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On Some Growth Relations of Composite Integral Functions Represented By Central Index

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Abstract
In the present paper, the growth rate of the central index of the composition of two integral functions in terms of the left and right factors are discussed.

Keywords: Integral Functions, Central Index, Maximum Term.

الخلاصة
في هذه الورقة البحثية تمت مناقشة معدل نمو الدليل المركزي لتكوين دالتين كليتين من حيث العوامل اليسرى واليمنى.

The introduction Basic definitions and Some notations

Suppose $g(z)$ is an integral function which is analytic in every finite region of the complex planeC, such that the function $g(z)$ can be written as [9]:

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (1.1)$$

Now let,

$$M(r,g) = \max_{|z|=r} g(z) \qquad (1.2)$$

Be a maximum absolute value of the function represented by (1.1) on the closed disc of the radius r, that is to say on $|z| = r$. It has been shown that the absolute value of an integral function $g(z)$ attains its maximum over a closed disc not at any interior point of the disc but on the circle serving as its boundary, also let,

$$\mu(r,g) = \max_{n\geq 0} |a_n|\, r^n \qquad (1.3)$$

denote the maximum term of the function represented by (1.1) on $|z| = r$, [9].
value of index n such that the specific term $|a_n|r^n$ becomes maximum is called central index of the maximum term and is indicated by $v(r,g)$. If more than one maximum term is obtained, the highest index is taken by agreement and call it the central index of the maximum term $\mu(r,g)$. If the series (1.1) is infinite series, then $v(r,g)$ is increasing function, unbounded function, and has only ordinary discontinuities. In another word the central index $v(r,g)$ is the greatest exponent n such that,

$$|a_n|r^n = \mu(r,g) \qquad (1.4)$$

The general theory of growth of composite integral functions is originated in the works of polya [7], he discussed the growth of composite integral functions in terms of maximum term and he obtained some results.
After that many results related to the growth of composite integral functions (e.g. Clunie [2],[3], Edrei and Fuchs[4], Mori[6], Yang[11]) have been obtained.
In this paper, the growth of composite integral functions in terms of central index is discussed. Also, in this paper some definitions, notations and lemmas are given which are used in the sequel.

Definition1.[1] Suppose $\mathbf{g(z)}$ is integral function, then the order $\boldsymbol{\rho_g}$ and the lower order $\boldsymbol{\lambda_g}$ of this function can be state as follows:

$$\lim_{r\to\infty}\sup\frac{\log^{[2]}M(r,g)}{\log r}$$

and

$$\lambda_g = \lim_{r\to\infty}\inf\frac{\log^{[2]}M(r,g)}{\log r} \qquad (1.5)$$

Definition2.[8] Let $\mathbf{K \geq 1}$ be an integer number. Then the generalized order $\boldsymbol{\rho_g^{[K]}}$ and generalized lower order $\boldsymbol{\lambda_g^{[K]}}$ of an integral function $\mathbf{g(z)}$ can state as:

$$\rho_g^{[K]} = \lim_{r\to\infty}\sup\frac{\log^{[K+1]}M(r,g)}{\log r}$$

and

$$\lambda_g^{[K+1]} = \lim_{r\to\infty}\inf\frac{\log^{[K+1]}M(r,g)}{\log r} \qquad (1.6)$$

Definition3. [5] Suppose $\mathbf{g(z)}$ is integral function, then the $(\mathbf{m},\mathbf{n})-$order, and the $(\mathbf{m},\mathbf{n})-$lower order of an integral function $\mathbf{g(z)}$ such that $\mathbf{m},\mathbf{n}$ are positive integers, $(\mathbf{m} \geq \mathbf{n})$ can state as follows:

$$\rho_g(m,n) = \lim_{r\to\infty}\sup\frac{\log^{[m+1]}M(r,g)}{\log^{[n]}r},$$

$$\lambda_g(m,n) = \lim_{r\to\infty}\inf\frac{\log^{[m+1]}M(r,g)}{\log^{[n]}r} \qquad (1.7)$$

Notation1. [10] $\log^{[s]}y = \log(\log^{[s-1]}y)$, for $s = 1,2,3,\dots$, also

$$\log^{[0]}y = y \qquad (1.8)$$

Now the following lemmas are used later.

Lemma1. [1] Let $g(z)$ is integral function of the $(m,n)-$ order $\rho_g(m,n)$, such that (m,n) are positive integers, $(m \geq n)$, and suppose that $v(r,g)$ be the central index of $g(z)$. Then

$$\rho_g(m,n) = \lim_{r\to\infty}\sup\frac{\log^{[m+1]}v(r,g)}{\log^{[n]}r} \qquad (1.9)$$

Lemma 2. [1] Let $g(z)$ is integral function of the $(m,n)-$ lower order $\lambda_g(m,n)$, such that (m,n) are positive
e integers, $(m \geq n)$, and suppose that $v(r,g)$ be the central index of $g(z)$. Then

$$\lambda_g(m,n) = \lim_{r\to\infty}\inf\frac{\log^{[m+1]}v(r,g)}{\log^{[n]}r} \qquad (1.10)$$

Theorems.

Now in the following theorems, the main results are given.

Theorem (2.1). Let $f(z)$ and $g(z)$ are integral functions such that fog is integral function, also let $0< \lambda_{f\circ g} \leq \rho_{f\circ g} < \infty$ and $0\leq \lambda_f \leq \rho_f < \infty$. Then for any arbitrary positive number A,

$$\frac{\lambda_{fog}}{A\rho_f} \leq \lim_{r\to\infty}\inf\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\lambda_{f\circ g}}{A\lambda_f} \leq \lim_{r\to\infty}\sup\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\rho_{f\circ g}}{A\lambda_f} \qquad (2.1)$$

Proof

Now from lemma 1 and lemma 2, it implies that for any $(\in > 0)$ and for a large value of r,

$$\log^{[m+1]}v(r,f\circ g) \geq (\lambda_{f\circ g}(m,n)-\in)\log^{[n]}r \qquad (2.2)$$

also

$$\log^{[m+1]}v(r^A,f) \leq (\rho_f(m,n)+\in)\log r^A$$

i.e.;

$$\log^{[m+1]}v(r^A,f) \leq A(\rho_f(m,n)+\in)\log^{[n]}r \qquad (2.3)$$

Gathering each of (2.2) and (2.3) in order to obtain the following for r is sufficiently large

$$\frac{\mathrm{Log}^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \geq \frac{\lambda_{f\circ g}(m,n)-\in}{A(\rho_f(m,n)+\in)}$$

Since $(\in > 0)$is arbitrary, implies that

$$\mathrm{Liminf}_{r\to\infty}\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \geq \frac{\lambda_{f\circ g}(m,n)}{A\rho_f(m,n)} \qquad (2.4)$$

Again, for a sequence $(r = r_L \to \infty)$as $L \to \infty$,

$$\mathrm{Log}^{[m+1]}v(r,f\circ g) \leq (\lambda_{f\circ g}(m,n)+\in)\log^{[n]}r \qquad (2.5)$$

and for a big number r,

$$\log^{[m+1]}v(r^A,f) \geq A(\lambda_f(m,n)-\in)\log^{[n]}r \qquad (2.6)$$

Combining (2.5) and (2.6) to get for a sequence $(r = r_L \to \infty)$ as $L \to \infty$,

$$\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\lambda_{f\circ g}(m,n)+\in}{A(\lambda_f(m,n)-\in)}$$

Since $(\in > 0)$is an arbitrary number, it implies that,

$$\lim_{r\to\infty}\inf\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\lambda_{f\circ g}(m,n)}{A\lambda_f(m,n)} \qquad (2.7)$$

Also, for a sequence $(r = r_L \to \infty)$ as $L \to \infty$

$$\log^{[m+1]}v(r^A, f) \leq A(\lambda_f(m,n)+\in)\log^{[n]}r \qquad (2.8)$$

Combining (2.2) and (2.8) to have for a sequence $(r = r_L \to \infty)$ as $L \to \infty$,

$$\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \geq \frac{\lambda_{f\circ g}(m,n)-\in}{A(\lambda_f(m,n)+\in)}$$

Since $(\in > 0)$is an arbitrary number, it implies that

$$\lim_{r\to\infty}\sup\frac{\log^{[m+1]}v(r, f\circ g)}{\log^{[m+1]}v(r^A, f)} \geq \frac{\lambda_{f\circ g}(m,n)}{A\lambda_f(m,n)} \qquad (2.9)$$

Also, for a sequence $(r = r_L \to \infty)$ as $L \to \infty$,

$$\log^{[m+1]}v(r, f\circ g) \leq \rho_{f\circ g}(m,n)+\in)\log^{[n]}r \qquad (2.10)$$

From (2.6) and (2.10), to get

$$\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\rho_{f\circ g}(m,n)+\in}{A(\lambda_f(m,n)-\in)}$$

Since $(\in > 0)$is an arbitrary, it implies that

$$\lim_{r\to\infty}\sup\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\rho_{f\circ g}(m,n)}{A\lambda_f(m,n)} \qquad (2.11)$$

Thus, the theorem is proved.

Example 2.1

Let $f(z)$ and $g(z)$ be integral functions, where $f(z) = e^{az}, a \neq 0, a \in R$.
and $g(z) = z$, $fog(z) = e^{az}$, $a \neq 0$.

$\rho_{f\circ g} = \lim_{r\to\infty}\sup\frac{\log\log M(r,f\circ g)}{\log r}$, $M(r, f\circ g) = e^{|a|r}$

$\rho_{fog} = \lim_{r\to\infty}\sup\frac{\frac{|a|}{|a|r}}{\frac{1}{r}} = 1,$ and $\rho_f = 1$.Then implies that $\lambda_{f\circ g} \leq \rho_{f\circ g} \leq 1$.

Now let $\lambda_{f\circ g} = 0.5, \lambda_f = 0.25, A = 2,$ and $m = 1$. Then according to Theorem 2.1 and by using simple calculation that,

$$0.25 \leq \lim_{r\to\infty}\inf\frac{\log^{[2]}v(r,f\circ g)}{\log^{[2]}v(r^A,f)} \leq 1 \leq \lim_{r\to\infty}\sup\frac{\log^{[2]}v(r,f\circ g)}{\log^{[2]}v(r^A,f)} \leq 2.$$

Theorem(2.2).

Let $f(z)$and $g(z)$ are integral functions such that fog is integral function, also let $0 < \lambda_{f\circ g} \leq \rho_{f\circ g} < \infty$ and$\rho_f < \infty$, and $0 < \lambda_g \leq \rho_g < \infty$ .Then for any arbitrary positive A,

$$\frac{\lambda_{f\circ g}}{A\rho_g} \leq \lim_{r\to\infty}\inf\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\lambda_{f\circ g}}{A\lambda_g} \leq \lim_{r\to\infty}\sup\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\rho_{f\circ g}}{A\lambda_g} \qquad (2.12)$$

Proof

Theorem 2.2 is proved as Theorem 2.1, therefore, any arbitrary positive number, then the proof is canceled.

Theorem (2.3).

Let $\mathbf{f(z)}$ and $\mathbf{g(z)}$**are** integral functions such that **fog** is integral function, also let $0 < \boldsymbol{\lambda_{f\circ g}} \leq \boldsymbol{\rho_{f\circ g}} < \infty$ and $0 < \boldsymbol{\rho_f} < \infty$. Then for any arbitrary positive **A,**

$$lim_{r\to\infty}\, inf\,\frac{log^{[m+1]}v(r,f\circ g)}{log^{[m+1]}v(r^A,f)} \leq \frac{\rho_{f\circ g}}{A\rho_f} \leq lim_{r\to\infty}\, sup\,\frac{log^{[m+1]}v(r,f\circ g)}{log^{[m+1]}v(r^A,f)} \qquad (2.13)$$

Proof

Now from lemma 1, it implies for any ($\epsilon > 0$) and arbitrary large r that,

$$log^{[m+1]}v(r^A, f) \geq A(\rho_f(m,n)-\in)log^{[n]}r \qquad (2.14)$$

Gathering each of (2.10) and (2.14) in order to obtain a sequence $(r = r_L \to \infty)$ as $L \to \infty$,

$$\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\rho_{f\circ g}(m,n)+\in}{A(\rho_f(m,n)-\in)}$$

Since $(\in > 0)$is arbitrary, it follows that

$$\lim_{r\to\infty}\inf\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq \frac{\rho_{f\circ g}(m,n)}{A\rho_f(m,n)} \qquad (2.15)$$

Also, for a sequence $(r = r_L \to \infty)$ as$L \to \infty$,

$$\log^{[m+1]}v(r, f\circ g) \geq (\rho_{f\circ g}(m,n)-\in)\log^{[n]}r \qquad (2.16)$$

Now from (2.3) and (2.16) to obtain,

$$\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \geq \frac{\rho_{f\circ g}(m,n)-\in}{A(\rho_f(m,n)+\in)}$$

Since $(\in > 0)$is an arbitrary, it implies that

$$\lim_{r\to\infty}\sup\frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \geq \frac{\rho_{f\circ g}(m,n)}{A\rho_f(m,n)} \qquad (2.17)$$

Thus from (2.15) and (2.17), Theorem 2.3 is proved.

Now, the next corollary is result of the Theorem 2.1 and Theorem 2.3.

Corollary (2.1)

Let $f(z)$and $g(z)$ are integral functions such that fog is integral function, also let $0 < \lambda_{f\circ g} \leq \rho_{f\circ g} < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for any arbitrary positive A,

$$\liminf_{r\to\infty} \frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq$$
$$\min\left\{\frac{\lambda_{f\circ g}(m,n)}{A\lambda_f(m,n)}, \frac{\rho_{f\circ g}(m,n)}{A\rho_f(m,n)}\right\} \leq$$
$$\max\left\{\frac{\lambda_{f\circ g}(m,n)}{A\lambda_f(m,n)}, \frac{\rho_{f\circ g}(m,n)}{A\rho_f(m,n)}\right\} \leq$$
$$\limsup_{r\to\infty} \frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)}$$

Combining Theorem 2.2 and Corollary 2.1 to get the next corollary

Corollary (2.2)

Suppose f(z) and g(z) be integral functions such that **fog** are integral function, also let $0< \lambda_{f\circ g} \leq \rho_{f\circ g} < \infty$ and $0< \lambda_f \leq \rho_f < \infty$ and$0< \lambda_g \leq \rho_g < \infty$. If A is arbitrary positive number

$$\liminf_{r\to\infty} \frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)} \leq$$
$$\min\left\{\frac{\lambda_{f\circ g}(m,n)}{A\lambda_g(m,n)}, \frac{\rho_{f\circ g}(m,n)}{A\rho_g(m,n)}\right\} \leq$$
$$\max\left\{\frac{\lambda_{fog}(m,n)}{A\lambda_g(m,n)}, \frac{\rho_{f\circ g}(m,n)}{A\rho_g(m,n)}\right\} \leq$$
$$\limsup_{r\to\infty} \frac{\log^{[m+1]}v(r,f\circ g)}{\log^{[m+1]}v(r^A,f)}$$

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