Research Article **Open Access**

The Solvability of the Continuous Classical Boundary Optimal Control of Couple Nonlinear Elliptic Partial Differential Equations with State Constraints

Jamil A. Ali Al-Hawasy*, Safaa J. Mohammed Al-Qaisi
Department of Mathematics, Faculty of Science, Mustansiriyah University, IRAQ
*Correspondent author email: jhawassy17@uomustansiriyah.edu.iq

Article Info

Received
19/02/2018

Accepted
12/06/2018

Published
15/08/2019

Abstract
This paper concerns with, the proof of the existence and the uniqueness theorem for the solution of the state vector of couple of nonlinear elliptic partial differential equations by using the Minty-Browder theorem, where the continuous classical boundary control vector is given. Also the existence theorem of a continuous classical boundary optimal control vector governing by the couple of nonlinear elliptic partial differential equation with equality and inequality constraints is proved. The existence of the uniqueness solution of the couple of adjoins equations which are associated with the couple of the state equations with equality and inequality constraints are studied. The necessary and sufficient conditions theorem for optimality of the couple of nonlinear elliptic equations with equality and inequality constraints are proved by using the Kuhn-Tucker-Lagrange multipliers theorems.

Keywords: Classical boundary optimal control, couple of nonlinear elliptic partial differential equations, necessary and sufficient conditions.

الخلاصة
يتناول هذا البحث مبرهنة وجود وحدانية الحل لمتجه الحالة لزوج من المعادلات التفاضلية الجزئية الغير خطية من النمط الاهليجي باستخدام مبرهنة مينتي- براودر عندما يكون متجه السيطرة الحدودية التقليدية المستمرة ثابتا". ايضا يتناول مبرهنة الوجود لمتجه سيطرة امثلية حدودية مستمرة تقليدية المسيطر بواسطة الزوجان من المعادلة التفاضلية الجزئية الغير خطية بوجود قيدي التساوي والتباين. تمت دراسة مسالة وجود وحدانية الحل للمعادلة المرافقة لزوج المعادلات التفاضلية الجزئية الغير خطية من النمط الاهليجي بوجود قيدي التساوي والتباين. استخدمت مبرهنتي كان-تاكر-لاكرانج لبرهان مبرهنتي الشرط الضروري والكافي لوجود السيطرة الأمثلية الحدودية التقليدية المستمرة بوجود قيدي التساوي والتباين.

Introduction
The optimal control problems play an important role in many fields in the real life problems, for examples in robotics [1], in an electric power [2], in civil engineering [3], in Aeronautics and Astronautics [4], in medicine [5], in economic [6], in heat conduction [7], in biology [8] and many others fields.
This importance of optimal control problems encouraged many researchers interested to study the optimal control problems of systems are governed either by nonlinear ordinary differential equations as in [9] and [10] or by linear partial differential equations as in [11] or are governed by nonlinear partial differential equations either of a hyperbolic type as in [12] or of a parabolic type as in [13] or by an elliptic type as in [14], or optimal control problem are governed either by a couple of nonlinear partial differential equations of a hyperbolic type as in [15] or of a parabolic type as in [16] or by an elliptic type as in [17], or of an elliptic type but involve a boundary control as in [18]. While the optimal control problem which, is considered in this work is an optimal boundary (Neumann boundary conditions NBCs) control problem governed by a couple of nonlinear partial differential equations of elliptic type.
This work is concerned at first with, the proof of existence and the uniqueness theorem of the state vector solution of a couple nonlinear

elliptic partial differential equations " CNLEPDEs" for a given continuous classical boundary control vector (CCBCV) using the Minty- Browder theorem. Second the existence theorem of a continuous classical boundary optimal control vector "CCBOCV" which is governing by the considered couple of nonlinear partial differential equation of elliptic type with equality and inequality constraints is proved. The existence and the uniqueness solution of the couple of adjoint vector equations associated with the couple of state equations with equality and inequality constraints are studied. The necessary conditions theorem for optimality and the sufficient conditions theorem for optimality of CNLEPDEs with equality and inequality constraints are proved via the Kuhn-Tucker-Lagrange multipliers theorems.

Description of the problem

Let $\Omega \subset \mathbb{R}^2$, with its boundary $\Gamma = \partial\Omega$ be Lipschitz. Consider the following continuous classical boundary optimal control consisting of CNLEPDEs "state equations" with NBCs

$$A_1 y_1 + a_0(x) y_1 - b(x) y_2 + f_1(x, y_1) = f_2(x), \text{ in } \Omega \quad (1)$$

$$A_2 y_2 + b_0(x) y_2 + b(x) y_1 + h_1(x, y_2) = h_2(x), \ in\, \Omega \quad (2)$$

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial y_1}{\partial n} = u_1, \ in\, \Gamma \quad (3)$$

$$\sum_{i,j=1}^{n} b_{ij} \frac{\partial y_2}{\partial n} = u_2, \ in\, \Gamma \quad (4)$$

With $A_1 y_1 = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial y_1}{\partial x_i}\right)$,
$A_2 y_2 = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(b_{ij}(x) \frac{\partial y_2}{\partial x_i}\right)$,
where $a_0(x), b_0(x), b(x), a_{ij}(x), b_{ij}(x) \in C^\infty(\Omega)$, and $(u_1, u_2) = \left(u_1(x), u_2(x)\right) \in \left(L^2(\Gamma)\right)^2$ is the classical boundary control vector, $(y_1, y_2) = \left(y_1(x), y_2(x)\right) \in \left(H^1(\Omega)\right)^2$ is the state vector, corresponding to the control vector, and $(f_1, h_1) = \left(f_1(x, y_1), h_1(x, y_2)\right) \in \left(L^2(\Omega)\right)^2$ and $(f_2, h_2) = \left(f_2(x), h_2(x)\right) \in \left(L^2(\Omega)\right)^2$ are a vector of functions.
The constraint on the controls is given by
$\vec{u} \in \vec{W}$, , $\vec{W} \subset \left(L^2(\Gamma)\right)^2$,
where $\vec{u} = (u_1, u_2)$ and $\vec{W} = W_1 \times W_2$ with
$\vec{W} = \vec{W}_{\vec{U}} = \{\vec{w} \in \left(L^2(\Gamma)\right)^2 \,|\vec{w} = (w_1, w_2) \in \vec{U} \text{ a.e. in } \Gamma\}$
where $\vec{U} = U_1 \times U_2$, and $\forall i = 1,2$, $U_i \subset \mathbb{R}$ is a convex and compact set, and
The cost functional is

$$G_0(\vec{u}) = \iint_\Omega [g_{01}(x, y_1) + g_{02}(x, y_2)] dx_1 dx_2 + \int_\Gamma [g_{03}(x, u_1) + g_{04}(x, u_2)] d\gamma \quad (5)$$

The state constraints are

$$G_1(\vec{u}) = \iint_\Omega [g_{11}(x, y_1) + g_{12}(x, y_2)] dx_1 dx_2 + \int_\Gamma [g_{13}(x, u_1) + g_{14}(x, u_2)] d\gamma = 0 \quad (6)$$

$$G_2(\vec{u}) = \iint_\Omega [g_{21}(x, y_1) + g_{22}(x, y_2)] dx_1 dx_2 + \int_\Gamma [g_{23}(x, u_1) + g_{24}(x, u_2)] d\gamma \le 0 \quad (7)$$

The set of admissible control is

$$\vec{W}_A = \left\{\vec{u} \in \vec{W} | G_1(\vec{u}) = 0, G_2(\vec{u}) \le 0\right\} \quad (8)$$

The CCBOCP is to find the minimum of (5) such that "s.t." the state constraints (6) and (7), i.e. to find $\vec{u}$
$\vec{u} \in \vec{W}_A$ and $G_0(\vec{u}) = \min_{\vec{w} \in \vec{W}_A} G_0(\vec{w})$.
Let $\vec{V} = V \times V = H^1(\Omega) \times H^1(\Omega)$. We denote to the $(v, v)_\Omega((v, v)_\Gamma)$ and $\|v\|_{L^2(\Omega)}(\|v\|_{L^2(\Gamma)})$ to be the inner product and the norm in $L^2(\Omega)\left(L^2(\Gamma)\right)$, by (v, v) and $\|v\|_{H^1(\Omega)}$ the inner product and the norm in $H^1(\Omega)$, by $(\vec{v}, \vec{v})_\Omega = \sum_{i=1}^{2}(v_i, v_i)$ and $\|\vec{v}\|_{(L^2(\Omega))^2} = \sum_{i=1}^{2}\|v_i\|_{L^2(\Omega)}$ the inner product and the norm in $L^2(\Omega) \times L^2(\Omega)$, by $(\vec{v}, \vec{v}) = \sum_{i=1}^{2}(v_i, v_i)$ and $\|\vec{v}\|_{(H^1(\Omega))^2} = \sum_{i=1}^{2}\|v_i\|_{H^1(\Omega)}$ the inner product and the norm in $\vec{V}$ and $\vec{V}^*$ is the dual of $\vec{V}$.

Weak Formulation of the State Equations

The weak form (WF) of problem (1- 4) is obtained by multiplying both sides of (1- 2) by $v_1 \in V$ and $v_2 \in V$ respectively, integrating both sides and then by using the generalize Green's theorem (in Hilbert Space) for the terms which have the 2^{nd} derivatives, once get.

$$a_1(y_1, v_1) + (a_0 y_1, v_1)_\Omega - (b y_2, v_1)_\Omega + (f_1(y_1), v_1)_\Omega = (f_2, v_1)_\Omega + (u_1, v_1)_\Gamma, \forall v_1 \in V \quad (9)$$

And

$$a_2(y_2, v_2) + (b_0 y_2, v_2)_\Omega + (b y_1, v_2)_\Omega + (h_1(y_2), v_2)_\Omega = (h_2, v_2)_\Omega + (u_2, v_2)_\Gamma, \forall v_2 \in V \quad (10)$$

Adding (9) with (10), get that

$$a(\vec{y},\vec{v}) + (f_1(y_1),v_1)_\Omega + (h_1(y_2),v_2)_\Omega = (f_2,v_1)_\Omega + (u_1,v_1)_\Gamma + (h_2,v_2)_\Omega + (u_2,v_2)_\Gamma \quad (11)$$
$$\forall (v_1,v_2) \in \vec{V}$$

where $a(\vec{y},\vec{v}) =$

$$a_1(y_1,v_1) + (a_0 y_1, v_1)_\Omega - (b y_2, v_1)_\Omega + a_2(y_2,v_2) + (b_0 y_2, v_2)_\Omega + (b y_1, v_2)_\Omega$$

with $a_1(y_1,v_1) = \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_1}{\partial x_i} . \frac{\partial v_1}{\partial x_j}$,

$a_2(y_2,v_2) = \sum_{i,j=1}^{n} b_{ij} \frac{\partial y_2}{\partial x_i} . \frac{\partial v_2}{\partial x_j}$,

$a_i(y_i,y_i) \geq c_i \|y_i\|^2_{H^1(\Omega)}$, where $c_i > 0$, $i = 1,2$

$|a_i(y_i,v_i)| \leq \bar{c}_i \|y_i\|_{H^1(\Omega)} \|v_i\|_{H^1(\Omega)}$,

where $\bar{c}_i > 0$, $i = 1,2$.

The following assumptions are useful to prove the existence theorem of a unique solution of the weak form (11).

Assumptions (A):

a) $a(\vec{y},\vec{v})$ is coercive,

i.e. $\frac{a(\vec{y},\vec{y})}{\|\vec{y}\|_{(H^1(\Omega))^2}} \geq c\|\vec{y}\|_{(H^1(\Omega))^2} > 0$, $\forall \vec{y} \in \vec{V}$

b) $|a(\vec{y},\vec{v})| \leq \ell_1 \|\vec{y}\|_{(H^1(\Omega))^2} \|\vec{v}\|_{(H^1(\Omega))^2}$, $\ell_1 > 0$, $\forall \vec{y},\vec{v} \in \vec{V}$

c) f_1 and h_1 are of Carathéodory type " **C.T.**" on $\Omega \times \mathbb{R}$ and satisfy the following conditions with respect to " **w.r.t.** " y_1 and y_2 respectively, i.e. for $\phi_1(x), \phi_2(x) \in L^2(\Omega)$, and $\bar{c}_1, \bar{c}_2 \geq 0$:

$|f_1| \leq \phi_1 + \bar{c}_1 |y_1|, \forall (x,y_1) \in \Omega \times \mathbb{R}$

$|h_1| \leq \phi_2 + \bar{c}_2 |y_2|, \forall (x,y_2) \in \Omega \times \mathbb{R}$

d) f_1 and h_1 are monotone for each $x \in \Omega$ w.r.t. y_1 and y_2 respectively, and $(x,0) = 0$, $h_1(x,0) = 0$, $\forall x \in \Omega$.

e) f_2 and h_2 are of C.T. on Ω and satisfy for $\phi_3(x), \phi_4(x) \in L^2(\Omega)$

$|f_2| \leq \phi_3$, and $|h_2(x)| \leq \phi_4$, $\forall x \in \Omega$.

Proposition (1)[19]: Let $f: \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of Carathéodory type, let F be a functional, s.t. $F(y) = \int_\Omega f(x, y(x)) dx$, where Ω is a measurable subset of $\mathbb{R}^n$, and suppose that

$\|f(x,y)\| \leq \zeta(x) + \eta(x)\|y\|^\alpha$,

$\forall (x,y) \in \Omega \times \mathbb{R}^n, y \in L^P(\Omega \times \mathbb{R}^n)$

where $\zeta \in L^1(\Omega \times \mathbb{R}), \eta \in L^{\frac{P}{P-\alpha}}(\Omega \times \mathbb{R})$, and $\alpha \in [1,P]$, if $P \in [1,\infty)$, and $\eta \equiv 0$, if $P = \infty$. Then F is continuous on $L^P(\Omega \times \mathbb{R}^n)$.

Proposition (2)[19]: Let f, $f_y: \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are of the Carathéodory type, let $F: L^p(\Omega) \longrightarrow \mathbb{R}$ be a functional, s.t. $F(y) = \int_\Omega f(x, y(x)) dx$, where **Ω** is a measurable subset of $\mathbb{R}^d$, and

$\|f_y(x,y)\| \leq \zeta(x) + \eta(x)\|y\|^{\frac{\beta}{q}}$,

$\forall (x,y) \in \Omega \times \mathbb{R}^n$, where $\zeta \in L^q(\Omega \times \mathbb{R})$, $\frac{1}{P} + \frac{1}{q} = 1$, $\eta \in L^{\frac{Pq}{P-\beta}}(\Omega \times \mathbb{R})$, $\beta \in [0,P]$, if $P \neq \infty$, and $\eta \equiv 0$, if $P = \infty$.

Then the Fréchet derivative of F exists for each $y \in L^P(\Omega \times \mathbb{R}^n)$ and is given by

$\Phi'(y)h = \int_\Omega f_y(x, y(x))\, h(x) dx$.

Theorem (1) (Minty-Browder) [20]: "Let V be a reflexive Banach space, and $A: V \longrightarrow V^*$ be a continuous nonlinear map s.t.

$\langle Av_1 - Av_2, v_1 - v_2 \rangle > 0, \forall v_1, v_2 \in V$,

$v_1 \neq v_2$ and $\lim_{\|v\|_{H^1(\Omega)} \to \infty} \frac{\langle Av, v \rangle}{\|v\|_{H^1(\Omega)}} = \infty$.

Then for every $f \in V^*$, there exists a unique solution$y \in V$ of the equation $Ay = f$".

Theorem (2) (Egorov's theorem) [18]: Let Ω be a measurable subset of $\mathbb{R}^d$, $\phi: \Omega \longrightarrow \mathbb{R}$ and $\phi \in L^1(\Omega, \mathbb{R})$, if the following inequality is satisfied $\int_S \phi(x) dx \geq 0$ (or ≤ 0 or $= 0$), for each measurable subset $S \subset \Omega$, then $\phi(x) \geq 0$ (or ≤ 0 or $= 0$), a.e. in **Ω**.

Theorem (3): If the assumptions A are hold, and if the function f_1 (or h_1) in (11) is strictly monotone, then for a given control $\vec{u} \in \vec{W}_A$, the w.f. (11) has a unique solution $\vec{y} \in \vec{V}$.

Proof: Let $\bar{A}: \vec{V} \longrightarrow \vec{V}^*$. Then the w.f. (11) can rewrite as

$$\langle \bar{A}(\vec{y}), \vec{v} \rangle = (\vec{F}(\vec{u}), \vec{v}) \quad (12)$$

where $\langle \bar{A}(\vec{y}), \vec{v} \rangle = a(\vec{y},\vec{v}) + (f_1(y_1),v_1)_\Omega + (h_1(y_2),v_2)_\Omega$

$(\vec{F}(\vec{u}), \vec{v}) = (f_2,v_1)_\Omega + (u_1,v_1)_\Gamma + (h_2,v_2)_\Omega + (u_2,v_2)_\Gamma$

i) From assumptions A-(a & d), $\bar{A}$ is coercive.
ii) From assumptions A-(b & c) and using
iii) Proposition (1) then $\bar{A}$ is continuous w.r.t. $\vec{y}$.

From assumptions A-(a & d) and part (i)$\bar{A}$ is strictly monotone w.r.t. $\vec{y}$.

Then by Theorem (1), the uniqueness solution $\vec{y} \in \vec{V}$ of the w.f. (12) is obtained.

Existence of a Classical Optimal Boundary Control of CCBOCV

This section deals with the state and proof the existence theorem of CCBOCV with the suitable assumptions. Therefore, the following lemmas and assumptions are useful.

Lemma (1): If the assumptions (A) are hold, the functions f_1, h_1 are Lipschitz w.r.t. y_1 and y_2 respectively, and if f_2, h_2 are bounded. Then the mapping $\vec{u} \mapsto \vec{y}_{\vec{u}}$ is Lipschitz continuous from $\vec{W}_A$ into $(L^2(\Omega))^2$, i.e.
$\|\overrightarrow{\Delta y}\|_{(L^2(\Omega))^2} \leq L\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2}$, with $L > 0$.

Proof: Let $\vec{u}, \vec{u}' \in \vec{W}$ be two given controls vectors, and $\vec{y}, \vec{y}'$ be their corresponding state solutions vectors (of the weak form (11)). Subtracting the above two obtained weak forms from (11), setting $\overrightarrow{\Delta y} = \vec{y}' - \vec{y}$ and $\overrightarrow{\Delta u} = \vec{u}' - \vec{u}$, with $\vec{v} = \overrightarrow{\Delta y}$, then adding the obtained two equations, once get

$$\begin{aligned} &a_1(\Delta y_1, \Delta y_1) + (a_0 \Delta y_1, \Delta y_1)_\Omega + \\ &a_2(\Delta y_2, \Delta y_2) + (b_0 \Delta y_2, \Delta y_2)_\Omega \\ &+(f_1(y_1 + \Delta y_1) - f_1(y_1), \Delta y_1)_\Omega + \\ &(h_1(y_2 + \Delta y_2) - h_1(y_2), \Delta y_2)_\Omega \\ &= (\Delta u_1, \Delta y_1)_\Gamma + (\Delta u_2, \Delta y_2)_\Gamma \end{aligned} \tag{13}$$

Using assumptions, A-(a, d), taking the absolute value for both sides of (13), it becomes

$$\begin{aligned} &c\|\overrightarrow{\Delta y}\|^2_{(H^1(\Omega))^2} \\ &\leq \alpha_1\|\Delta y_1\|^2_{H^1(\Omega)} + \alpha_2\|\Delta y_2\|^2_{H^1(\Omega)} + 0 + 0 \\ &\leq |(\Delta u_1, \Delta y_1)_\Gamma| + |(\Delta u_2, \Delta y_2)_\Gamma| \end{aligned} \tag{14}$$

Using the Cauchy-Schwartz inequality and then the trace operator in (14), to get

$$\begin{aligned} &c\|\overrightarrow{\Delta y}\|^2_{(H^1(\Omega))^2} \leq 2c_1\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2}\|\overrightarrow{\Delta y}\|_{(H^1(\Omega))^2} \\ &\Rightarrow \|\overrightarrow{\Delta y}\|_{(H^1(\Omega))^2} \leq c_2\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} \end{aligned} \tag{15}$$

where $c_2 = \frac{2c_1}{c}$,which gives

$$\|\overrightarrow{\Delta y}\|_{(L^2(\Omega))^2} \leq L\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2}, \text{ with } L = cc_2 \tag{16}$$

Assumption (B):

Assume that g_{l1}, g_{l2}, g_{l3} and g_{l4} are of C.T. on $\Omega \times \mathbb{R}$, $\Omega \times \mathbb{R}$, $\Omega \times U_1$ and $\Omega \times U_2$ respectively, and $\forall l = 0,1,2$, are satisfy
$|g_{l1}(x, y_1)| \leq \gamma_{l1}(x) + c_{l1}y_1^2$,
$|g_{l2}(x, y_2)| \leq \gamma_{l2}(x) + c_{l2}y_2^2$
$|g_{l3}(x, u_1)| \leq \gamma_{l3}(x) + c_{l3}u_1^2$, and
$|g_{l4}(x, u_2)| \leq \gamma_{l4}(x) + c_{l4}u_2^2$
where $\gamma_{l1}, \gamma_{l2}, \gamma_{l3}, \gamma_{l4} \in L^1(\Gamma)$ and $c_{l1}, c_{l2}, c_{l3}, c_{l4} \geq 0$

Lemma (2): If assumptions (B) are hold, then $(\forall l = 0,1,2)$ the functional $G_l(\vec{u})$ is continuous on $(L^2(\Gamma))^2$.

Proof: Set $\forall l = 0,1,2$,
$p_{l1}(x, \vec{y}) = g_{l1}(x, y_1) + g_{l2}(x, y_2)$, and
$p_{l2}(x, \vec{u}) = g_{l3}(x, u_1) + g_{l4}(x, u_2)$,
From assumptions (B), and by using Proposition (1) on each of the functional $\iint_\Omega p_{l1}(x, \vec{y})dx_1dx_2$, and $\int_\Gamma p_{l2}(x, \vec{u})d\gamma$ are continuous on $(L^2(\Omega))^2$ and on $(L^2(\Gamma))^2$ respectively. Hence
$G_l(\vec{u}) = \iint_\Omega p_{l1}(x, \vec{y})dx_1dx_2 + \int_\Gamma p_{l2}(x, \vec{u})d\gamma$
is continuous on $(L^2(\Gamma))^2$.

Theorem (4): If the assumptions (A) and (B) are hold, $\vec{W}_A \neq \emptyset$, $f_1(h_1)$ is independent of u_1 (u_2), and $f_2(h_2)$ is bounded functions, s.t. for $\phi_1(x) \in L^2(\Omega)$, $(\phi_2(x) \in L^2(\Omega))$ and $\bar{c}_1 \geq 0$, $(\bar{c}_2 \geq 0)$
$|f_1| \leq \phi_1 + \bar{c}_1|y_1|$, $(|h_1| \leq \phi_2 + \bar{c}_2|y_2|)$
$|f_2| \leq \kappa_1, \kappa_1 \geq 0$ $(|h_2| \leq \kappa_2, \kappa_2 \geq 0)$.
g_{11} (g_{12}) is independent of u_1 (u_2), $g_{l3}(g_{l4})(\forall l = 0,2)$ is convex w.r.t. u_1 (w.r.t. u_2). Then there exists a CCBOCV.

Proof: The set W_i is convex and bounded $\forall i = 1,2$, since U_i it is, then so is $W_1 \times W_2$. On the other hand, and by Egorov's theorem, W_i $\forall i = 1,2$ is closed since U_i it is, then $W_1 \times W_2$ is closed, hence it is weakly compact " **w.c.".**

From the assumption on $\vec{W}_A$, there is an element $\vec{w} \in \vec{W}_A$ with $G_1(\vec{w}) = 0$, $G_2(\vec{w}) \leq 0$ and a minimum sequence $\{\vec{u}_n\} = \{(u_{1n}, u_{2n})\} \in \vec{W}_A$, for each n, s.t. $\lim_{n\to\infty} G_0(\vec{u}_n) = \inf_{\vec{w}\in\vec{W}} G_0(\vec{w})$.

But $\vec{W}$ is w.c., this means that $\{\vec{u}_n\}$ has a subsequence say again $\{\vec{u}_n\}$ which converges weakly to $\vec{u}$ in $\vec{W}$.

Then from the proof of Theorem (3), corresponding to this sequence $\{\vec{u}_n\}$ there is a sequence of solutions $\{\vec{y}_n\}$ of the sequence of weak form:

Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online)
Volume 30, Issue 1, 2019
DOI: http://doi.org/10.23851/mjs.v30i1.464

$$a_1(y_{1n},v_1)+(a_0y_{1n},v_1)_\Omega-(by_{2n},v_1)_\Omega+a_2(y_{2n},v_2)+(b_0y_{2n},v_2)_\Omega+(by_{1n},v_2)_\Omega+(f_1(y_{1n}),v_1)_\Omega+(h_1(y_{2n}),v_2)_\Omega=(f_2,v_1)_\Omega+(u_{1n},v_1)_\Gamma+(h_2,v_2)_\Omega+(u_{2n},v_2)_\Gamma \quad (17)$$

s.t. $\|\vec{y}_n\|_{(H^1(\Omega))^2}$ is bounded, for each n. Then $\{\vec{y}_n\}$ has a subsequence say again $\{\vec{y}_n\}$ s.t. $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\vec{V}$ (Alaoglu theorem [22]).
To prove that (17) converges to

$$a_1(y_1,v_1)+(a_0y_1,v_1)_\Omega-(by_2,v_1)_\Omega+a_2(y_2,v_2)+(b_0y_2,v_2)_\Omega+(by_1,v_2)_\Omega+(f_1(y_1),v_1)_\Omega+(h_1(y_2),v_2)_\Omega=(f_2,v_1)_\Omega+(u_1,v_1)_\Gamma+(h_2,v_2)_\Omega+(u_2,v_2)_\Gamma \quad (18)$$

Let $(v_1,v_2)\in(C(\bar{\Omega}))^2$, and first for the left hand sides, since $y_{in}\longrightarrow y_i$ weakly in V_i, i.e. $y_{in}\longrightarrow y_i$ weakly in $L^2(\Omega)$, for each $i=1,2$.
Then from the left hand sides of (17), (18) and by using Cauchy- Schwarz inequality, one has

$$\begin{aligned}|a_1(y_{1n},v_1)&+(a_0y_{1n},v_1)_\Omega-(by_{2n},v_1)_\Omega+a_2(y_{2n},v_2)+(b_0y_{2n},v_2)_\Omega+(by_{1n},v_2)_\Omega-a_1(y_1,v_1)\\&-(a_0y_1,v_1)_\Omega+(by_2,v_1)_\Omega-a_2(y_2,v_2)-(b_0y_2,v_2)_\Omega-(by_1,v_2)_\Omega|\\ \le\ & c_1\|y_{1n}-y_1\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)}+c_2\|y_{1n}-y_1\|_{L^2(\Omega)}\|v_1\|_{L^2(\Omega)}+c_3\|y_{2n}-y_2\|_{L^2(\Omega)}\|v_1\|_{L^2(\Omega)}\\&+c_4\|y_{2n}-y_2\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)}+c_5\|y_{2n}-y_2\|_{L^2(\Omega)}\|v_2\|_{L^2(\Omega)}+c_6\|y_{1n}-y_1\|_{L^2(\Omega)}\|v_2\|_{L^2(\Omega)}\longrightarrow 0\end{aligned} \quad (19)$$

From assumptions (B), and proposition (1) the functional $\iint_\Omega f_1(x,y_{1n})v_1dx_1dx_2$ and $\iint_\Omega h_1(x,y_{2n})v_2dx_1dx_2$ are continuous with respect to y_{1n} and y_{2n} respectively. But $\vec{y}_n\longrightarrow\vec{y}$ weakly in $(L^2(\Omega))^2$ (since $\vec{y}_n\longrightarrow\vec{y}$ weakly in $\vec{V}$), then by using the compactness theorem (Rellich-Kondrachov theorem) in [21], once get that $\vec{y}_n\longrightarrow\vec{y}$ strongly in $(L^2(\Omega))^2$, and $\forall(v_1,v_2)\in(C(\bar{\Omega}))^2$, we have

$$(f_1(y_{1n}),v_1)_\Omega+(h_1(y_{2n}),v_2)_\Omega\longrightarrow(f_1(y_1),v_1)_\Omega+(h_1(y_2),v_2)_\Omega, \quad (20a)$$

i.e. the left-hand side of (17) $\longrightarrow$ the left-hand side of (18)
Second, but $u_{1n}\longrightarrow u_1$ weakly in $L^2(\Gamma)$ and so as $u_{2n}\longrightarrow u_2$, then

$$(u_{1n}-u_1,v_1)_\Gamma+(u_{2n}-u_2,v_2)_\Gamma\longrightarrow 0 \quad (20a)$$

From (20a) and (20b) give us that (17) converges to (18).
But $(C(\bar{\Omega}))^2$ is dense in $\vec{V}$, then these convergences hold $\forall(v_1,v_2)\in\vec{V}$, which gives $\vec{y}=\vec{y}_{\vec{u}}$ satisfies the w.f. of the state equations.
From Lemma (2), the functional $G_l(\vec{u})$ is continuous on $(L^2(\Gamma))^2$, $\forall l=0,1,2$.
From the assumptions on g_{11}, g_{12}, $G_1(\vec{u}_n)$ is continuous, and the strongly converges of $y_{1n}\longrightarrow y_1$, $y_{2n}\longrightarrow y_2$ in $L^2(\Omega)$, once get

$$G_1(\vec{u})=\lim_{n\to\infty}G_1(\vec{u}_n)=0$$

Also, from the assumptions on $g_{l1}(x,y_1)$ and $g_{l3}(x,u_1)$ ($\forall l=0,1,2$) and Lemma (2), the integrals $\iint_\Omega g_{l1}(x,y_1)dx_1dx_2$ and $\int_\Gamma g_{l3}(x,u_1)d\gamma$ are continuous w.r.t. y_1 and u_1 respectively, but $g_{l3}(x,u_1)$, (for each $l=0,2$) is convex w.r.t. u_1, then $\int_\Gamma g_{l3}(x,u_1)d\gamma$ is weakly lower semicontinuous "w.l.sc." w.r.t. u_1, and then

$$\begin{aligned}&\iint_\Omega g_{l1}(x,y_1)dx_1dx_2+\int_\Gamma g_{l3}(x,u_1)d\gamma\le\\&\iint_\Omega g_{l1}(x,y_1)dx_1dx_2+\underline{\lim}_{n\to\infty}\int_\Gamma g_{l3}(x,u_1)d\gamma\\&=\lim_{n\to\infty}\iint_\Omega[g_{l1}(x,y_{1n})-g_{l1}(x,y_1)]dx_1dx_2+\iint_\Omega g_{l1}(x,y_1)dx_1dx_2+\underline{\lim}_{n\to\infty}\int_\Gamma g_{l3}(x,u_1)d\gamma\\&=\underline{\lim}_{n\to\infty}\iint_\Omega g_{l1}(x,y_{1n})dx_1dx_2+\underline{\lim}_{n\to\infty}\int_\Gamma g_{l3}(x,u_1)d\gamma\end{aligned}$$

By the same way one can get

$$\iint_\Omega g_{l2}(x,y_2)dx_1dx_2+\int_\Gamma g_{l4}(x,u_2)d\gamma\le\underline{\lim}_{n\to\infty}\iint_\Omega g_{l2}(x,y_{2n})dx_1dx_2+\underline{\lim}_{n\to\infty}\int_\Gamma g_{l4}(x,u_2)d\gamma,\ (\text{for each } l=0,2)$$

From the above inequalities, one gets $G_l(\vec{u})$ ($\forall l=0,2$) is w.l.sc. with respect to $(\vec{y},\vec{u})$.
But $G_2(\vec{u}_n)\le 0,\forall n$, then

$$G_2(\vec{u})\le\underline{\lim}_{n\to\infty}G_2(\vec{u}_n)=0$$

Finally,

$$G_0(\vec{u})\le\underline{\lim}_{n\to\infty}G_0(\vec{u}_n)=\lim_{n\to\infty}G_0(\vec{u}_n)=\inf_{\vec{w}\in\vec{W}}G_0(\vec{w})$$

Which implies that $\vec{u}$ is a CCBOCV.

The NCFO "necessary conditions for optimality " of CCBOCV

To find the derivatives of the Hamiltonian" Fréchet derivatives" The following assumption is useful.

Assumptions(C):

a) f_{1y_1}, h_{1y_2} are of the C.T. on $\Omega \times \mathbb{R}$, and satisfy for $(x, y_1), (x, y_2) \in \Omega$ the conditions

$|f_{1y_1}| \leq \breve{c}_1$, $|h_{1y_2}| \leq \breve{c}_2$, with $\breve{c}_1, \breve{c}_2 \geq 0$.

$f_{1y_1} \geq 0$ and $h_{1y_2} \geq 0$.

b) f_2, h_2 are of the C. T. on Ω and for $x \in \Omega$ and $\breve{c}_3, \breve{c}_4 \geq 0$ satisfy

$|f_2(x)| \leq \breve{c}_3$, $|h_2(x)| \leq \breve{c}_4$, .

c) g_{l1y_1}, g_{l2y_2}, g_{l3u_1}, g_{l4u_2} $(\forall l = 0,1,2)$ are of the C. T. on $\Omega \times \mathbb{R} \times \mathbb{R}$ and satisfy

$|g_{l1y_1}| \leq \gamma_{l1} + c_{l1}|y_1|$, $|g_{l2y_2}| \leq \gamma_{l2} + c_{l2}|y_2|$, $|g_{l3u_1}| \leq \gamma_{l3} + c_{l3}|u_1|$, $|g_{l4u_2}| \leq \gamma_{l4} + c_{l4}|u_2|$

$c_{l1}, c_{l2}, c_{l3}, c_{l4} \geq 0$, $\gamma_{l1}, \gamma_{l2}, \gamma_{l3}, \gamma_{l4} \in L^2(\Omega)$.

Theorem (5): If the assumptions (A), (B) and (C) are hold, the Hamiltonian is defined by:

$$
\begin{aligned}
H &= H(x, y_{1u_1}, y_{2u_2}, z_1, z_2, u_1, u_2) \\
&= z_1\big(f_2(x) - f_1(x, y_1)\big) + g_{01}(x, y_1) \\
&\quad + g_{03}(x, u_1) + z_2\big(h_2(x) - h_1(x, y_2)\big) \\
&\quad + g_{02}(x, y_2) + g_{04}(x, u_2)
\end{aligned}
$$

The adjoint equations of state equations (1- 4) are given by

$$A_1 z_1 + a_0(x) z_1 + b(x) z_2 + z_1 f_{1y_1}(x, y_1) = g_{01y_1}(x, y_1),\ in\, \Omega \tag{21}$$

$$A_2 z_2 + b_0(x) z_2 - b(x) z_1 + z_2 h_{1y_2}(x, y_2) = g_{02y_2}(x, y_2),\ in\ \Omega \tag{22}$$

$$\frac{\partial z_1}{\partial n} = 0,\ in\, \Gamma \tag{23}$$

$$\frac{\partial z_2}{\partial n} = 0,\ in\, \Gamma \tag{24}$$

Then the Fréchet derivatives of G_0 are given by $\acute{\vec{G}}_0(\vec{u}) \cdot \overrightarrow{\Delta u} = \int_\Gamma H'^{\mathrm{T}}_{\vec{u}} \cdot \overrightarrow{\Delta u}\ d\gamma$, where $H'_{\vec{u}} = \begin{pmatrix} H'_{u_1} \\ H'_{u_2} \end{pmatrix} = \begin{pmatrix} z_1 + g_{03u_1} \\ z_2 + g_{04u_2} \end{pmatrix}$ and $\vec{z} = \vec{z}_{\vec{u}}$ is the adjoint of the state $\vec{y}_{\vec{u}}$.

Proof: Writing the couple of the adjoint equations (21-24) by their w.f., then adding them, and then substituting $\vec{v} = \overrightarrow{\Delta y}$ in the obtained equation to get

$$
\begin{aligned}
& a_1(z_1, \Delta y_1) + (a_0 z_1, \Delta y_1)_\Omega + (b z_2, \Delta y_1)_\Omega + \\
& a_2(z_2, \Delta y_2) + (b_0 z_2, \Delta y_2)_\Omega - (b z_1, \Delta y_2)_\Omega + \\
& \big(z_1 f_{1y_1}(y_1), \Delta y_1\big)_\Omega + \big(z_2 h_{1y_2}(y_2), \Delta y_2\big)_\Omega \\
& = \big(g_{01y_1}(y_1), \Delta y_1\big)_\Omega + \big(g_{02y_2}(y_2), \Delta y_2\big)_\Omega
\end{aligned} \tag{25}
$$

One can easily prove that the w.f. (25) "for a given control $\vec{u} \in \overrightarrow{W}$" has a unique solution $\vec{z} = \vec{z}_{\vec{u}}$ using a similar way which is used in proof of theorem (3).

Now, substituting once the solutions y_1 in the weak form of the state equations (9) and once again the solution $y_1 + \Delta y_1$, then subtracting the 1st obtained weak form from the other one, to obtain

$$
\begin{aligned}
& a_1(\Delta y_1, v_1) + (a_0 \Delta y_1, v_1)_\Omega - (b \Delta y_2, v_1)_\Omega + \\
& (f_1(y_1 + \Delta y_1) - f_1(y_1), v_1)_\Omega \\
& = (\Delta u_1, v_1)_\Gamma,\ \forall v_1 \in V_1
\end{aligned} \tag{26}
$$

The above substituting and subtracting are repeated with the solutions y_2 and $y_2 + \Delta y_2$, and in the weak form of the state equations (10), to obtain

$$
\begin{aligned}
& a_2(\Delta y_2, v_2) + (b_0 \Delta y_2, v_2)_\Omega + (b \Delta y_1, v_2)_\Omega + \\
& (h_1(y_2 + \Delta y_2) - h_1(y_2), v_2)_\Omega \\
& = (\Delta u_2, v_2)_\Gamma,\ \forall v_2 \in V_2
\end{aligned} \tag{27}
$$

Adding (26) with (27), then substituting $\vec{v} = (z_1, z_2)$ in the resulting equation, to get

$$
\begin{aligned}
& a_1(\Delta y_1, z_1) + (a_0 \Delta y_1, z_1)_\Omega - (b \Delta y_2, z_1)_\Omega + \\
& a_2(\Delta y_2, z_2) + (b_0 \Delta y_2, z_2)_\Omega + (b \Delta y_1, z_2)_\Omega \\
& + (f_1(y_1 + \Delta y_1) - f_1(y_1), z_1)_\Omega + \\
& (h_1(y_2 + \Delta y_2) - h_1(y_2), z_2)_\Omega \\
& = (\Delta u_1, z_1)_\Gamma + (\Delta u_2, z_2)_\Gamma,\ \forall (z_1, z_2) \in \vec{V}
\end{aligned} \tag{28}
$$

From the assumptions on $f_1(h_1)$, and by using Proposition (2), the Fréchet derivative of $f_1(h_1)$ exists, and hence from Lemma (1) and the Minkowski inequality, (28) becomes

$$
\begin{aligned}
& a_1(\Delta y_1, z_1) + (a_0 \Delta y_1, z_1)_\Omega - (b \Delta y_2, z_1)_\Omega \\
& + a_2(\Delta y_2, z_2) \\
& + (b_0 \Delta y_2, z_2)_\Omega \\
& + (b \Delta y_1, z_2)_\Omega \\
& + \big(f_{1y_1} \Delta y_1, z_1\big)_\Omega + \tilde{\varepsilon}_1(\overrightarrow{\Delta u}) \|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} + \\
& \big(h_{1y_2} \Delta y_2, z_2\big)_\Omega + \tilde{\varepsilon}_2(\overrightarrow{\Delta u}) \|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} \\
& = (\Delta u_1, z_1)_\Gamma + (\Delta u_2, z_2)_\Gamma
\end{aligned} \tag{29}
$$

where $\tilde{\varepsilon}_1(\overrightarrow{\Delta u}), \tilde{\varepsilon}_2(\overrightarrow{\Delta u}) \longrightarrow 0$ as $\overrightarrow{\Delta u} \longrightarrow 0$.

Subtracting (25) from (29), to get

$$
\begin{aligned}
& \big(g_{01y_1}(x, y_1), \Delta y_1\big)_\Omega + \big(g_{02y_2}(x, y_2), \Delta y_2\big)_\Omega \\
& + \tilde{\varepsilon}_1(\overrightarrow{\Delta u}) \|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} \\
& + \tilde{\varepsilon}_2(\overrightarrow{\Delta u}) \|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} \\
& = (\Delta u_1, z_1)_\Gamma + (\Delta u_2, z_2)_\Gamma
\end{aligned} \tag{30}
$$

Now, from the assumptions on g_{01}, g_{02}, g_{03} and g_{04}, the definition of the Fréchet derivative and then using the result of Lemma (1), we have

$$G_0(\vec{u}+\overrightarrow{\Delta u}) - G_0(\vec{u}) = \int_\Gamma (z_1 + g_{03u_1})\Delta u_1 d\gamma + \int_\Gamma (z_2 + g_{04u_2})\Delta u_2 d\gamma + \tilde{\varepsilon}_8(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} \quad (31)$$

where

$$\tilde{\varepsilon}_8(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} = \tilde{\varepsilon}_7(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} - \tilde{\varepsilon}_1(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2} - \tilde{\varepsilon}_2(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_{(L^2(\Gamma))^2}$$

But from the definition of the Fréchet derivative of G_0, once get

$$\acute{G}_0(\vec{u}) \cdot \overrightarrow{\Delta u} = \int_\Gamma H'^{\mathrm{T}}_{\vec{u}} \cdot \overrightarrow{\Delta u}\, d\gamma.$$

Note: In the prove of the above theorem, we have found the Fréchet derivative for the functional G_0, so the same technique is used to find the Fréchet derivative for G_1 and G_2.

Theorem (6):

(a) If assumptions (A), (B) and (C) are hold, $\vec{W}$ is convex, and if $\vec{u} \in \vec{W}_A$ is a classical optimal, then $\forall l = 0,1,2$, there exists multipliers $\lambda_l \in \mathbb{R}$, with λ_0, λ_2 are nonnegative, $\sum_{l=0}^{2}|\lambda_l| = 1$, s.t. the following Kuhn- Tucker- Lagrange(K.T.L.)conditions are satisfied:

$$\int_\Gamma H'^{T}_{\vec{u}} \cdot \overrightarrow{\Delta u}\, d\gamma \geq 0, \quad (32a)$$

For each $\vec{w} \in \vec{W}$, with $\overrightarrow{\Delta u} = \vec{w} - \vec{u}$
where $g_i = \sum_{l=0}^{2}\lambda_l g_{liu_j}$ and $z_j = \sum_{l=0}^{2}\lambda_l z_{lj}$, $(for\ j = 1,2, i = 3,4)$ in (Theorem (5)),

$$\lambda_2 G_2(\vec{u}) = 0, \quad (32b)$$

(b) (Minimum Principle in point wise weak form): The inequality (32a) is equivalent to

$$H'^{T}_{\vec{u}} \cdot \vec{u} = \min_{\vec{w}\in\vec{U}} H'^{T}_{\vec{u}} \cdot \vec{w}\ a.e.\ on\ \Gamma \quad (33)$$

Proof: (a) From Theorem (5), $G_l(\vec{u})$ (for each $l = 0,1,2$ and at each $\vec{u} \in \vec{W}$) has a continuous Fréchet derivative, since the control (classical) $\vec{u} \in \vec{W}_A$ is optimal, then using the K.T.L. theorem $\forall l = 0,1,2$, there exists multipliers $\lambda_l \in \mathbb{R}$, with λ_0, λ_2 are nonnegative, and $\sum_{l=0}^{2}|\lambda_l| = 1$, s.t.

$$(\textstyle\sum_{l=0}^{2}\lambda_l \acute{G}_{l\vec{u}}(\vec{u}))_\Gamma \cdot (\vec{w} - \vec{u}) \geq 0, \forall \vec{w} \in \vec{W} \quad (34a)$$

$$\lambda_2 G_2(\vec{u}) = 0 \quad (34b)$$

Then from Theorem 5, (34a) with setting $\Delta u_1 = w_1 - u_1$, $\Delta u_2 = w_2 - u_2$, can be written $\forall \vec{w} \in \vec{W}$ as

$$\int_\Gamma [(z_1 + g_{3u_1})\Delta u_1 + (z_2 + g_{4u_2})\Delta u_2] d\gamma \geq 0$$

where $z_j = \sum_{l=0}^{2}\lambda_l z_{jl}$, $g_{iu_j} = \sum_{l=0}^{2}\lambda_l g_{liu_j}$, for $j = 1,2$, $i = 3,4$

$$\Rightarrow \int_\Gamma H'^{\mathrm{T}}_{\vec{u}} \cdot \overrightarrow{\Delta u}\, d\gamma \geq 0, \forall \vec{w} \in \vec{W}, \overrightarrow{\Delta u} = \vec{w} - \vec{u}.$$

(b) Let $\{\vec{u}_n\}$ be a dense sequence in $\vec{W}_{\vec{U}}$, and $S \subset \Gamma$ be a measurable set s.t.

$$\vec{w}(x) = \begin{cases} \vec{u}_n(x), \text{if } x \text{ belongs in } S \\ \vec{u}(x), \text{if } x \text{ not belong in } S \end{cases}$$

Then (32a), gives

$$\int_S H'^{\mathrm{T}}_{\vec{u}} \cdot (\vec{u}_n - \vec{u}) ds \geq 0, \forall S \subset \Gamma$$

Then using Theorem (2) once get that

$$H'^{\mathrm{T}}_{\vec{u}} \cdot (\vec{u}_n - \vec{u}) \geq 0, \text{a.e. on } \Gamma,$$

The above inequality holds on the boundary Γ of the region Ω except in a subset Γ_n with $\mu(\Gamma_n) = 0$, for each n, where μ is a Lebesgue measure, then this equality satisfies on the boundary Γ except in the union of $\bigcup_n \Gamma_n$ with $\mu(\bigcup_n \Gamma_n) = 0$, but $\{\vec{u}_n\}$ is a dense sequence in the control set $\vec{W}$, then there exists $\vec{u} \in \vec{W}$ s.t.

$$H'^{\mathrm{T}}_{\vec{u}} \cdot \vec{u} = \min_{\vec{w}\in\vec{U}} H_{\vec{u}}^{\mathrm{T}} \cdot \vec{w}, \text{ a.e.on } \Gamma.$$

The converse of the proof is obtained directly.

SCFO "Sufficient Conditions for Optimality" of CCBOCV

Theorem (7): If assumptions (A), (B) and (C) are hold, if f_1 and g_{11}(h_1, g_{12}) are affine w.r.t. $y_1(y_2)$, g_{13}(g_{14}) is affine w.r.t. $u_1(u_2)$, f_2 and h_2 are bounded functional for x, and if $\forall l = 0,2$ g_{l1}, g_{l2}, g_{l3}, g_{l4} are convex w.r.t. y_1, y_2, u_1, u_2 respectively. Then the NCFO in Theorem 6, are also sufficient if λ_0 is positive.

Proof: Assume $\vec{u} \in \vec{W}_A$, $\vec{u}$ is satisfied the conditions(32a) and(32b).

Let $G(\vec{u}) = \sum_{l=0}^{2}\lambda_l G_l(\vec{u}) \Longrightarrow$

$$\acute{G}(\vec{u})\overrightarrow{\Delta u} = \sum_{l=0}^{2}\lambda_l \int_\Gamma [\left((z_{l1} + g_{l3u_1})\Delta u_1 + (z_{l2} + g_{l4u_2})\Delta u_2\right)] d\gamma$$

$$= \int_\Gamma H'_{\vec{u}}(x, z_1, z_2, u_1, u_2) \cdot \overrightarrow{\Delta u}\, d\gamma \geq 0.$$

Since

$$f_1(x, y_1) = f_{11}(x)y_1 + f_{12}(x), f_2(x) = f_{21}(x)$$

$h_1(x, y_2) = h_{11}(x)y_2 + h_{12}(x), \qquad h_2(x) = h_{21}(x)$

Let $\vec{u} = (u_1, u_2)$, $\vec{\bar{u}} = (\bar{u}_1, \bar{u}_2)$ be a given controls then $\vec{y} = (y_1, y_2)$, $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2)$ are their corresponding solutions, substituting the pair $(\vec{y}, \vec{u})$ in (1-4) and multiplying all the obtained equations by $\beta \in [0,1]$ once, and then substituting the pair $(\vec{\bar{y}}, \vec{\bar{u}})$ in (1-4) and multiplying all the obtained equations by $\bar{\beta} = (1 - \beta)$, finally adding each pair from the corresponding equations together one gets:

$$A_1(\beta y_1 + \bar{\beta}\bar{y}_1) + a_0(x)(\beta y_1 + \bar{\beta}\bar{y}_1) - b(x)(\beta y_2 + \bar{\beta}\bar{y}_2) + f_{11}(x)(\beta y_1 + \bar{\beta}\bar{y}_1) + f_{12}(x) = f_{21}(x) \qquad (35a)$$

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial n}(\beta y_1 + \bar{\beta}\bar{y}_1) = \alpha u_1 + \bar{\beta}\bar{u}_1 \qquad (35b)$$

And

$$A_2(\beta y_2 + \bar{\beta}\bar{y}_2) + b_0(x)(\beta y_2 + \bar{\beta}\bar{y}_2) + b(x)(\beta y_1 + \bar{\beta}\bar{y}_1) + h_{11}(x)(\beta y_2 + \bar{\beta}\bar{y}_2) + h_{12}(x) = h_{21}(x) \qquad (36a)$$

$$\sum_{i,j=1}^{n} b_{ij} \frac{\partial}{\partial n}(\beta y_2 + \bar{\beta}\bar{y}_2) = \beta u_2 + \bar{\beta}\bar{u}_2 \qquad (36b)$$

Now, if we have the control vector $\vec{\bar{\bar{u}}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2)$ with

$\bar{\bar{u}}_1 = \beta u_1 + \bar{\beta}\bar{u}_1$ and $\bar{\bar{u}}_2 = \beta u_2 + \bar{\beta}\bar{u}_2$

Then from (35a, 35b) , (36a, 36b), once get that the state vector $(\bar{\bar{y}}_1 = y_{1\bar{\bar{u}}_1}, \bar{\bar{y}}_2 = y_{2\bar{\bar{u}}_2})$ with

$\bar{\bar{y}}_1 = \beta y_{1u_1} + \bar{\beta} y_{1\bar{u}_1} = \beta y_1 + \bar{\beta}\bar{y}_1$

and $\bar{\bar{y}}_2 = \beta y_{2u_2} + \bar{\beta} y_{2\bar{u}_2} = \beta y_2 + \bar{\beta}\bar{y}_2$,

are their corresponding solution, i.e. are satisfied (1-4) respectively. So, the operators $u_1 \longmapsto y_{1u_1}$, $u_2 \longmapsto y_{2u_2}$ are convex- linear w.r.t. (y_1, u_1) and (y_2, u_2) respectively.

Now, from this result and since g_{11}, g_{12}, g_{13} , g_{14} are affine w. r. t. y_1, y_2, u_1 , u_2 respectively, on Ω, once get that $\forall x \in \Omega$, $G_1(\vec{u})$ is convex-linear w.r.t. $(\vec{y}, \vec{u})$,.

Also, since $(\forall l = 0,2 \ \& \ \forall x \in \Omega)$ g_{l1}, g_{l2}, g_{l3} , g_{l4}, are convex w.r.t. y_1, y_2, u_1, and u_2 respectively, i.e. $G(\vec{u})$ is convex w.r.t. $\vec{y}$ and $\vec{u}$. Then $G(\vec{u})$ is convex w.r.t. $\vec{y}$,and $\vec{u}$, in "the convex set" $\vec{W}$, and has a " continuous" Fréchet derivative satisfies

$\acute{\vec{G}}(\vec{u})\overrightarrow{\Delta u} \geq 0 \Longrightarrow \vec{u}$ minimize $G(\vec{u})$, i.e. for any $\vec{w}$ in $\vec{W}$, we have

$$\sum_{l=0}^{2} \lambda_l G_l(\vec{u}) \leq \sum_{l=0}^{2} \lambda_l G_l(\vec{w}) \qquad (37)$$

Now, let $\vec{w}$ is also admissible and satisfies the Transversality condition, then (37) becomes $G_0(\vec{u}) \leq G_0(\vec{w})$, $\forall \vec{w} \in \vec{W}$, i.e. $\vec{u}$ is a CCBOCV continuous classical optimal control for the problem.

Conclusions

The Minty-Browder theorem can be used successfully to prove the existence and uniqueness solution of the continuous state vector of the couple nonlinear elliptic partial differential equations when the continuous classical boundary control vector is given. The existence theorem of a continuous classical boundary optimal control vector which is governing by the considered couple of nonlinear partial differential equation of elliptic type with equality and inequality constraints is proved. The existence and uniqueness solution of the couple of adjoint equations which is associated with the considered couple equations with equality and inequality constraints of the state are studied. The necessary conditions theorem so as the sufficient conditions theorem of optimality of a CNLEPDEs with equality and inequality constraints are proved via Kuhn- Tucker-Lagrange's Multipliers theorems.

References

[1] D. J. Braun, F. Petit, F. Huber, S. Haddadin, A. Albu-Schaffer, and S. Vijayakumar, "Robots Driven by Compliant Actuators: Optimal Control under Actuation Constraints", Ieee Transactions On Robotics, VOL. 29, NO.5, 2013.

[2] Y. Wang, X. Lin, S. Park, and N. Chang, "Optimal Control of a Grid-Connected Hybrid Electrical Energy Storage System for Homes", 978-3-9815370-0-0, Date 13, Edaa, 2013.

[3] F. Amini and M. Afshar, "Modified Predictive Optimal linear Control of Structures in Seismic Region", Iranian Journal of Science & Technology, Transaction B, Engineering, Vol.32, No.B2, 2008, pp: 91-106.

[4] L. Lessard, L and S. Lall, "Optimal Controller Synthesis for the Decentralized Two-Player Problem with output Feedback", American Control Conference, 2012, pp. 6314–6321.

[5] A. Di. Liddo, "Optimal Control and Treatment of Infectious Diseases", the case of huge treatment Costs, Mathematics doi: 10.3390/math4020021, 2016.

[6] M. Derakhshan, "Control Theory and Economic Policy Optimization: The Origin, Achievements and the

Fading Optimism from a Historical Standpoint", International Journal of Business and Development Studies Vol. 7, No. 1, 2015, pp 5-29.

[7] A. Yilmaz, I. Mahariq, and F. Yilmaz, "Numerical Solutions of Optimal Control Problems for microwave heating", International Journal of Advances in Science Engineering and Technology, ISSN: 2321-9009 Vol.4, Issue3, 2016.

[8] M. Chalak, "Optimal Control for a Dispersing Biological Agent", Journal of Agricultural and Resource Economics, 39(2):271–289, 2014, ISSN 1068-5502.

[9] J. Warga, "Optimal Control of Differential and Functional Equations", Academic Press: New York, and London, 1972.

[10] A. Orpel, "Optimal Control Problems with Higher order Constraints", Folia Mathematical, Vol.16, No.1, 2009, pp: 31-44.

[11] J. L. Lions, "Optimal Control of Systems Governed by partial Differential Equations", Springer-Verlag, New York, 1972.

[12] J. Al-Hawasy, "The Continuous Classical Optimal Control of a nonlinear Hyperbolic Equation (CCOCP)", Mustansiriyah Journal of Science., Vol.19, No.8, 2008, pp.96-110.

[13] I. Chryssoverghi and J. Al-Hawasy, "The Continuous Classical Optimal Control Problem of a semi linear Parabolic Equations (CCOCP)", Journal of Karbala University, Vol.8, No.3, 2010, pp: 57-70.

[14] D. Bors and S. Walczak, "Optimal control elliptic system with distributed and boundary controls", Nonlinear Analysis 63, 2005, e1367- e1376.

[15] J. Al-Hawasy, "The Continuous Classical Optimal Control of a Couple Nonlinear Hyperbolic Partial Differential Equations with equality and inequality Constraints", Iraqi Journal of Science, Vol.57, No.2C, 2016, pp: 1528-1538.

[16] J. Al-Hawasy and G. Kadhem, "The Continuous Classical Optimal Control of a coupled of nonlinear parabolic Equations", Al-Nahrain Journal of Sciences, Vol.19, No.1, 2016, pp: 173-186.

[17] J. Al-Hawasy and E. Al-Rawdanee, "The Continuous Classical Optimal Control of a Coupled of non-linear Elliptic Equations", Mathematical Theory and modeling, Vol.4, No.14, 2014, pp: 211-219.

[18] B. Vexler, "Finite Element Approximation of elliptic Dirichlet Optimal Control Problems", Numer, Funct, Anal., Optim., 28, 2007, 957–973.

[19] I. Chryssoverghi and A. Bacopoulos, "Approximation of Relaxed Nonlinear Parabolic Optimal Control Problems", Journal of Optimization Theory and Applications, Vol.77, No.1, 1993.

[20] A. H. Borzabadi, A. V. Kamyad, and M. H. G. Farahi, "Optimal control of the heat equation in an inhomogeneous body", J. Appl. Math., and Computing, Vol.15, No.1-2, 2004, pp.127-146.

[21] R. Temam, "Navier-Stokes Equations", North-Holand Publishing Company, 1977.

[22] A. Bacopoulos and I. Chryssoverghi, "Numerical solutions of partial differential equations by finite elements methods", Symeom publishing co, Athens, 2003.