frac{h\theta}{2}+h\theta}dt\right|^p dx\right\}^{\frac{1}{p}}$$

$\leq\ h\theta.\|f'\|_{p,\emptyset_n} \leq C\delta\|f'\|_{p,\emptyset_n}$. ■

Lemma 2.4:

For $f \in L_{P,\emptyset_n}(B)$,$1 \leq p < \infty$, then:

$$\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} \leq C\delta^r\left\|f^{(r)}\right\|_{p,\emptyset_n},$$

where C is a constant depends on k.

Proof:

By using lemma 2.3 (1) we get:

$\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} \leq A_1\delta\omega_k^{r-1,\theta}(f',\delta)_{p,\emptyset_n} \leq A_2\delta\omega_k^{r-2,\theta}(f'',\delta)_{p,\emptyset_n} \leq \cdots A_{r-1}\delta^{r-1}\,\omega_k^{1,\theta}\left(f^{(r-1)},\delta\right)_{p,\emptyset_n}$.

Now using lemma 2.3(2) we get:

$$\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} \leq A_{r-1}\delta^{r-1}\,\omega_k^{1,\theta}\left(f^{(r-1)},\delta\right)_{p,\emptyset_n} \leq C\delta^r\left\|f^{(r)}\right\|_{p,\emptyset_n}.$$ ■

Lemma 2.5:

For the kernel q_{2n}, which is defined in definition (1.6), we have:

$\frac{1}{\pi}\int_{-\pi}^{\pi} q_{2n}(t)dt = 1$ For each $n \in N$.

Proof: From definition (1.6), we see that

$\frac{1}{\pi}\int_{-\pi}^{\pi} q_{2n}(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{n+1}\sum_{k=0}^{n} D_{n+k}(t)dt$

$= \frac{1}{n+1}\left[\int_{-\pi}^{\pi}\frac{1}{\pi}[D_n(t) + D_{n+1}(t) + \cdots D_{2n}(t)]dt\right]$

$= \frac{1}{n+1}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t) + \frac{1}{\pi}\int_{-\pi}^{\pi} D_{n+1}(t) + \cdots\frac{1}{\pi}\int_{-\pi}^{\pi} D_{2n}(t)\right]$

$= \frac{1}{n+1}[1 + 1 + \cdots 1_{n+1-time}] = 1$, since

$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t) = 1$ For each $n \in N$. ■

Lemma 2.6:

For $f \in L_{P,\emptyset_n}(B)$,$1 \leq p < \infty$, there is a constant $A(p)$ depends on p such that:

$$\left\|\mathcal{V}_{3n,2n}(f)\right\|_{p,\theta_n} \leq A(p)\|f\|_{p,\theta_n}.$$

Proof:

By lemma (2.2) we have:

$\mathcal{V}_{3n,2n}(f,x) = \frac{1}{\pi n}\int_{-\pi}^{\pi} f\emptyset_n(x)q_{2n}(x-t)dt.$

Thus,

$$\left\|\mathcal{V}_{3n,2n}(f)\right\|_{p,\theta_n} = sup\left\{\int_B \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f\emptyset_n(t)q_{2n}(x-t)dt\right|^p\right\}^{\frac{1}{p}}$$

$$= sup\left\{\int_B |f\emptyset_n(t)|^p dt.\ \frac{1}{\pi}\int_{-\pi}^{\pi} q_{2n}(x-t)dx\right\}^{\frac{1}{p}}$$

(Jensen inequality)

$$\leq sup\left\{\int_B |f\emptyset_n(t)|^p dt.\ A\right\}^{\frac{1}{p}} \leq A(p)\|f\|_{p,\theta_n}$$

where $A = \frac{1}{\pi}\int_{-\pi}^{\pi} q_{2n}(x-t)dx.$

Then $\left\|\mathcal{V}_{3n,2n}(f)\right\|_{p,\theta_n} \leq A(p)\|f\|_{p,\theta_n}$. ■

3. Main results:

In this section, we present the following main results.

Theorem 3.1:

For $f \in L_{P,\emptyset_n}(B)$,$1 \leq p < \infty, B = [-\pi,\pi]$ and $k \geq 1$ we have:

$$E_n(f)_{p,\emptyset_n} \leq A(k)\omega_k^{r,\theta}(f,\delta)_{p,\theta_n},,$$

where $A(k)$ is a constant depends on k .

Proof:

Since $\delta^r\left\|T_n^{(r)}\right\|_{p,\emptyset_n} \geq 0$ where $T_n \in \Pi_n$ the best multiplier approximation of $f, 0 \leq \theta(x) = \sqrt{x-x^2}, \leq \frac{1}{2}$ for $x \in [0,1]$ then we have

$E_n(f)_{p,\emptyset_n} = inf\|f - T_n\|_{p,\emptyset_n}$

$\leq \|f - T_n\|_{p,\emptyset_n} + \delta^r\left\|T_n^{(r)}\right\|_{p,\emptyset_n}$

$\leq 2^k\|\theta^k(.)(f - T_n)\|_{p,\emptyset_n} + 2^k\delta^r\left\|\theta^k(.)T_n^{(r)}\right\|_{p,\emptyset_n} \leq 2^k K^{r,k,\theta}(f,\delta)_{p,\emptyset_n}$

$\cong 2^k \omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}$, by (2). Thus:
$E_n(f)_{p,\emptyset_n} \le A(k)\omega_k^{r,\theta}(f,\delta)_{p,\theta_n}$ ■

Theorem 3.2:

For $f \in L_{P,\emptyset_n}(B)$,$1 \le p < \infty, B = [-\pi,\pi]$. Then there is a constant $A(p,k)$ depends on p and k such that the following inequality hold:

$$\left\|f(.) - \mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n} \le A(p,k)\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}.$$

Proof:

By using lemma (2.6), definition (1.7) and theorem (3.1) we get that:

$$\left\|f(.) - \mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n} = sup\left\{\int_B \left|\left(f(x) - \mathcal{V}_{3n,2n}(f,x)\right)\emptyset_n\right|^p dx\right\}^{\frac{1}{p}}$$
$$= sup\left\{\int_B \left|\left(f(x) - T_n + T_n - \mathcal{V}_{3n,2n}(f,x)\right)\emptyset_n\right|^p dx\right\}^{\frac{1}{p}}$$
$$\le sup\left\{\int_B |(f(x) - T_n)\emptyset_n|^p dx\right\}^{\frac{1}{p}} + sup\left\{\int_B \left|\left(T_n - \mathcal{V}_{3n,2n}(f,x)\right)\emptyset_n\right|^p dx\right\}^{\frac{1}{p}}$$
$$= sup\left\{\int_B |(f(x) - T_n)\emptyset_n|^p dx\right\}^{\frac{1}{p}} + sup\left\{\int_B \left|\left(\mathcal{V}_{3n,2n}(T_n,x) - \mathcal{V}_{3n,2n}(f,x)\right)\emptyset_n\right|^p dx\right\}^{\frac{1}{p}} = \|f - T_n\|_{p,\emptyset_n} + \left\|\mathcal{V}_{3n,2n}(T_n,x) - \mathcal{V}_{3n,2n}(f,x)\right\|_{p,\emptyset_n}$$
$$= \|f - T_n\|_{p,\emptyset_n} + \left\|\mathcal{V}_{3n,2n}(T_n - f)\right\|_{p,\emptyset_n}$$
$$\le \|f - T_n\|_{p,\emptyset_n} + A_1(p)\|T_n - f\|_{p,\emptyset_n}$$
$$\le E_n(f)_{p,\emptyset_n} + A_1(p)E_n(f)_{p,\emptyset_n}$$
$$= A(p)E_n(f)_{p,\emptyset_n} \le A(p)2^k\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}$$

where T_n be the best multiplier approximation of f and $\mathcal{V}_{3n,2n}(T_n) = T_n$, thus:

$$\left\|f(.) - \mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n} \le A(p,k)\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n} \quad ■$$

Corollary 3.3:

For $f \in L_{P,\emptyset_n}(B)$,$1 \le p < \infty$, using theorem (3.2) and lemma (2.4) then there is a constant $A(p,k)$ depends on p *and* k such that the following inequality hold:

$$\left\|f(.) - \mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n} \le A(p,k)\delta^r\left\|f^{(r)}\right\|_{p,\emptyset_n}.$$

Theorem 3.4:

For $f \in L_{P,\emptyset_n}(B)$,$1 \le p < \infty$, then there is a constant $A(p,k)$ depends on p and k such that the following inequality hold:

$$\|f(.) - S_n^{**}(f,.)\|_{p,\emptyset_n} \le A(p,k)\frac{2n+1}{n}\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}.$$

Proof:

$$\|f(.) - S_n^{**}(f,.)\|_{p,\emptyset_n} = sup\left\{\int_B |(f(x) - S_n^{**}(f,x))\emptyset_n|^p dx\right\}^{\frac{1}{p}} = sup\left\{\int_B \left|\left(f(x) - \mathcal{V}_{3n,2n}(f,x) + \mathcal{V}_{3n,2n}(f,x) - S_n^{**}(f,x)\right)\emptyset_n\right|^p dx\right\}^{\frac{1}{p}} \le sup\left\{\int_B \left|\left(f(x) - \mathcal{V}_{3n,2n}(f,x)\right)\emptyset_n\right|^p dx\right\}^{\frac{1}{p}} +$$
$$sup\left\{\int_B \left|\left(\mathcal{V}_{3n,2n}(f,x) - S_n^{**}(f,.)\right)\emptyset_n\right|^p dx\right\}^{\frac{1}{p}}$$
$$= \left\|f(.) - \mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n} + sup\left\{\int_B \left|\left(\mathcal{V}_{3n,2n}(f,x) - \tfrac{1}{n}\textstyle\sum_{i=1}^{2n} f(x_i)\, D_n(x - x_i)\right)\emptyset_n\right|^p dx\right\}^{\frac{1}{p}} \le \left\|f(.) - \mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n} +$$
$$\frac{2n+1}{n}\sum_{i=1}^{2n} sup\left\{\int_B \left|(\mathcal{V}_{3n,2n}(f,x) - f(x_i))\emptyset_n\right|^p\right\}^{\frac{1}{p}}.$$

Since, $D_n(x - x_i) \le 2n + 1$, for each $i = 1,2,\dots,2n$, and using theorem (3.2) we get:

$$\|f(.) - S_n^{**}(f,.)\|_{p,\emptyset_n} \le \left\|f(.) - \mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n} +$$

$$\frac{2n+1}{n}\sum_{i=1}^{2n}\left\|\mathcal{V}_{3n,2n}(f,x)-f(x_i)\right\|_{p,\emptyset_n}$$
$$\le B(p)2^k\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}$$
$$+\frac{2n+1}{n}\sum_{i=1}^{2n}C(p)2^k\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}$$
$$\le\frac{2n+1}{n}A(p)2^k\,\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}.$$

Then $\|f(.)-S_n^{**}(f,.)\|_{p,\emptyset_n}$
$$\le A(p,k)\frac{2n+1}{n}\omega_k^{r,\theta}(f,\delta)_{p,\emptyset_n}\quad\blacksquare$$

Corollary 3.5:

For $f\in L_{P,\emptyset_n}(B)$,$1\le p<\infty$, using theorem (3.2) and lemma (2.4) then there is a constant $A(p,k)$ depends on p *and* k such that the following inequality hold:

$$\|f(.)-S_n^{**}(f,.)\|_{p,\emptyset_n}\le\frac{2n+1}{n}A(p,k)\delta^r\left\|f^{(r)}\right\|_{p,\emptyset_n}.$$

Theorem 3.6:

For $f\in L_{P,\emptyset_n}(B)$,$1\le p<\infty, B=[-\pi,\pi]$ there is a constant $A(p)$ depends on p such that the following inequality hold:

$$\left\|f(.)-\mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n}$$

$$\le(1+A(p))E_n(f)_{p,\emptyset_n}$$

Proof:

Let T_n^* be the best multiplier approximation of f then $\left\|f(.)-\mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n}$

$$=\left\|f(.)-T_n^*+T_n^*-\mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n}$$
$$\le\|f(.)-T_n^*\|_{p,\emptyset_n}+\left\|T_n^*-\mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n}$$
$$=\|f(.)-T_n^*\|_{p,\emptyset_n}+\left\|\mathcal{V}_{3n,2n}(T_n^*-f)\right\|_{p,\emptyset_n},$$

this by linearity of $\mathcal{V}_{3n,2n}$ and by $\mathcal{V}_{3n,2n}(T_n^*)=T_n^*$ Then by boundedness of $\mathcal{V}_{3n,2n}$ we get

$$\left\|f(.)-\mathcal{V}_{3n,2n}(f,.)\right\|_{p,\emptyset_n}=\|f(.)-T_n^*\|_{p,\emptyset_n}+\left\|\mathcal{V}_{3n,2n}(T_n^*-f)\right\|_{p,\emptyset_n}$$
$$\le E_n(f)_{p,\emptyset_n}+A(p)\|T_n^*-f\|_{p,\emptyset_n}$$
$$=E_n(f)_{p,\emptyset_n}+A(p)E_n(f)_{p,\emptyset_n}$$
$$=\left(1+A(p)\right)E_n(f)_{p,\emptyset_n}.\quad\blacksquare$$

References:

[1] Ahmed K., Mohammad A. and Suleiman M.,"Approximation Properties of de la Vallee -Poussin sums in Morrey spaces" SQU Journal for science, Tishreen University Vol.122, pp.89-95, (2017).

[2] Sadulla Z. Jafarov,"Approximation of functions by de la Vallee-Poussin sums in weighted Orlicz spaces ", Arabian Journal of Math. 5: p.125-137, (2016). https://doi.org/10.1007/s40065-016-0148-y

[3] Kamel Al-Khaled and Rateb Albtoush ,"Approximation of periodic functions by Vallee-Poussin sums ",Hokkaido Math.Jounrnal ,Jordan University, (2001).

[4] Habeeb A. Kadhim, "On best Trigonometric approximation in L_p (μ) -space ", Dep. of Math. Journal of Education College of computer and Math. University of Thi-Qar, No.3, Vo.2, (2012).

[5] G.H. Hardy, "divergent series", Clarendon press, Oxford, (1949).

[6] VASIL A. POPOV, "Averaged moduli of smoothness", Sofa (1983).