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On solutions of the combined KdV-nKdV equation

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Abstract

The aim of this work is to deal with a new integrable nonlinear equation of wave propagation, the combined of the Korteweg-de vries equation and the negative order Korteweg-de vries equation (combined KdV-nKdV) equation, which was more recently proposed by Wazwaz. Upon using wave reduction variable, it turns out that the reduced combined KdV-nKdV equation is alike the reduced (3+1)-dimensional Jimbo Miwa (JM) equation, the reduced (3+1)-dimensional Potential Yu-Toda-Sasa-Fukuyama (PYTSF) equation and the reduced (3 + 1)dimensional generalized shallow water (GSW) equation in the travelling wave. In fact, the four transformed equations belong to the same class of ordinary differential equation. With the benefit of well-known general solutions for the reduced equation, we show that subjects to some change of parameters, a variety of families of solutions are constructed for the combined KdV-nKdV equation which can be expressed in terms of rational functions, exponential functions and periodic solutions of trigonometric functions and hyperbolic functions. In addition to that the equation admits solitary waves, and double periodic waves in terms of special functions such as Jacobian elliptic functions and Weierstrass elliptic functions

Keywords: Combined KdV-nKdV equation, JM equation, PYTSF equation, GSW equation, travelling wave solutions, Jacobian and Weierstrass elliptic functions

الخلاصة

الهدف من هذا العمل التعامل مع معادلة جديده غير خطيه من انتشار الموجه, المعادله المركبة لمعادلتي كورتيويغ-دي فريس و كورتيويغ-دي فريس الرتبة السالبة (معادلة KdV-nKdV المركبة) التي اقترحت مؤخرا من قبل وزواز. باستخدام متغير خفض الموجة اتضح ان معادلة KdV-nKdV المركبة المحولة تشابة معادلة جيمبو ميوا المحولة (JM) في ٣+١ بعد, معادلة يوتو ساساسي فوكوياما (PYTSF) المحولة في ٣+١ بعد و معادلة المياة الضحلة المعممة المحولة (GWS)في ٣+١ بعد. في الحقيقة المعادلات الاربعة المحولة تنتمي الى نفس صنف المعادلة التفاضلية الاعتيادية وبالاستفاده من الحلول العامة المعروفة للمعادلة المحولة, بينا اخذين بنظر االعتبار اخضاع بعض المعاملات للتغيير عائلة متنوعه من الحلول يمكن تشكيلها لمعادلة KdV- nKdV المركبة والتي يمكن التعبير عنها بدلالة دوال نسبية , دوال اسية, حلول دورية من دوال مثلثية ودوال مثلثية زائدية. بالاضافة الى ذلك المعادلة تملك حلول موجية منعزلة وموجية دورية ثنائية بدلالة الدوال اليعقوبية الأهليلجية ودوال ويرستراس الأهليلجية.

Introduction

Newly, Wawaz [1] introduced the combined KdV-nKdV equation, and that by unifying the KdV recursion operator and the negative order KdV recursion operator to form a new equation, this is given by

$$u_{xt} + 6u_x u_{xx} + u_{xxxx} + u_{xxxt} + 4u_x u_{xt} + 2u_{xx} u_t = 0 \quad (1)$$

where u is the amplitude of the wave in one field x plus time t. Clearly, it is a scalar fourth order nonlinear equation and it has three linear and three nonlinear terms; the terms u_{xxxx}, u_{xxxt} and u_{xt} represent the linear terms while $u_x u_{xt}$, $u_{xx} u_t$ and $u_x u_{xx}$ represent the nonlinear terms. The equation is an integrable equation in the sense that it passes Painlev´e test [1]. The tan /cot, tanh / coth, and a modified version of Hirota's formula were used to get solutions for this equation see for more details [1].

The other three equations, we consider in this work, are the JM equation in (3 + 1)-dimensions which was proposed by Jimbo and Miwa [2], that is

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0, \qquad (2)$$

The PYTSF equation in (3 + 1)-dimensions was introduced by Yu and his collaborators [3], reads

$$-4u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z + 3u_{yy} = 0, \qquad (3)$$

and the GSW equation in (3 + 1)-dimensions was given by Tian and Gao [4], this is given by

$$u_{xxxy} - 3u_y - 3u_x u_{xy} + u_{yt} - u_{xz} = 0, \qquad (4)$$

The JM equation (2), the PYTSF equation (3) and the GSW equation (4) are all of fourth order nonlinear equations they describe waves propagate in three spatial dimensions. These equations have attracted researchers from many fields and many approaches have been used to study and analyze the behavior of the solutions of these equations see for instances [5-13].
This work is lined up as follows. In section two, a wave reduction variable is used to reduce the equations in higher dimensions, the combined KdV-nKdV, JM, PYTSF and GSW equations, to an ordinary differential equation in one variable, also in the same section we address how these equations are related and the relations for the parameters are stated. We move next to section three where solutions for the combined KdV-nKdV equation are established by showing how the solutions are built on the solutions of other equations. Conclusion is given in the last section.

The connections among the reduced combined KdV-nKdV equation and other reduced nonlinear equations

The combined KdV-nKdV equation 1, that is

$$u_{xt} + 6u_x u_{xx} + u_{xxxx} + u_{xxxt} + 4u_x u_{xt} + 2u_{xx} u_t = 0, \qquad (5)$$

where u is a function of one field x plus time t, can be transformed to an ordinary differential equation and that by reducing the number of independent variables of the equation to only one wave reduction variable by taking

$$u(x,y,z,t) = U(\zeta), \qquad \zeta = \alpha x + \delta t, \qquad (6)$$

where α and δ are arbitrary constants.
Substituting into equation (5), after using chain rule, yields

$$\alpha\delta U_{\zeta\zeta} + 6\alpha^3 U_\zeta U_{\zeta\zeta} + \alpha^4 U_{\zeta\zeta\zeta\zeta} + \alpha^3\delta U_{\zeta\zeta\zeta\zeta} + 4\alpha^2\delta U_\zeta U_{\zeta\zeta} + 2\alpha^2\delta U_\zeta U_{\zeta\zeta} = 0,$$

Integrating once

$$\alpha^3(\alpha+\delta)U_{\zeta\zeta\zeta} + 3\alpha^2(\alpha+\delta){U_\zeta}^2 \quad \alpha\delta U_\zeta = c^* \qquad (7)$$

where c^* is a constant of integration.
Rewriting the equation

$$U_{\zeta\zeta\zeta} + A_1 {U_\zeta}^2 + B_1 U_\zeta + C_1 = 0, \quad \zeta = \alpha x + \delta t, \ (8)$$

where $A_1 = \dfrac{3}{\alpha}, \quad B_1 = \dfrac{\delta}{\alpha^2(\alpha+\delta)}$ and

$$C_1 = \frac{1}{\alpha^3(\alpha+\delta)} c^*.$$

The equation under reduction (8) by a wave reduction variable is similar to the transformed J M equation (2) in the travelling wave [5], that is

$$U_{\zeta\zeta\zeta} + A_2 {U_\zeta}^2 + B_2 U_\zeta + C_2 = 0,$$

$$\zeta = ax + by + cz - wt, \qquad (9)$$

where $A_2 = \dfrac{3}{a}, \quad B_2 = \dfrac{-2bw - 3ac}{a^3 b}$ and

$$C_2 = 0,$$

although the roles of the parameters are not the same. By comparing the coefficients of the equation (8) and the equation (9), one can deduce the following relations of the parameters

$$\alpha = a, \quad \text{and} \quad \delta = \frac{-a(2bw+3ac)}{ab+2bw+3ac}, \qquad (10)$$

where a, b, c and w are arbitrary constants.

Another form for (3+1)-dimensional Jimbo-Miwa equation can be found in [14], this is given by

$$u_{xxxy} + 3(uu_y)_x + 3u_{xx} + 3u_{xx}\partial_x^{-1}u_y + 3u_x u_y - 2u_{yt} - 3u_{xz} = 0, \qquad (11)$$

Putting $u = v_x$ the equation becomes

$$v_{xxxxy} + 3(v_x v_{xy})_x + 3v_{xxx} + 3(v_{xx}v_y)_x - 2v_{xyt} - 3v_{xxz} = 0,$$

integrating once and setting the constant of integration to zero yields

$$v_{xxxy} + 3v_x v_{xy} + 3v_{xx} + 3v_{xx}v_y - 2v_{yt} - 3v_{xz} = 0.$$

Taking $v(x,t) = U(\zeta)$ and

$\zeta = \mathrm{x} + \mathrm{cy} + \mathrm{dz} + \mathrm{et},$

where c, d and e are constants, the equation transforms

$$cU_{\zeta\zeta\zeta\zeta} + 6cU_\zeta U_{\zeta\zeta} + (3-2ce-3d)U_{\zeta\zeta} = 0,$$

integrating once gives

$$cU_{\zeta\zeta\zeta} + 3cU_\zeta^2 + (3-2ce-3d)U_\zeta = c^*,$$

where c^* is a constant of integration.

Rewriting the equation

$$U_{\zeta\zeta\zeta} + A_2^* U_\zeta{}^2 + B_2^* U_\zeta + C_2^* = 0, \quad \zeta = x + cy + dz + et, \qquad (12)$$

where

$$A_2^* = 3, \quad B_2^* = \frac{(3-2ce-3d)}{c} \quad \text{and} \quad C_2^* = -\frac{c^*}{c}.$$

Comparing equation (12) and the equation (8) leads to the relations

$$\alpha = 1 \quad \text{and} \quad \delta = -\frac{\alpha^3(2ce+3d-3)}{c+\alpha^2(2ce+3d-3)}. \qquad (13)$$

The connection of the reduced combined KdV-nKdV equation (8) and the reduced (3 + 1)-dimensional PYTSF equation in [9], that is

$$U_{\zeta\zeta\zeta} + A_3 U_\zeta{}^2 + B_3 U_\zeta + C_3 = 0, \quad \zeta = ax + by + cz - wt, \qquad (14)$$

where

$$A_3 = \frac{3}{\alpha}, \quad B_3 = \frac{3b^2+4aw}{a^3c} \quad \text{and} \quad C_3 = 0.$$

By comparing the coefficients of the equation (8) and the equation (14) leads to the relations

$$\alpha = a \quad \text{and} \quad \delta = \frac{a(3b^2+4aw)}{ac-3b^2-4aw} \qquad (15)$$

where a, b, c, and w are constants.

The link of the reduced combined KdV-nKdV equation (8) and the reduced (3 + 1)-dimensional GSW equation in [10] this given by

$$U_{\zeta\zeta\zeta} + A_4 U_\zeta{}^2 + B_4 U_\zeta + C_4 = 0, \quad \zeta = x + y + z - Vt, \qquad (16)$$

where

$$A_4 = -3, \quad B_4 = -(V+1) \quad \text{and} \quad C_4 = C_4.$$

Comparing the coefficients of the equation (8) and the equation (16) gives the relations

$$\alpha = -1 \quad \text{and} \quad \delta = \frac{V+1}{V+2}, \qquad (17)$$

where V is a constant.

In the next section, we shall use these relations among these equations to construct solutions for the combined KdV-nKdV equation.

Solutions for the combined KdV-nKdV equation

This section is dedicated to get solutions for the combined KdV-nKdV equation. The strategy for obtaining exact solutions is built on the links among the reduced combined KdV-nKdV equation and the solutions of other reduced nonlinear equations in the travelling wave.
According to a transformed rational function method which was applied by Wen and Jhn [5], and based on the relations of the parameters in (10), we obtain the following solutions for the combined KdV-nKdV equation

Solution I

$$u(x,t)=\frac{-2\alpha}{1+A\exp(\zeta)}+D,$$
$$\zeta=\alpha x-\frac{\alpha^3}{1+\alpha^2}t,$$

where all the constants are arbitrary.

Solution II

$$u(x,t)=\frac{24B^2}{4B-C\exp(\zeta)}+B\zeta+D,$$
$$\zeta=-3Bx+\frac{27B^3}{-1+9B^2}t,$$

where all the constants are arbitrary.

Solution III

$$u(x,t)=-2\alpha\tan(\zeta)+D,$$
$$\zeta=\alpha x-\frac{4\alpha^3}{-1+4\alpha^2}t,$$

where all the constants are arbitrary.

Solution IV

$$u(x,t)=-2\alpha\tan(\zeta)+\frac{4}{3}\alpha\zeta+D,$$
$$\zeta=\alpha x-\frac{4\alpha^3}{1+4\alpha^2}t,$$

where all the constants are arbitrary.

Solution V

$$u(x,t)=2\alpha\cot(\zeta)+D,$$
$$\zeta=\alpha x-\frac{4\alpha^3}{-1+4\alpha^2}t,$$

where all the constants are arbitrary.

Solution VI

$$u(x,t)=2\alpha\cot(\zeta)+\frac{4}{3}\alpha\zeta+D,$$
$$\zeta=\alpha x-\frac{4\alpha^3}{1+4\alpha^2}t,$$

where all the constants are arbitrary.

Solution VII

$$u(x,t)=2\alpha\tan(\zeta)+D,$$
$$\zeta=\alpha x-\frac{4\alpha^3}{1+4\alpha^2}t,$$

where all the constants are arbitrary.

Solution VIII

$$u(x,t)=2\alpha\tan(\zeta)-\frac{4}{3}\alpha\zeta+D,$$
$$\zeta=\alpha x-\frac{4\alpha^3}{-1+4\alpha^2}t,$$

where all the constants are arbitrary.

Solution VIIII

$$u(x,t)=2\alpha\coth(\zeta)+D\ ,$$
$$\zeta=ax-\frac{4\alpha^3}{1+4\alpha^2}t,$$

where all the constants are arbitrary

Solution IX

$$u(x,t)=2\alpha\coth(\zeta)-\frac{4}{3}\alpha\zeta+D\ ,$$
$$\zeta=ax-\frac{4\alpha^3}{-1+4\alpha^2}t,$$

where all the constants are arbitrary.

According to a generalization of the $\frac{G'}{G}$ expansion method which was used by Zhang

[13], and taking into account the relation in (10), the solutions for the combined KdV – nKdV equation

Solution XI

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\tanh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta + \ln C) - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D$$

and

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\left\{\tanh(\sqrt{\lambda^2-4\mu}\zeta + \ln C) \pm i\mathrm{sech}\left(\sqrt{\lambda^2-4\mu}\zeta + \ln C\right)\right\} - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary.

Solution XII

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\coth(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta + \frac{1}{2}\ln(-C)) - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D$$

and

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\left\{\tanh(\sqrt{\lambda^2-4\mu}\zeta + \ln(-C)) \pm i\mathrm{sech}\left(\sqrt{\lambda^2-4\mu}\zeta + \ln(-C)\right)\right\}^{-1} - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{1-\alpha^2(\lambda^2-4\mu)}t, C < 0$ and all the constants are arbitrary.

solution XIII

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta\right) - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D$$

and

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\left\{\mathrm{tanth}\,(\sqrt{\lambda^2-4\mu}\zeta) \pm i\mathrm{sech}\left(\sqrt{\lambda^2-4\mu}\zeta\right)\right\}^{-1} - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary

Solution XIV

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta\right) - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D,$$

and

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\left\{\tanh(\sqrt{\lambda^2-4\mu}\zeta) \pm i\mathrm{sech}\left(\sqrt{\lambda^2-4\mu}\zeta\right)\right\} - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary.

Solution XV

$$u(x,t) = -\alpha\sqrt{4\mu-\lambda^2}\tan\left\{\frac{1}{2}\sqrt{4\mu-\lambda^2}\zeta - \tan^{-1}(C)\right\} - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D,$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary.

Solution XVI

$$u(x,t) = \alpha\sqrt{4\mu-\lambda^2}\cot\left\{\frac{1}{2}\sqrt{4\mu-\lambda^2}\zeta + \tan^{-1}(C)\right\} - \frac{1}{3}\alpha(\lambda^2-4\mu)\zeta + D,$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary .

Solution XVII

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta + \frac{1}{2}\ln C\right) + D,$$

and

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\left\{\tanh(\sqrt{\lambda^2-4\mu}\zeta + \ln C) \pm i\,\text{sech}\left(\sqrt{\lambda^2-4\mu}\zeta + \ln C\right)\right\} + D,$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{-1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary.

Solution XVII

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\coth\left\{\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta + \frac{1}{2}\ln(-C)\right\} + D,$$

and

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\left\{\text{tanth}(\sqrt{\lambda^2-4\mu}\zeta + \ln(-C)) \pm i\,\text{sech}\left(\sqrt{\lambda^2-4\mu}\zeta + \ln(-C)\right)\right\}^{-1} + D$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{-1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary .

Solution XIX

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\coth\left\{\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta\right\} + D,$$

and

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\left\{\tanh(\sqrt{\lambda^2-4\mu}\zeta) \pm i\,\text{sech}\left(\sqrt{\lambda^2-4\mu}\zeta\right)\right\}^{-1} + D\ ,$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{-1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary .

Solution XX

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\,tanh\left\{\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta\right\} + D,$$

and

$$u(x,t) = \alpha\sqrt{\lambda^2-4\mu}\left\{\tanh(\sqrt{\lambda^2-4\mu}\zeta) \pm i\,\text{sech}\left(\sqrt{\lambda^2-4\mu}\zeta\right)\right\} + D,$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{-1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary.

Solution XXI

$$u(x,t) = -\alpha\sqrt{4\mu-\lambda^2}\,tan\left\{\frac{1}{2}\sqrt{4\mu-\lambda^2}\zeta - \tan^{-1}(C)\right\} + D,$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{-1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary.

Solution XXII

$$u(x,t) = \alpha\sqrt{4\mu-\lambda^2}\,cot\left\{\frac{1}{2}\sqrt{4\mu-\lambda^2}\zeta - \tan^{-1}(C)\right\} + D,$$

where $\zeta = \alpha x + \frac{\alpha^3(\lambda^2-4\mu)}{-1-\alpha^2(\lambda^2-4\mu)}t$ and all the constants are arbitrary.

According to [9] and based on the relations of the parameters in (15), the solutions for the combined KdV – nKdV equation

Solution XXII

$$u(x,t) = \frac{-2\alpha}{1+2D^* \exp(\zeta)} + D,$$

$$\zeta = \alpha x - \frac{\alpha^3}{1+\alpha^2} t,$$

where all the constants are arbitrary .

Solution XXIV

$$u(x,t) = \frac{-24A^{*2}}{4A^* - B^* \exp(\zeta)} + A^*\zeta + D,$$

$$\zeta = -3A^*x + \frac{27A^{*3}}{-1+9A^{*2}} t,$$

where all the constants are arbitrary .

Solution XXV

$$u(x,t) = 2\alpha\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\zeta\right) + D, \zeta = \alpha x - \frac{4\alpha^3\sigma}{-1+4\alpha^2\sigma} t, \sigma < 0,$$

where all the constants are arbitrary .

Solution XXVI

$$u(x,t) = 2\alpha\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\zeta\right) + D,$$

$$\zeta = \alpha x - \frac{4\alpha^3\sigma}{-1+4\alpha^2\sigma} t \,, \sigma < 0$$

where all the constants are arbitrary .

Solution XXVII

$$u(x,t) = 2\alpha\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\zeta\right) + 2\alpha\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\zeta\right) + D,$$

Where $\zeta = \alpha x - \frac{16\alpha^3\sigma}{-1+16\alpha^2\sigma} t \,, \sigma < 0$ and all the constants are arbitrary .

Solution XXVIII

$$u(x,t) = 2\alpha\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\zeta\right) + \frac{4}{3}\alpha\sigma\zeta + D,$$

$$\zeta = \alpha x - \frac{4\alpha^3\sigma}{1+4\alpha^2\sigma} t \,, \sigma < 0$$

where all the constants are arbitrary .

Solution XXIX

$$u(x,t) = 2\alpha\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\zeta\right) + \frac{4}{3}\alpha\sigma + D,$$

$$\zeta = \alpha x - \frac{4\alpha^3\sigma}{1+4\alpha^2\sigma} t \,, \sigma < 0$$

where all the constants are arbitrary .

Solution XXX

$$u(x,t) = 2\alpha\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\zeta\right) + 2\alpha\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\zeta\right) + \frac{16}{3}\alpha\sigma\zeta + D,$$

where $\zeta = \alpha x - \frac{16\alpha^3\sigma}{1+16\alpha^2\sigma} t \,, \sigma < 0$ all the constants are arbitrary .

Solution XXXI

$$u(x,t) = -2\alpha\sqrt{\sigma}\tan\left(\sqrt{\sigma}\zeta\right) + D,$$

$$\zeta = \alpha x - \frac{4\alpha^3\sigma}{1+4\alpha^2\sigma} t \,, \sigma > 0$$

Where all the constants are arbitrary .

Solution XXXII

$$u(x,t) = 2\alpha\sqrt{\sigma}\cot\left(\sqrt{\sigma}\zeta\right) + D,$$

$$\zeta = \alpha x - \frac{4\alpha^3\sigma}{-1+4\alpha^2\sigma} t \,, \sigma > 0$$

where all the constants are arbitrary .

Solution XXXIII

$$u(x,t) = -2\alpha\sqrt{\sigma}\tan\left(\sqrt{\sigma}\zeta\right) + 2\alpha\sqrt{\sigma}\cot\left(\sqrt{\sigma}\zeta\right) + D,$$

where $\zeta = \alpha x - \frac{16\alpha^3\sigma}{-1+16\alpha^2\sigma} t \,, \sigma > 0$ and all the constants are arbitrary .

Solution XXXIV

$$u(x,t) = -2\alpha\sqrt{\sigma}\tan(\sqrt{\sigma}\zeta) + \frac{4}{3}\alpha\sigma\zeta + D,$$

$$\zeta = \alpha x - \frac{4\alpha^3\sigma}{1+4\alpha^2\sigma}t, \sigma > 0$$

where all the constants are arbitrary .

Solution XXXV

$$u(x,t) = 2\alpha\sqrt{\sigma}\cot(\sqrt{\sigma}\zeta) + \frac{4}{3}\alpha\sigma\zeta + D,$$

$$\zeta = \alpha x - \frac{4\alpha^3\sigma}{1+4\alpha^2\sigma}t, \sigma > 0$$

where all the constants are arbitrary .

Solution XXXVI

$$u(x,t) = -2\alpha\sqrt{\sigma}\tanh(\sqrt{\sigma}\zeta) + 2\alpha\sqrt{\sigma}\coth(\sqrt{\sigma}\zeta) + \frac{16}{3}\alpha\sigma\zeta + D,$$

where $\zeta = \alpha x - \frac{4\alpha^3\sigma}{1+4\alpha^2\sigma}t, \sigma > 0$ and all the constants are arbitrary .

Solution XXXVII

$$u(x,t) = -2\alpha s^2\int_0^\zeta \mathrm{sn}^2(\zeta_1,s)d\zeta_1 - \frac{2}{3}\alpha\left(-1 - s^2 \pm \sqrt{(s^4 - s^2 + 1)}\right)\zeta + D,$$

$$\zeta = \alpha x + \frac{2\alpha^3\sqrt{1+26s^2+s^4}}{-1+4\alpha^2\sqrt{1+26s^2+s^4}}t,$$

where all the constants are arbitrary .

Solution XXXVIII

$$u(x,t) = -2\alpha\int_0^\zeta \left(s^2\mathrm{sn}^2(\zeta_1,s) + \mathrm{ns}^2(\zeta_1,s)\right)d\zeta_1 - \frac{2}{3}\alpha\left(\begin{matrix}-1 - s^2\\ \pm\sqrt{(s^4 - s^2 + 1)}\end{matrix}\right)\zeta + D,$$

$$\zeta = \alpha x + \frac{2\alpha^3\sqrt{1+11s^2+s^4}}{-1+4\alpha^2\sqrt{1+11s^2+s^4}}t,$$

where all the constants are arbitrary .

Solution XXXVIX

$$u(x,t) = -\alpha s\int_0^\zeta \left(s\,\mathrm{sn}^2(\zeta_1,s) \pm \mathrm{cn}(\zeta_1,s)\mathrm{dn}(\zeta_1,s)\right)d\zeta_1 - \frac{1}{6}\alpha\left(-1 - s^2 \pm \sqrt{(s^4 + 14s^2 + 1)}\right)\zeta + D,$$

$$\zeta = \alpha x + \frac{2\alpha^3\sqrt{1+26s^2+s^4}}{-1+4\alpha^2\sqrt{1+26s^2+s^4}}t,$$

where all the constants are arbitrary .

Solution XL

$$u(x,t) = -\alpha\int_0^\zeta \left(s^2\mathrm{sn}^2(\zeta_1,s) + \mathrm{ns}^2(\zeta_1,s) \pm s\,\mathrm{cn}(\zeta_1,s)\mathrm{dn}(\zeta_1,s) \pm \mathrm{cs}(\zeta_1,s)\mathrm{ds}(\zeta_1,s)\right)d\zeta_1 - \frac{1}{6}\alpha\left(-1 - s^2 \pm 6s \pm \sqrt{(s^4 \pm 60s(1+s^2) + 134s^2 + 1)}\right)\zeta + D,$$

$$\zeta = \alpha x + \frac{\sqrt{1+60s+134s^2+60s^3+s^4}}{-1+4\alpha^2\sqrt{1+60s+134s^2+60s^3+s^4}}t,$$

where all the constants are arbitrary .

According to [8] and by applying the relations of the parameters in (15), the solutions for combined KdV – nKdV equation

Solution XLI

$$u(x,t) = \frac{-4\sqrt{A}\exp(\pm 2\sqrt{A}\zeta)}{\pm 16\sqrt{AB} \mp \exp \pm 2\sqrt{A}\zeta} + \alpha_0,$$

where $\zeta = x - \frac{4A}{1+4A}t$ and all the constants are arbitrary .

Solution XLII

$$u(x,t) = \frac{-4\sqrt{A}\begin{Bmatrix}-2\cosh(\sqrt{A}\zeta)\sinh(\sqrt{A}\zeta)\\ +2cosh^2(\sqrt{A}\zeta)-1\end{Bmatrix}}{2cosh^2(\sqrt{A}\zeta)-2\cosh(\sqrt{A}\zeta)\sinh(\sqrt{A}\zeta)-3} + B,$$

where $\zeta = x - \frac{4A}{1+4A}t$ and all the constants are arbitrary .

According to [7] and by calling the relations in (15), the solutions for combined KdV – nKdV equation

Solution XLIII

$$u(x,t) = A + 2\sqrt{-\mu}(1+\mu\lambda^2)\left(\frac{\tanh(B+\sqrt{-\mu}\zeta)}{1+\lambda\sqrt{-\mu}\tanh(B+\sqrt{-\mu}\zeta)}\right),$$

and

$$u(x,t) = A + 2\sqrt{-\mu}(1+\mu\lambda^2)\left(\frac{\coth(B+\sqrt{-\mu}\zeta)}{1+\lambda\sqrt{-\mu}\coth(B+\sqrt{-\mu}\zeta)}\right),$$

where$\zeta = x - \frac{4\mu}{-1+4\mu}t,\ \ \mu < 0$ and all the constants are arbitrary .

Solution XLIV

$$u(x,t) = A + \sqrt{-\mu}\left(\tanh(B+\sqrt{-\mu}\zeta) \pm isech(B+\sqrt{-\mu}\zeta)\right),$$

and

$$u(x,t) = A + \sqrt{-\mu}\left(\coth(B+\sqrt{-\mu}\zeta) \pm csch(B+\sqrt{-\mu}\zeta)\right),$$

where $\zeta = x - \frac{\mu}{-1+\mu}t,\ \mu < 0$ and all the constants are arbitrary .

Solution XLV

$$u(x,t) = A + 2\sqrt{-\mu}\left(\lambda\sqrt{-\mu} + \coth(B+\sqrt{-\mu}\zeta)\right),$$

and

$$u(x,t) = A + 2\sqrt{-\mu}\left(\lambda\sqrt{-\mu} + \tanh(B+\sqrt{-\mu}\zeta)\right),$$

where $\zeta = x - \frac{4\mu}{-1+4\mu}t, \mu < 0$ and all the constants are arbitrary .

Solution XLVI

$$u(x,t) = A + 2\sqrt{-\mu}\left(\tanh(B+\sqrt{-\mu}\zeta) + \coth(B+\sqrt{-\mu}\zeta)\right),$$

where $\zeta = x - \frac{16\mu}{-1+16\mu}t, \mu < 0$ and all the constants are arbitrary .

Solution XLVII

$$u(x,t) = A + \sqrt{-\mu}\left(\tanh(B+\sqrt{-\mu}\zeta) \pm isech(B+\sqrt{-\mu}\zeta)\right) + \lambda\sqrt{-\mu}$$

and

$$u(x,t) = A + \sqrt{-\mu}\left(\coth(B+\sqrt{-\mu}\zeta) \pm csch(B+\sqrt{-\mu}\zeta)\right) + \lambda\sqrt{-\mu},$$

where $\zeta = x - \frac{\mu}{-1+\mu}t, \mu < 0$ and all the constants are arbitrary .

Solution XLVIII

$$u(x,t) = A + 2\sqrt{\mu}(1+\mu\lambda^2)\left(\frac{\tan(B-\sqrt{\mu}\zeta)}{1+\lambda\sqrt{\mu}\tan(B-\sqrt{\mu}\zeta)}\right)$$

and

$$u(x,t) = A + 2\sqrt{\mu}(1+\mu\lambda^2)$$

$$\left(\frac{\cot(B+\sqrt{\mu}\zeta)}{1+\lambda\sqrt{\mu}\cot(B+\sqrt{\mu}\zeta)}\right)$$

Where $\zeta = x - \frac{4\mu}{-1+4\mu}t, \mu < 0$ and all the constants are arbitrary .

Solution XLIX

$$u(x,t) = A + \sqrt{\mu}\left(\tan\left(B-\sqrt{\mu}\zeta\right) \pm sec(B - \sqrt{\mu}\zeta)\right)$$

and

$$u(x,t) = A + \sqrt{\mu}\left(\cot\left(B+\sqrt{\mu}\zeta\right) \pm csc(B + \sqrt{\mu}\zeta)\right)$$

where ,$\zeta = x - \frac{\mu}{-1+\mu}t$ and all the constants are arbitrary .

Solution L

$$u(x,t) = A + 2\sqrt{\mu}\left(\lambda\sqrt{\mu} + \cot\left(B + \sqrt{\mu}\zeta\right)\right)$$

and

$$u(x,t) = A + 2\sqrt{\mu}\left(\lambda\sqrt{\mu} + \tan\left(B - \sqrt{\mu}\zeta\right)\right),$$

where $\zeta = x - \frac{4\mu}{-1+4\mu}t$ and all the constants are arbitrary .

Solution LI

$$u(x,t) = A + 2\sqrt{\mu}\left(\tan\left(B-\sqrt{\mu}\zeta\right) - \cot\left(B-\sqrt{\mu}\zeta\right)\right),$$

Where $\zeta = x - \frac{16\mu}{-1+16\mu}t$ and all the constants are arbitrary .

Solution LII

$$u(x,t) = A - \sqrt{\mu}\left(\tan\left(B+\sqrt{\mu}\zeta\right) \pm sec\left(B+\sqrt{\mu}\zeta\right)\right) + \lambda\sqrt{\mu} \qquad (84)$$

and

$$u(x,t) = A - \sqrt{\mu}\left(\cot\left(B-\sqrt{\mu}\zeta\right) \pm csc\left(B+\sqrt{\mu}\zeta\right)\right) + \lambda\sqrt{\mu}$$

where $\zeta = x - \frac{\mu}{-1+\mu}t$ and all the constants are arbitrary .

According to [10] and based on the relations in (17), we have the following for solutions for the combined KdV – nKdV equation

Solution LII, when $\lambda^2 - 4\mu > 0$

$$u(x,t) = -\sqrt{\lambda^2-4\mu}\left\{\frac{-A^* cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta\right)+B^* sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta\right)}{A^* sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta\right)+B^* cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\zeta\right)}\right\} + +\lambda,$$

where $\zeta = \alpha x - \frac{\boldsymbol{\lambda^2-4\mu}}{\boldsymbol{\lambda^2-4\mu+1}}t,\ \alpha = -1$ and all the constants are arbitrary.

Solution LIII, when $\lambda^2 - 4\mu < 0$

$$u(x,t) = -\sqrt{4\mu-\lambda^2}\left\{\frac{-A^*\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\zeta\right) + B^* cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\zeta\right)}{A^*\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\zeta\right) + B^* sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\zeta\right)}\right\} + \alpha_0 + \lambda,$$

where $\zeta = \alpha x - \frac{\boldsymbol{\lambda^2-4\mu}}{\boldsymbol{\lambda^2-4\mu+1}}t, \alpha = -1$ and all the constants are arbitrary .

According to modification of Fan sub– equation method which was applied by Zhang and Peng [6] ,the solutions for the combined KdV – nKdV equation

Solution LIV

$$u(x,t) = \frac{2\alpha k_2 s^2}{s^2+1}\int_0^{\zeta} cn^2\left(\sqrt{\frac{k_2}{2s^2-1}}\zeta_1, s\right)d\zeta_1 - \frac{2}{3}\left(\alpha k_2 \pm \sqrt{\alpha^2 k_2^2 - 3\alpha^2 k_0 k_4}\right)\zeta + D,$$

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$$\zeta = \alpha x - \frac{4\alpha^2\sqrt{-\alpha^2(-k_2^2+3k_0k_4)}}{\pm 1 + 4\alpha\sqrt{-\alpha^2(-k_2^2+3k_0k_4)}}t,$$

where $k_0 = \alpha x - \frac{1-s^2}{(2s^2-1)^2}$, $k_4 < 0, k_2 > 0$ and all the constants are arbitrary .

Solution LV

$$u(x,t) = \frac{2\alpha k_2 s^2}{s^2+1}\int_0^{\zeta} sn^2(\sqrt{\frac{-k_2}{2s^2-1}}\zeta_1, s)d\zeta_1 - \frac{2}{3}(\alpha k_2 \pm \sqrt{\alpha^2 k_2^2 - 3\alpha^2 k_0 k_4})\zeta + D,$$

$$\zeta = \alpha x - \frac{4\alpha^2\sqrt{-\alpha^2(-k_2^2+3k_0k_4)}}{\pm 1 + 4\alpha\sqrt{-\alpha^2(-k_2^2+3k_0k_4)}}t,$$

where $k_0 = \frac{\boldsymbol{k_2^2 s^2}}{2k_4(s^2+1)}$, $k_4 < 0, k_2 > 0$ and all the constants are arbitrary .

Now , going back to the reduced combined KdV – nKdV equation (7), that is

$$\alpha^3(\alpha+\delta)U_{\zeta\zeta\zeta} + 3\alpha^2(\alpha+\delta)U_\zeta^2 + \alpha\delta U_\zeta = c^*.$$

Taking $U_\zeta = V$ the equation becomes

$$\alpha^3(\alpha+\delta)V_{\zeta\zeta} + 3\alpha^2(\alpha+\delta)V^2 + \alpha\delta V = c^*,$$

multiplying by V_ζ yields

$$\alpha^3(\alpha+\delta)V_\zeta V_{\zeta\zeta} + 3\alpha^2(\alpha+\delta)V_\zeta V^2 + \alpha\delta V_\zeta V = c^* V_\zeta,$$

integrating once

$$\frac{1}{2}\alpha^3(\alpha+\delta){V_\zeta}^2 + \alpha^2(\alpha+\delta)V^3 + \frac{1}{2}\alpha\delta V^2 = c^* V + c^{**},$$

where c^{**} is a constant of integration.

Rewriting the equation

$${V_\zeta}^2 = -\frac{2}{\alpha}V^3 - \frac{\delta}{\alpha^2(\alpha+\delta)}V^2 + \frac{2c^*}{\alpha^3(\alpha+\delta)}V + \frac{2c^{**}}{\alpha^3(\alpha+\delta)}.$$

The solution for the last equation is well known in the literature see for instance the classic book of Whittaker and Watson [15], and can be given in terms of Weierstrass double periodic waves in the following form

$$V(\zeta) = -2\alpha\wp(\zeta; g2, g3) - \frac{\delta}{6\alpha(\alpha+\delta)},$$

where Weierstrass elliptic function $\wp$ satisfies the differential equation

$$\wp_\zeta^2 = 4\wp^3 - g_2\wp - g_3$$

and the invariants of the Weierstrass function are given by

$$g_2 = \frac{18\alpha^2\delta^3 + 18\alpha^3\delta^2 + 432c^*\alpha^3\delta + 216c^*\alpha^2\delta^2 + 216c^*\alpha^4}{54\alpha^4(\alpha+\delta)^3}$$

and

$$g_3 = \frac{-218c^{**}\alpha^2\delta - 108c^{**}\alpha\delta^2 + \delta^3 + 18c^*\delta^2 - 108c^{**}\alpha^3 + 18c^*\alpha}{54\alpha^4(\alpha+\delta)^3},$$

where all the constants are arbitrary. As a result the solutions in terms of Weierstrass double periodic waves for the combined KdV-nKdV equation are established.

the equation becomes $U_\zeta = V$ Taking

Conclusions

To sum up, families of solutions for the combined KdV-nKdV equation have been found. The results were built on the connections to some nonlinear equations in higher dimensions. We have shown that the solutions of the new integrable nonlinear equation, the combined KdV-nKdV equation, can be obtained by some analogues of solutions of the JM , PYTSF and GSW equations. These nonlinear equations actually are related to each other in the travelling wave; they belong to the same class of nonlinear ordinary differential equation. The exact solutions for combined KdV-nKdV equation are represented in terms of solitary wave, periodic wave solutions by means of trigonometric functions and hyperbolic functions, and double periodic wave solutions by means of Jacobian elliptic functions and Weierstrass elliptic functions, although the solutions in terms of elliptic functions were not given in simple forms.

One may notice that each method that have been used in solving the nonlinear equations provide variety types of solutions, and by examining these approaches one may recognize the efficiency and reliability of such methods; however some methods give more general solutions than others. Finally we should say that some solutions may repeated or looked the same after using some equivalent relations, but we have written them to show the similarities and differences of the methods that have been used to solve these types of equations.

References

[1] Wazwaz A. M. A new integrable equation that combines the KdV equation with the negative-order KdV equation. Math. Mech. Appl. Sci., vol. 41: pp. 80–87, 2018.

[2] Jimbo M. and Miwa T. Solitons and infinite-dimensional Lie algerbras. Publ. Res. Inst. Math. Sci., vol. 19: pp. 943–1001, 1983.

[3] Yu S. J.; Toda K.; Sasa N. and Fukuyama. N-soliton solutions to Bogoyavknskii-Schiff equation and a guest for the soliton solutions in (3 + 1)-dimensions. J. Phys. A: Math. and Gen., vol. 31: pp. 3337–3347, 1998.

[4] Tian B. and Gao Y.T. Beyond traveling waves: a new algorithm for solving nonlinear evolution equations. Comput. Phys. Comu., vol. 95: pp. 139– 142, 1996.

[5] Wen-Xiu M. and Jhy-Hao L. A transformed rational function method and exact solutions to the (3 + 1)-dimensional Jimbo-Miwa equation. Chaos , Soliton and Fractals, vol. 42: pp. 1356–1363, 2009.

[6] Sheng Z. and Ao-Xue P. A modification of Fan sub-equation method for nonlinear partial differential equations. IAENG International Journal of Applied Mathematics, vol. 44: pp. 1-5, 2014.

[7] Kamruzzaman K. and Akbar M. A. The two variable $(\frac{G'}{G},\frac{1}{G})-$ expansion method for finding exact travelling wave solutions of the (3 + 1)-dimensional nonlinear potential Yu-Toda-Sasa-Fukuyama equation. Interntail Conference on Advanced Computer Science and electronics Information, vol. 4: pp. 181–197, 2014.

[8] Shakeel M. and Mohyud-Din S. T. Exact soltuions of (3 + 1)-dimensionalpotential-YTSF equation by improved $\frac{G'}{G}$ expansion method and the extended Tanh-method. Annals of Pure and Applied Mathematics, vol. 4: pp. 160–171, 2013.

[9] Sheng Z. and Hong-Qing Z. A transformed rational function method for (3+1)-dimensional potential Yu-Toda-Sasa-Fukuyama equation. J. Phys, vol. 76: pp. 561–671, 2011.

[10] Zayed E. M. E. Travelling wave solutions for higher dimensional nonlinear evolution equations using the $\frac{G'}{G}$ expansion method. J. Appl. Math. and informatics, vol. 28: pp. 383–395, 2010.

[11] Zayed E. M. E. and Arnous A. H. Exact solutions of the nonlinear ZKMEW and the potential Yu-Toda-Sasa-Fukuyama equations using the modified simple equation method. AIP Conf. Proc., vol. 1479: pp. 2044–2048, 2012.

[12] Zayed E. M. E. and S. A. Hoda Ibrahim. The two variable $(\frac{G'}{G},\frac{1}{G})-$ expansion method for finding exact travelling wave solutions of the (3 + 1)-dimensional nonlinear potential Yu-Toda-Sasa-Fukuyama equation. Interntail Conference on Advanced Computer Science and electronics Information, pp. 88–391, Atlantis Press, 2013.

[13] Sheng Z. A generalization of the $\frac{G'}{G}$ expansion method and its application to Jimbo-Miwa equation. Bull. Malays. Math. Sci. Soc., vol. 36: pp. 699– 708, 2013.
[14] Engui F. An algebraic method for finding a series of exact solutions to integrable and nonintegrable nonlinear evolution equations. J. Phys. A: Math. Gen., vol. 36: pp. 7009–7026, 2003.
[15] Whittaker E. T. and Watson G. N. A course of modern analysis. Cambridge University Press, UK, 1927.