Research Article

Existence and Stability of Solutions for Bilinear Control System with Rieman-Leovel Initial Condition

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Abstract
In this paper, we shall study the existence of a new class called fractional Caputo type of order $0 < \alpha \leq 1$ sobolev type fractional order differential Equations motion in separable Banach spaces. The class of impulsive nonlinear fractional order bilinear control differential Equations with Riemann Leovel differential initial value studied and discussed also given the important results for the almost periodic mild solution to be sTable by using Menards function and granwal fractional inequality.

Keywords: fractional Caputo differential, Riemann Leovel differential, Menards function, granwal fractional inequality.

الخلاصة
في هذا البحث سوف تدرس الوجودية لصنف جديد يدعى معادلات كسرية سبولوفية في فضاء قابل للفصل بناخ ذات مشتقة من نوع كابوتو. ان صنف المعادلات النبضية غير الخطية ذات رتبة كسرية ودالة سيطرة ثنائية مع قيمة ابتدائية من نوع ريمان ليوفل درست ونوقشت وايضا اعطيت نتائج مهمة في دراسة الحل الشبه دوري المعلول ومتى يكون مستقر باستخدام دالة ميناردس ومتراجحة كرانول الكسرية.

Introduction

Sobolev type differential Equations have been investigated by many authors, as examples, [1] and [2]. They established the existence of solutions of nonlinear impulsive fractional integro- differential Equations of sobolev type with nonlocal condition. Zhou, Wang and Feckan [1][2] investigated a class of sobolev type semilinear fractional evolution systems in separable Banach space. The existence of the mild solution to the Problem in this section by using Laplace transform of the Caputo function integral to define and find the mild solution of problem (3.1)-(3.2)

Preliminaries

In this section, some basic definitions and properties concepts are for the almost periodic solutions.

Definition (2.1) [4]
A number $\tau \in$ □ is called an $\varepsilon -$ translation number of the function $\phi \in PC(R,X)$ if for all $\tau \in$ □ which satisfies $|t - t_i| > \varepsilon$, for all $i \in$ □.

Denote by $T(\phi,\varepsilon)$ the set of all $\varepsilon -$ translation number of ϕ.

Definition (2.2) [4]:
A function $\phi \in PC(R,X)$ is said to be piecewise almost periodic if the following conditions are satisfied:

(1) $\{t_i^j = t_{i+j} - t_i\}$, $i \in$ □, $j = 0, \pm 1, \pm 2, \ldots,$ are equipotentially almost periodic; that is, for any $\varepsilon > 0$, there exists a relatively dense set $\varepsilon -$ almost periodic that are common to all the sequences $\{t_i^j\}$.

(2) For any $\varepsilon > 0$, there exist a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to a same interval of continuity of: ϕ and $|t' - t''| < \delta$, then $\|\phi(t') - \phi(t'')\| < \varepsilon$.

(3) For every $\varepsilon > 0$, $T(\phi,\varepsilon)$ is a relatively dense set in □ .

We denote by $AP_T(R,X)$ the space of all piecewise almost periodic functions.

Theorem (2.3): "Banach Contraction Principle Theorem" [3]

Let *f* be a contraction on a complete metric space *X*. Then *f* has a unique fixed point $\overline{x} \in X$.

Definition (2.4), [5]:

The Mainardi's function is defined by:

$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma(-\alpha(\alpha+1)+1)}$, where $0 < \alpha < 1, z \in ₵$

Definition (2.5), [5]:

The Mittag - Leffler function is defined by:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

where $\alpha, \beta > 0, z \in ₵$.

Lemma (2.6) [1]:

$\beta > 0$, $a(t)$ is nonnegative function locally integrable on $0 \le t < T$ (some $T \le +\infty$) and g(t) is nonnegative, nondecreasing continuous function with $\|g(t)\| < N$ and u(t) is nonnegative and locally integrable on $0 \le t < T$ with:

$$u(t) \le a(t) + g(t)\int_o^t (t-s)^{\beta-1} u(s)ds$$

Then on $0 \le t < T$:

$$u(t) \le a(t) + \int_o^t [\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s)]ds$$

If a(t) be a nondecreasing function on [0,T], then:

$$u(t) \le a(t) E_\beta(g(t)\Gamma(\beta)t^\beta)$$

Where E_β is the Mittag Leffler function:

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)}.$$

Existence of the Fractional Order impulsive Bilinear control

Consider the Bilinear Control System with Rieman-Leovel initial condition:

$${}^cD^\alpha(Ex - g(t,x)) = (A+\Delta A)x(t) + (B+\Delta B)u_1(t) + (C+\Delta C)x(t)u_2(t) + F(t,x) \quad ...(3.1)$$

$${}^LD^{1-\alpha}x(t) = x_0 + h(x(t)),\ \ h: C(J:X) \to X,$$

$$J = [0,T]\ ,\ 0 < \alpha \le 1 \qquad ...(3.2)$$

$$\Delta x(t_i) = I_i(x(t_i)), i = 1,2,...,m.\ 0 < \alpha < 1.$$

1. $x(t) \in C([0,T]; L^2([0,T], X))$ equipped with the sup norm $\|x\|_C = (\sup_{t\in(0,T]} \|x(t)\|^2)^{\frac{1}{2}}$ such that X is a real separable Banach.
2. The operators E and $A + \Delta A$ are defined on domains contained in X, $D(E) \subset D(A+\Delta A)$ and ranges contained in a real separable Banach space Z such that E is bijective linear operator, E^{-1} is compact and $A + \Delta A$ is a closed linear operator.
3. The control function $u_1, u_2 \in L^2([0,T]; U)$, U is Banach space and the operator $B + \Delta B$ from U into Z is a bounded linear operator.
4. The functions:
$$\begin{cases} F: [0,T] \times X \to Z,\ C + \Delta C: X \to X, \\ h: C(J,X) \to X \\ g: [0,T] \times X \to Z \end{cases}$$
are continuous functions.
5. I_i and t_i satisfy suitable conditions that will be established later and the symbol $\Delta x(t_i)$,i=1...m represents the jump of the function x at (t_i), which is defined by $\Delta x(t) = x(t^+) - x(t^-)$.

Conclusion

Remark (3.1):

1. The operator $D(E) \subseteq X \to Z$ is a bijective linear operator then $E^{-1}: Z \to D(E) \subseteq X$ is a bijective linear.

2. E^{-1} is compact linear operator we obtain that E^{-1} is bounded.
3. E^{-1} is a bounded and L is a closed linear operator by (closed graph theorem)we obtain the boundedness of the linear operator $(A+\Delta A)E^{-1}: Z \to Z$.
4. The operator $(A+\Delta A)E^{-1}$ is bounded then $(A+\Delta A)E^{-1}$ is an infinitesimal generator of semi group $\{T(t)\}_{t\geq 0}$
5. Suppose that $\sup_{t\geq 0}\|T(t)\|_z = M < \infty$.

Hypotheses (3.2):
To establish our results, we introduce the following assumptions:

a. The semi group T(t), $t > 0$ which is generated by linear operator. $(A+\Delta A)E^{-1}$ is a strongly continuous and t is compact for any $t > 0$.
b. The functions $f:[0,T]\times X \to Z$, $g:[0,T]\times X \to Z$, and $(C+\Delta C):[0,T]\times X \to L_2(K,Z)$, $h(s):[0,T]\times X \to L_2(K,X)$ satisfy linear growth and lipschitz conditions this means that, for any $x, y \in X$, there exists positive constant $k_1, k_2 > 0$, $k_3, k_4 > 0$, $k_5, k_6 > 0$, and $k_7, k_8 > 0$ such that

i. $\|F(t,x)-F(t,y)\|_Z \leq k_1\|x-y\|_X$, $\|F(t,x)\|_Z \leq k_2(1+\|x\|_X)$

ii. $\|h(x(t))-h(y(t))\|_{L_2} \leq k_5\|x-y\|_X$, $\|h(x(t))\|_{L_2} \leq k_6(1+\|x\|_X)$

iii. $\|g(t,x(t))-g(t,y(t))\|_Z \leq k_7\|x-y\|_X$, $\|g(x(t))\|_Z \leq k_8(1+\|x\|_X)$

iv. $\|I_i(x)-I_i(y)\| \leq k_1\|x-y\|$

c. $$\|T_\alpha(t_2-t_1)-I\| < \min\left[\frac{\Gamma(\alpha)(w+m\|\Delta A(t)\|)\varepsilon}{4M_1C_1L_0}, \frac{\Gamma(\alpha)(1-e^{-(w+m\|\Delta A(t)\|)\alpha})\varepsilon}{4MC_1}\right]$$

and $\|E^{-1}\|k_7 < 1$.

Lemma (3.3):
If $(A+\Delta A)E^{-1}$ is a strongly continuous semi group T(t), $t\geq 0$ in z and λ^α belong to $p((A+\Delta A)E^{-1})$ is a resolvent set of $(A+\Delta A)E^{-1}$ then:

i- For any function $f \in([0,T]; Z)$,

$$E^{-1}\left(\lambda^\alpha I-(A+\Delta A)E^{-1}\right)f(t)\,dt = \int_0^\infty e^{-\lambda t}\left[\int_0^\infty T_\alpha(t-s)(t-s)^{\alpha-1}f(s)ds\right]dt \ldots(3.3)$$

For any $x_0 \in X$, we have

$$E^{-1}\left(\lambda^\alpha I-(A+\Delta A)E^{-1}\right)E\ x(0) = \int_0^\infty e^{-\lambda t}\tilde{T}_\alpha(t)E\ x(0)dt \ \ldots(3.4)$$

The operator $T_\alpha(t)$ and $\breve{T}_\alpha(t)$ are given by

$$\breve{T}_\alpha(t) = \int_0^\infty E^{-1}M_\alpha(r)\ T(t^\alpha r)dr \ \ldots(3.5)$$

$$T_\alpha(t) = \int_0^\infty \alpha E^{-1}M_\alpha(r)\ T(t^\alpha r)dr \ \ldots(3.6)$$

$M_\alpha(r)$ is Mainardis function in definition (2.4).

Proof:

$$\big(L\ x(t)\big)(\lambda)=\int_0^{\infty} e^{-\lambda t}\breve{T}_{\alpha}[E\ x(0)-g(0,x(0))]dt+\int_0^{\infty} e^{-\lambda t}E^{-1}g(t,x(t))dt+$$

$$\int_0^{\infty} e^{-\lambda t}[\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds]dt+\int_0^{\infty} e^{-\lambda t}[\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}$$

$$F(s,x(s))ds]dt+\int_0^{\infty} e^{-\lambda t}[\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}$$

$$(C+\Delta C)x(s)u_2(s)ds]dt+\sum_{t_i<t}\int_0^{\infty} e^{-\lambda t}[\int_0^t T_{\alpha}(t-s)$$

$$(t-s)^{\alpha-1}(t-t_i)I_i x(t_i)]dt$$

Taking Laplace inverse for both sides, we get

$$x(t)=\breve{T}_{\alpha}(t)[E\ x(0)-g(0,x(0))]+E^{-1}g(t,x(t))+$$

$$\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds+\int_0^t T_{\alpha}(t-s)$$

$$(t-s)^{\alpha-1}F(s,x(s))ds+\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}$$

$$(C+\Delta C)x(s)u_2(s)ds+\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}$$

$$(t-t_i)I_i x(t_i)$$

we have:

$$x(t)=\breve{T}_{\alpha}(t)E[x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}h\big(x(s)\big)ds]$$

$$-\breve{T}_{\alpha}(t)g(0,x(0))-E^{-1}g(t,x(t))+\int_0^t T_{\alpha}(t-s)$$

$$(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds+\int_0^t T_{\alpha-1}(t-s)ds$$

$$t-s)^{\alpha-1}F(s,x(s))ds+\sum_{t_i<t}T_{\alpha}(t-t_i)E\ I_i x(t_i)+$$

$$\int_0^t T_{\alpha-1}(t-s)(t-s)^{\alpha-1}(C+\Delta C)x(s)u_2(s)ds+$$

$$\sum_{t_i<t}T_{\alpha}(t-t_i)E\ I_i x(t_i)$$

...(3.7)

The Equation (3.7) is an Equation to the Equation (3.1)-(3.2) provided the integrals in (3.7) exists. From the properties of perturbation generator in [6] we presented the following lemma.

Lemma (3.4):

If $\{\breve{T}(t),t\geq 0\}$ is a strongly continuous semigroup by linear operator $(A+\Delta A)\ E^{-1}:\square\to\square$ then the operators $\{\breve{T}_{\alpha}(t),t\geq 0\}$ having the following properties:

i- For any fixed t ≥ 0, the operator $\breve{T}_{\alpha}(t)$ and $T_{\alpha}(t)$ are linear and bounded

i.e. for any $z\in\square$ there exists m > 1 such that $\|T_{\alpha}(t)z\|\leq\frac{c_1 m}{\Gamma(\alpha)}\|z\|_z$ and $\|\breve{T}_{\alpha}(t)z\|\leq c_1 m\|z\|_z$, where $\|E\|_x^{-1}=c_1$.

ii- The operators $\{\tilde{T}(t),t\geq 0\}$ and $\{\breve{T}_{\alpha}(t),t\geq 0\}$ are strongly continuous, which mean that $0\leq t_1<t_2\leq T$ for every $z\in\square$ and we have

$\|\breve{T}_{\alpha}(t_2)z-\breve{T}_{\alpha}(t_1)z\|_x\to 0$ and

$\|T_{\alpha}(t_2)z-T_{\alpha}(t_1)z\|_x\to 0$ if $t_2\to t_1$

iii- $T_{\alpha}(t)$ is compact operator in X for each t > 0.

Existences of Almost Periodic Solution For Fractional Impulsive Soblev Type:

In this section, the existence of the mild solution to the Problem formulation has been developed.

Definition (3.5):

An X-valued process $x(t)\in C([0,T];L^2([0,T],X))$ is almost periodic mild solution of bilinear control system motion in Equation (3.1) with Rieman-leuovel non local condition and implosive condition in Equation (3.2), for each control function $u_1(0) \in; L^2([0,T),u)$, it satisfies the integral Equation:

$$x(t)=\breve{T}_\alpha(t)E[x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}h(x(s))ds]-\breve{T}_\alpha(t)g(0,x(0))-E^{-1}g(t,x(t))+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}F(s,x(s))ds+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C+\Delta C)x(s)u_2(s)ds+\sum_{t_i<t}T_\alpha(t-t_i)EI_i(x(t_i)). \quad ...(3.8)$$

Lemma (3.6):

For any $x\in C([0,T];L^2([0,T],X))$, the operator $(\Psi x)(t)$ is continuous on [0,T] in the space $L^2([0,T],X))$

Proof:

Let $t_1,t_2\in[0,T]$ such that $t_1<t_2$ then for any $x\in C([0,T];L^2([0,T],X))$

From Equation (3.8) we have

$$\|\Psi x(t_2)-\Psi x(t_1)\|=\|\breve{T}_\alpha(t_2)E[x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t_2-s)^{-\alpha}h(x(s))ds]-\breve{T}_\alpha(t_2)g(0,x(0))+E^{-1}g(t_2,x(t_2))+\int_0^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}(B+\Delta B)u_1(s)ds+\int_0^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}F(s,x(s))ds+\int_0^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}(C+\Delta C)x(s)u_2(s)ds+\sum_{t_2<t_i}T_\alpha(t_2-t_i)E\ I_i x(t_i)-\breve{T}_\alpha(t_1)E\left[x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-s)^{-\alpha}h(x(s))ds\right]+\breve{T}_\alpha(t_1)g(0,x(0))-E^{-1}g(t_1,x(t_1))-\int_0^{t_1}T_\alpha(t_1-s)(t_1-s)^{\alpha-1}(B+\Delta B)u_1(s)ds-\int_0^{t_1}T_\alpha(t_1-s)(t_1-s)^{\alpha-1}F(s,x(s))ds-\int_0^{t_1}T_\alpha(t_1-s)(t_1-s)^{\alpha-1}(C+\Delta C)x(s)u_2(s)ds-\sum_{t_1<t_i}T_\alpha(t_1-t_i)E\ I_i x(t_i)\Big\|_x$$

$$=\Big\|(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1))Ex_0+(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1))E\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-s)^{-\alpha}h(x(s))ds+\breve{T}_\alpha(t_2)E\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}[(t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}]h(x(s))ds+\breve{T}_\alpha(t_2)E\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}h(x(s))ds+[\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)]g(0,x(0))+E^{-1}[g(t_2,x(t_2))-g(t_1,x(t_1))]+\int_0^{t_1}[T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-T_\alpha(t_1-s)(t_1-s)^{\alpha-1}](B+\Delta B)u_1(s)ds+\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}$$

$$(B+\Delta B)u_1(s)ds+$$
$$\int_0^{t_1}\Big[T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-T_\alpha(t_1-s)(t_1-s)^{\alpha-1}\Big]$$
$$F(s,x(s))ds+$$
$$\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}F(s,x(s))ds+$$
$$\int_0^{t_1}[T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-T_\alpha(t_1-s)(t_1-s)^{\alpha-1}]$$
$$(C+\Delta C)x(s)u_2(s)ds+$$
$$\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}(C+\Delta C)x(s)u_2(s)ds+$$
$$\sum_{t_i<t_1}[T_\alpha(t_2-t_i)-T_\alpha(t_1-t_i)]EI_i(x(t_i))$$
$$+\sum_{t_1<t_i<t_2}T_\alpha(t_2-t_i)EI_i(x(t_i))\Bigg\|.$$
$$\le\left\|\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)Ex_0\right\|+$$
$$\left\|\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)E\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-s)^{-\alpha}h(x(s))ds\right\|+$$
$$\left\|\breve{T}_\alpha(t_2)E\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}h(x(s))ds\right\|+$$
$$\left\|\breve{T}_\alpha(t_2)E\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}h(x(s))ds\right\|+$$
$$\left\|\breve{T}_\alpha(t_2)E\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}h(x(s))ds\right\|+$$
$$\left\|E^{-1}[g(t_2,x(t_2))-g(t_1,x(t_1))\right\|+\Bigg\|\int_0^{t_1}[T_\alpha(t_2-s)$$
$$(t_2-s)^{\alpha-1}-T_\alpha(t_1-s)(t_1-s)^{\alpha-1}](B+\Delta B)u_1(s)ds\Big\|+$$
$$\left\|\int_{t_1}^{t_2}T_\alpha(t_1-s)(t_1-s)^{\alpha-1}(B+\Delta B)u_1(s)ds\right\|+$$
$$\Bigg\|\int_0^{t_1}[T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-T_\alpha(t_1-s)(t_1-s)^{\alpha-1}]$$
$$F(s,x(s))ds\Big\|+\left\|\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}F(s,x(s))ds\right\|+$$
$$\Bigg\|\int_0^{t_1}[T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-T_\alpha(t_1-s)(t_1-s)^{\alpha-1}]$$
$$(c+\Delta c)x(s)u_2(s)ds\Big\|+$$
$$\left\|\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}(c+\Delta c)x(s)u_2(s)ds\right\|+$$
$$\sum_{t_i<t_1}\left\|[T_\alpha(t_2-t_i)-T_\alpha(t_1-t_i)]EI_i(x(t_i))\right\|+$$
$$\sum_{t_1<t_i<t_2}\left\|[T_\alpha(t_2-t_i)EI_i(x(t_i))\right\|.$$

Moreover,

$$\Bigg\|(\breve{T}_\alpha(t_2)E[x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}(t_2-s)^{-\alpha}h(x(s))ds-$$
$$\breve{T}_\alpha(t_2)g(0,x(0))+E^{-1}g(0,x(0))+$$
$$E^{-1}g(t_2,x(t_2))-(\breve{T}_\alpha(t_1)E[x_0+\frac{1}{\Gamma(1-\alpha)}$$
$$\int_0^{t_1}(t_1-s)^{-\alpha}h(x(s))ds-\breve{T}_\alpha(t_1)g(0,x(0))-$$
$$E^{-1}g(t_1,x(t_1))\ \Big\|$$
$$\le\left\|(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1))Ex_0\right\|_x+$$
$$\left\|(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1))E\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-s)^{-\alpha}h(x(s))ds\right\|+$$
$$\left\|(\breve{T}_\alpha(t_2))E\frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}(t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}]h(x(s))ds\right\|$$
$$+\left\|(\breve{T}_\alpha(t_2))E\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}h(x(s))ds\right\|$$
$$+\left\|(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1))g(0,x(0))\right\|+$$
$$\left\|E^{-1}[g(t_2,x(t_2))-g(t_1,x(t_1))]\right\|.$$

$$\leq \|\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\|\|E\|\|x_0\|+\|\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\|$$
$$\frac{\|E\|T^{-2\alpha+2}k_6}{(\Gamma(1-\alpha))^2}(1+\|x\|_c)+\frac{c_1^2m^2\|E\|k_6T}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}$$
$$(\int_0^{t_1}[(t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}]^2ds(1+\|x\|_c)+$$
$$\frac{c_1^2m^2\|E\|}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}(\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}ds)(\int_{t_1}^{t_2}\|h(x(s))\|ds)+$$
$$\|\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\|.\|g(0,x(0))\|+$$
$$G_1^2\|g(t_2,x(t_2))-g(t_1,x(t_1))\|$$

Hence, by using strong continuity of $\breve{T}_\alpha(t)$ and $T_\alpha(t)$ in lemma (3.4),(ii) and lebesgues dominated convergence theorem, we conclude that the right-hand side of the above inequalities tends to zero as $t_2 \to t_1$, thus, we conclude $(\psi x)(t)$ is continuous from the right in [0,T]. A similar argument shows that it is also continuous from the left in [0,T] thus $(\psi x)(t)$ is continuous on [0,T] in the $L^2([0,T],X)$.

$$\left\|\int_0^{t_1}[T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-T_\alpha(t_1-s)(t_1-s)^{\alpha-1}]\right.$$
$$\left.(B+\Delta B)u_1(s)ds\right\|$$
$$\leq \left\|\int_0^{t_1}[T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-T_\alpha(t_1-s)\right.$$
$$\left.(t_1-s)^{\alpha-1}]ds\right\|\|B+\Delta B\|\|u_1(s)\|$$
$$\leq \int_0^{t_1}\|T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-(t_2-s)^{\alpha-1}T_\alpha(t_1-s)$$
$$+(t_2-s)^{\alpha-1}T_\alpha(t_1-s)-T_\alpha(t_1-s)\|$$
$$(t_1-s)^{\alpha-1}\|\|B+\Delta B\|.\|u_1(s)\|ds$$
$$\leq \int_0^{t_1}\|(t_2-s)^{\alpha-1}[T_\alpha(t_2-s)-T_\alpha(t_1-s)]+$$
$$T_\alpha(t_1-s)[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\|$$
$$\|B+\Delta B\|.\|u_1(s)\|ds$$
$$\leq \int_0^{t_1}[(t_2-s)^{\alpha-1}\|T_\alpha(t_2-s)-T_\alpha(t_1-s)\|+$$
$$\|T_\alpha(t_1-s)\|.\|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\|].$$
$$\|B+\Delta B\|.\|u_1(s)\|ds$$
$$\leq [\int_0^{t_1}T_\alpha(t_2-s)^{\alpha-1}.\|T_\alpha(t_2-s)-T_\alpha(t_1-s)\|+$$
$$\int_0^{t_1}\|T_\alpha(t_1-s)\|.\|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\|].$$
$$\|B+\Delta B\|.\|u_1(s)\|ds$$
$$\leq [\frac{(t_2-s)^\alpha}{\alpha}\|T_\alpha(t_2-s)-T_\alpha(t_1-s)\|+$$
$$t_1.\frac{c_1m}{\Gamma(\alpha)}.\|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\|.F_1.k_1$$
$$<\varepsilon$$

from

$$|t_2-s|>0,\ |t_2-s|^{\alpha-1}>0,\ -|t_2-s|^{\alpha-1}<0,$$

then

$$(|t_1-s|^{\alpha-1}-|t_2-s|^{\alpha-1})<|t_1-s|^{\alpha-1}<\frac{1}{n}<\varepsilon$$

then

$$\left||t_1-s|^{\alpha-1}-|t_2-s|^{\alpha-1}\right|<\varepsilon.$$

And

$$\left\|\int_0^{t_1}[T_\alpha(t_2-s)(t_2-s)^{\alpha-1}-\right.$$
$$\left.T_\alpha(t_1-s)(t_1-s)^{\alpha-1}]F(s,x(s))ds\right\|$$

$$\leq \int_0^{t_1} \left\| T_\alpha(t_2-s)(t_2-s)^{\alpha-1} - T_\alpha(t_1-s)(t_1-s)^{\alpha-1} \right\| \left\| F(s,x(s)) \right\| ds$$

$$\leq \int_0^{t_1} \left\| T_\alpha(t_2-s)(t_2-s)^{\alpha-1} - (t_2-s)^{\alpha-1} T_\alpha(t_1-s) + (t_2-s)^{\alpha-1} T_\alpha(t_1-s) - T_\alpha(t_1-s) \right\| (t_1-s)^{\alpha-1} \| \; \left\| F(s,x(s)) \right\| ds$$

$$\leq \int_0^{t_1} \left\| (t_2-s)^{\alpha-1} [T_\alpha(t_2-s) - T_\alpha(t_1-s)] + T_\alpha(t_1-s) [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \right\| \left\| F(s,x(s)) \right\| ds$$

$$\leq \int_0^{t_1} [(t_2-s)^{\alpha-1}. \left\| T_\alpha(t_2-s) - T_\alpha(t_1-s) \right\| + \left\| T_\alpha(t_1-s) \right\| \left\| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right\|] \left\| F(s,x(s)) \right\| ds$$

$$\leq [\int_0^{t_1} (t_2-s)^{\alpha-1}. \left\| T_\alpha(t_2-s) - T_\alpha(t_1-s) \right\| + \int_0^{t_1} \left\| T_\alpha(t_1-s) \right\| . \left\| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right\|]. \left\| F(s,x(s)) \right\| ds$$

$$\leq [\frac{(t_2-s)^\alpha}{\alpha}. \left\| T_\alpha(t_2-s) - T_\alpha(t_1-s) \right\| + t_1. \frac{c_1 m}{\Gamma(\alpha)}. \left\| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right\|]. k_2(1+\|x\|)$$

$$< \varepsilon.$$

from

$$|t_2-s| > 0, \; |t_2-s|^{\alpha-1} > 0, -|t_2-s|^{\alpha-1} < 0,$$

then

$$(|t_1-s|^{\alpha-1} - |t_2-s|^{\alpha-1}) < |t_1-s|^{\alpha-1} < \frac{1}{n} < \varepsilon$$

then

$$\left| |t_1-s|^{\alpha-1} - |t_2-s|^{\alpha-1} \right| < \varepsilon .$$

And

$$\left\| \int_0^{t_1} [T_\alpha(t_2-s)(t_2-s)^{\alpha-1} - T_\alpha(t_1-s)(t_1-s)^{\alpha-1}] (C+\Delta C) x(s) u_2(s) ds \right\|$$

$$\leq \int_0^{t_1} \left\| T_\alpha(t_2-s)(t_2-s)^{\alpha-1} - T_\alpha(t_1-s)(t_1-s)^{\alpha-1} \right\| \left\| C+\Delta C \right\| . \left\| x(s) \right\| . \left\| u_2(s) \right\| ds$$

$$\leq \int_0^{t_1} \left\| T_\alpha(t_2-s)(t_2-s)^{\alpha-1} - (t_2-s)^{\alpha-1} T_\alpha(t_1-s) + (t_2-s)^{\alpha-1} T_\alpha(t_1-s) - T_\alpha(t_1-s) \right\| (t_1-s)^{\alpha-1} \| \; \left\| C+\Delta C \right\| . \left\| x(s) \right\| . \left\| u_2(s) \right\| ds$$

$$\leq \int_0^{t_1} \left\| (t_2-s)^{\alpha-1} [T_\alpha(t_2-s) - T_\alpha(t_1-s)] + T_\alpha(t_1-s) [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] \right\| \left\| C+\Delta C \right\| . \left\| x(s) \right\| . \left\| u_2(s) \right\| ds$$

$$\leq \int_0^{t_1} \left\| [(t_2-s)^{\alpha-1}. \left\| T_\alpha(t_2-s) - T_\alpha(t_1-s) \right\| + \left\| T_\alpha(t_1-s) \right\| . \left\| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right\| \right\|] \left\| C+\Delta C \right\| . \left\| x(s) \right\| . \left\| u_2(s) \right\| ds$$

$$\leq [\int_0^{t_1} (t_2-s)^{\alpha-1}. \left\| T_\alpha(t_2-s) - T_\alpha(t_1-s) \right\| + \int_0^{t_1} \left\| T_\alpha(t_1-s) \right\| . \left\| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right\|]. \left\| C+\Delta C \right\| . \left\| x(s) \right\| . \left\| u_2(s) \right\| ds$$

$$\leq [\frac{(t_2-s)^\alpha}{\alpha}. \left\| T_\alpha(t_2-s) - T_\alpha(t_1-s) \right\| + t_1. \frac{c_1 m}{\Gamma(\alpha)}. \left\| (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right\|]. F_2. L_0. k_2$$

$$< \varepsilon.$$

from

$$|t_2-s| > 0, \; |t_2-s|^{\alpha-1} > 0, \; -|t_2-s|^{\alpha-1} < 0,$$

then

$$(|t_1-s|^{\alpha-1}-|t_2-s|^{\alpha-1})<|t_1-s|^{\alpha-1}<\frac{1}{n}<\varepsilon$$

then

$$\left||t_1-s|^{\alpha-1}-|t_2-s|^{\alpha-1}\right|<\varepsilon.$$

And

$$\left\|\sum_{t_i<t_1}[T_\alpha(t_2-t_i)-T_\alpha(t_1-t_i)]E\,I_i(x(t_i))\right\|=$$

$$\left\|\sum_{t_i<t_1}[T_\alpha(t_2-t_1)T_\alpha(t_1-t_i)-T_\alpha(t_1-t_i)]E\,I_i(x(t_i))\right\|$$

$$=\left\|\sum_{t_i<t_1}[T_\alpha(t_2-t_1)-I]T_\alpha(t_1-t_i).E\,I_i(x(t_i))\right\|$$

$$\leq\sum_{t_i<t_1}\|T_\alpha(t_2-t_1)-I\|.\|T_\alpha(t_1-t_i)\|.\|E\,I_i(x(t_i))\|$$

By condition (c) then

$$\leq\sum_{t_i<t_1}\frac{\Gamma(\alpha)(1-e^{-(w+m\|\Delta A(t)\|)\alpha})\varepsilon}{4MC_1}.\frac{C_1M}{\Gamma(\alpha)}.C<\varepsilon$$

Moreover,

$$\left\|\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}(B+\Delta B)u_1(s)ds\right\|$$

$$\leq\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|T_\alpha(t_2-s)\|.\|B+\Delta B\|.\|u_1(s)\|ds$$

$$\leq\frac{(t_2-t_1)^\alpha}{\alpha}.\frac{C_1M}{\Gamma(\alpha)}.F_1.k_1<\varepsilon.$$

And

$$\left\|\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}F(s,x(s))ds\right\|$$

$$\leq\frac{(t_2-t_1)^\alpha}{\alpha}.\|T_\alpha(t_2-s)\|.\|F(s,x(s))\|ds$$

$$\leq\frac{(t_2-t_1)^\alpha}{\alpha}.\frac{C_1M}{\Gamma(\alpha)}.k_2(1+\|x\|)$$

$$\leq\frac{(t_2-t_1)^\alpha}{\alpha}.\frac{C_1M}{\Gamma(\alpha)}.k_2+k_2.L_0<\varepsilon$$

And

$$\left\|\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}(c+\Delta c)x(s)u_2(s)ds\right\|$$

$$\leq\frac{(t_2-t_1)^\alpha}{\alpha}.\|T_\alpha(t_2-s)\|.\|c+\Delta c\|.\|x(s)\|.\|u_2(s)\|$$

$$\leq\frac{(t_2-t_1)^\alpha}{\alpha}.\frac{C_1M}{\Gamma(\alpha)}.F_2.L_0.k_2<\varepsilon$$

Similarly,

$$\left\|\sum_{t_1<t_i<t_2}[T_\alpha(t_2-t_i)]EI_i(x(t_i))\right\|$$

$$\leq\sum_{t_1<t_i<t_2}\|T_\alpha(t_2-t_i)\|.\|EI_i(x(t_i))\|$$

$$\leq\frac{c_1m}{\Gamma(\alpha)(1-e^{-(w+m\|\Delta A\|)})\alpha}.\|E\|.C<\varepsilon$$

Lemma (3.7):
If the operator Ψ is defined in Equation (3.8) then

$$\psi(C[0,T];L^2([0,T],X)\subset C([0,T];L^2([0,T],X))$$

Proof:
From lemma (3.6), for any $L^2([0,T],X)\subset C([0,T];L^2([0,T],X))$
The operator $(\psi_x)(t)$ is continuous on [0,T] in the space $L^2([0,T],X)$.
To prove that for $x\in C([0,T];L^2([0,T],X))$ implies $\|\psi x(t)\|_x<\infty$.
Let $x\in C([0,T];L^2([0,T],X))$. From Equation (3.8) we have:

$$\|\psi x(t)\|_x=\Big\|\breve{T}_\alpha(t)E[x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}h(x(s))ds$$
$$-\breve{T}_\alpha(t)g(0,x(0))+E^{-1}g(t,x(t))\Big\|$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds +$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}F(s,x(s))ds +$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C+\Delta C)u_2(s)ds +$$

$$\left.\sum_{t_i<t} T_\alpha(t-t_i)EI_i(x(t_i))\right\|_x$$

From the equality above, we have

$$\|\psi x(t)\|_x \le \left\|\breve{T}_\alpha(t)E[x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}h(x(s))ds\right\|_x$$

$$+\left\|\breve{T}_\alpha(t)g(0,x(0))+E^{-1}g(t,x(t))\right\|_x +$$

$$\left\|\breve{T}_\alpha(t)g(0,x(0))+E^{-1}g(t,x(t))\right\|_x +$$

$$\left\|\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds\right\|_x +$$

$$\left\|\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}F(s,x(s))ds\right\|_x +$$

$$\left\|\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C+\Delta C)u_2(s)ds\right\|_x +$$

$$\left\|\sum_{t_i<t} T_\alpha(t-t_i)EI_i(x(t_i))\right\|_x$$

We have,

$$\|\psi x(t)\|_x \le \frac{c_1^2 m^2\|E\|^2}{(\Gamma(\alpha))^2}[\|x_0\|_x + \frac{T^{2-2\alpha}k_6}{(\Gamma(1-\alpha))(1-\alpha))^2}$$

$$(1+\|x\|_c)+\frac{c_1^2m^2k_8}{(\Gamma(\alpha))^2}(1+\|x(0)\|_x)+$$

$$c_1k_8(1+\|x\|_c+\frac{c_1^2m^2L_B^2T^2}{|2\alpha-1|}\|u_1(s)\|+$$

$$\frac{c_1^2m^2T^{2\alpha}k_2}{|2\alpha-1|}(1+\|x\|_c)+\frac{c_1^2m^2T^{2\alpha-1}}{|2\alpha-1|}$$

$$\|C+\Delta C\|.\|x\|.\|u_2(s)\|+$$

$$\sum_{t_i<t}\|T_\alpha(t-t_i)\|.E.\|I_i(x(t_i))\|$$

$$\le \frac{c_1^2 m^2\|E\|^2}{(\Gamma(\alpha))^2}[L_0 + \frac{T^{2-2\alpha}k_6}{(\Gamma(1-\alpha))(1-\alpha))^2}$$

$$(1+L_0)+\frac{c_1^2m^2k_8}{(\Gamma(\alpha))^2}(1+L_0)+$$

$$c_1k_8(1+L_0+\frac{c_1^2m^2L_B^2T^2}{|2\alpha-1|}k_1+$$

$$\frac{c_1^2m^2T^{2\alpha}k_2}{|2\alpha-1|}F_2L_0k_2(1+L_0)+$$

$$\frac{c_1^2m^2T^{2\alpha-1}}{|2\alpha-1|}\|C+\Delta C\|.\|x\|.\|u_2(s)\|+$$

$$\sum_{t_i<t}\|T_\alpha(t-t_i)\|.E.\|I_i(x(t_i))\|$$

$$\le \frac{c_1^2 m^2\|E\|^2}{(\Gamma(\alpha))^2}\left[L_0 + \frac{T^{2-2\alpha}k_6}{(\Gamma(1-\alpha))(1-\alpha))^2}\right]$$

$$(1+L_0)+\frac{c_1^2m^2k_8}{(\Gamma(\alpha))^2}(1+L_0)+$$

$$c_1k_8(1+L_0+\frac{c_1^2m^2L_B^2T^2}{|2\alpha-1|}k_1+$$

$$\frac{c_1^2m^2T^{2\alpha}k_2}{|2\alpha-1|}F_2L_0k_2(1+L_0)+$$

$$\frac{c_1m}{\Gamma(\alpha)}.\frac{1}{(1-e^{-(w+m\|\Delta A\|)})\alpha}.\ c$$

$$< L_0$$

Hence, the last inequality implies that $\|\Psi x(t)\| < \infty$. Moreover,

for $x \in B_{\frac{1}{n}}C([0,T];L^2([0,T],X))$.

Lemma(3.8): Let $w \ge 0,\ m \ge 1$ and $\|\Delta A(t)\|$ is bounded linear operator then

$$\sum_{t_i<t} e^{-(w+M\|\Delta A(t)\|)} = \frac{M}{1-e^{-(w+M\|\Delta A(t)\|)\alpha}}.$$

proof :
In order to estimate the second term on the right-hand of the above formala we assume $t_j \le t < t_{j+1},\ j \in \mathbb{Z}$ so

$$t - t_i = (t - t_j) + (t_j - t_i) \ge (j - i)\alpha$$

and

$$\sum_{t_i < t} M e^{-(w + M\|\Delta A(t)\|)\ (t - t_i)} C \ \le$$

$$\sum_{-\infty < i \le j} M e^{-(w + M\|\Delta A(t)\|)\ (j - i)\alpha}\ C.$$

$$= \sum_{0 \le k = j - i < \infty} M e^{-(w + M\|\Delta A(t)\|)\ k\alpha}\ C$$

$$= \frac{M}{1 - e^{-(w + M\|\Delta A(t)\|)\alpha}}\ C.$$

Notation (3.9):
For any $n \in N$, let

$B_{\frac{1}{n}} = \{x \in AP_T(R, X) : \|x\| \le L_0,$

$\|x(t+\tau) - x(t)\| \le \frac{1}{n},$ and

$|t - t_i| > \frac{1}{n}, i \in Z, \tau \in \Omega_{\frac{1}{n}, f, I_i, k_0, T}\},$

$B = \bigcap_{n \in N} B_{\frac{1}{n}}$. The set $B_{\frac{1}{n}}$ .is relatively compact and so is B.Obviously, B is a nonempty closed convex set.

Lemma (3.10):
For any $\|x(t+\tau) - x(t)\| \le \frac{1}{n} (= \delta)$.
$x \in B \subset B_{\frac{1}{n}} (\forall n \in \mathbb{N})$ and $t \in \Omega_{\frac{1}{n}, f, I_i, K_0, T}$, If $|t - t_i| > \frac{1}{n}$ then

i. $\|F(t, x(t+\tau)) - F(t, x(t))\| \le \frac{k}{n}$

ii. $\|I_{i+q}(x(t_{i+q})) - I_i(x(t_i))\| \le \frac{k_1}{n}$ for $q \in \mathbb{Z}$ with k $|t_{i+q} - t_i - \tau| < \frac{1}{n}$

then $\|\Psi x(t+\tau) - \Psi x(t)\| \le \frac{1}{n}$

Proof:

$$(\Psi x(t+\tau)) - (\Psi x)(t) = \breve{T}_\alpha(t+\tau)\ E \left[x_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^{t+\tau} (t+\tau-s)^{-\alpha}\ h(x(s))\ ds\right] -$$

$$\breve{T}_\alpha(t+\tau)\ g(0, x(0)) + E^{-1}\ g(t+\tau, x(t+\tau))$$

$$+ \int_0^{t+\tau} T_\alpha(t+\tau-s)(t+\tau-s)^{\alpha-1}(B + \Delta B)u_1(s)\ ds +$$

$$\int_0^{t+\tau} T_\alpha(t+\tau-s)(t+\tau-s)^{\alpha-1}\ F(s, x(s))\ ds +$$

$$\int_0^{t+\tau} T_\alpha(t+\tau-s)(t+\tau-s)^{\alpha-1}\ (C + \Delta C)$$

$$x(s)\ u_2(s) ds + \breve{T}_\alpha(t) E \left[x_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\ h(x(s))\ ds\right] +$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} F(s, x(s))\ ds -$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C + \Delta C)\ x(s)\ u_2(s) ds -$$

$$\breve{T}_\alpha(t)\ g(0,x(0)) - E^{-1}\ g(t,x(t)) -$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ (B+\Delta B)u_1(s)\ ds -$$

$$\sum_{t_i<t} T_\alpha(t-t_i)\ E\ I_i(x(t_i))$$

$$= \breve{T}_\alpha(t+\tau)E\left[x_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\ h(x(s+\tau))\ ds\right] - \breve{T}_\alpha(t+\tau)\ g(0,x(0)) +$$

$$E^{1} g(t+\tau, x(t+\tau)) + \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ (B+\Delta B)u_1(s+\tau)\ ds +$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ F(s+\tau, x(s+\tau))\ ds$$

$$+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ (C+\Delta C)$$

$$x(s+\tau)\ u_2(s+\tau)ds + \sum_{t_i<t} T_\alpha(t-t_i)$$

$$E\ I_{i+q}(x(t_{i+q})) - \breve{T}_\alpha(t)E$$

$$\left[x_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} h(x(s))\ ds\right]$$

$$+\breve{T}_\alpha(t)\ g(0,x(0)) - E^{-1}\ g(t,x(t)) -$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ (B+\Delta B)u_1(s)\ ds -$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ F(s,x(s))\ ds -$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ (C+\Delta C)\ x(s)\ u_2(s)ds$$

$$-\sum_{t_i<t} T_\alpha(t-t_i)\ E\ I_i(x(t_i))$$

$$= \left[\breve{T}_\alpha(t+\tau) - \breve{T}_\alpha(t)\right]$$

$$E\left[x_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} h(x(s))\ ds\right]$$

$$+\breve{T}_\alpha(t+\tau)$$

$$(B+\Delta B)ds + \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}$$

$$\left[F(s+\tau, x(s+\tau)) - F(s,x(s))\right]ds +$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\left[x(s+\tau)\ u_2(s+\tau) - x(s)\ u_2(s)\right]$$

$$(C+\Delta C)ds + \sum_{t_i<t} T_\alpha(t-t_i)$$

$$E\ \left[I_{i+q}(x(t_{i+q})) - I_i(x(t_i))\right]$$

$$\left\|\breve{T}_\alpha(t+\tau) - \breve{T}_\alpha(t)\right\|\ \|E\|\ \|x_0\| + \|E\|$$

$$\left\|\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} h(x(s))\ ds\right\| + \left\|\breve{T}_\alpha(t+\tau)\right\|$$

$$\|E\|\left\|\frac{1}{\Gamma(1-\alpha)}\int_t^{t+\tau} (t+\tau-s)^{-\alpha} h(x(s+\tau))\ ds\right\|$$

Then for $t+\tau \to t$ then $\tau \to 0$, we get

$$\le C_1 m\ \|E\|\ L_0 + \|E\|\ \frac{1}{\Gamma(1-\alpha)}$$

$$\int_0^t (t-s)^{-\alpha}\ \|h(x(s))\| ds + C_1 m\ \|E\|$$

$$\frac{1}{\Gamma(1-\alpha)}\int_t^{t+\tau} (t+\tau-s)^{-\alpha}\ \|h(x(s+\tau))\| ds$$

$$\le \varepsilon_1\ \|E\|\ L_0 + \|E\|\ \frac{T^{1-\alpha}}{1-\alpha} k_6 (1+L_0) +$$

$$C_1 m\ \|E\|\ \frac{T^{1-\alpha}}{1-\alpha} k_6 (1+L_0)$$

Suppose $k_6 = \dfrac{(1-\alpha)\ \varepsilon_1}{\|E\| T^{1-\alpha}(1+L_0)}$

$$\le \varepsilon_1 \|E\| L_0 + \|E\| \frac{T^{1-\alpha}}{1-\alpha} \frac{(1-\alpha)\ \varepsilon_1}{\|E\| T^{1-\alpha}(1+L_0)}$$

$$(1+L_0) + C_1 m \|E\| \frac{T^{1-\alpha}}{1-\alpha}$$

$$\frac{(1-\alpha)\ \varepsilon_1}{\|E\| T^{1-\alpha}(1+L_0)}(1+L_0)$$

$$\le \varepsilon_1 \|E\| L_0 + \varepsilon_1 + C_1 m\ \varepsilon_1$$

$$\le (\|E\| L_0 + 1 + C_1 m)\ \varepsilon_1$$

Let $\varepsilon_1 = \dfrac{1}{n(\|E\| L_0 + 1 + C_1 m)}$

$$\|\breve{T}_\alpha(t+\tau) - \breve{T}_\alpha(t)\| \|g(0, x(0))\|$$

$$\le \varepsilon\ k_8(1+\|x\|) \qquad \le \varepsilon\ (k_8 + L_0\ k_8)$$

$$\|E^{-1}\| \|g(t+\tau, x(t+\tau)) - g(t, x(t))\|$$

$$\le L^* \|t - \tau - t\| + L^{**} \|x(t+\tau) - x(t)\|$$

$$\le L^* \|\tau\| + \frac{L^{**}}{n}$$

$$\left\| \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} \right.$$

$$\left. [u_1(s+\tau) - u_1(s)]\ (B + \Delta B) \right\|$$

$$\le \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\|$$

$$\|u_1(s+\tau) - u_1(s)\| \|(B + \Delta B)\|$$

$$\le \frac{(t-s)^\alpha}{\alpha} \frac{c_1 m}{\Gamma(\alpha)} \frac{k_2}{n} F_1$$

$$\left\| \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} \right.$$

$$\left. [F(s+\tau, x(s+\tau)) - F(s, x(s))] \right\|$$

$$\le \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\|$$

$$\|F(s+\tau, x(s+\tau)) - F(s, x(s))\|$$

$$\le \frac{(t-s)^\alpha}{\alpha} \frac{c_1 m}{\Gamma(\alpha)} \frac{k}{n}$$

$$\left\| \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} \right.$$

$$\left. \left[x(s+\tau)\ u_2(s+\tau) - x(s)\ u_2(s) \right] (C + \Delta C) \right\|$$

$$\le \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\|$$

$$\|x(s+\tau)\ u_2(s+\tau) - x(s)\ u_2(s)\| \|(C + \Delta C)\|$$

$$\le \frac{(t-s)^\alpha}{\alpha} \Big]_0^t \frac{c_1 m}{\Gamma(\alpha)} \|x(s+\tau)\ u_2(s+\tau) - x(s+\tau)\ u_2(s)$$

$$+ x(s+\tau)\ u_2(s) - x(s)\ u_2(s)\| \|(C + \Delta C)\|$$

$$\le \frac{(t-s)^\alpha}{\alpha} \Big]_0^t \frac{c_1 m}{\Gamma(\alpha)} [\|x(s+\tau)\| \|u_2(s+\tau) - u_2(s)\|$$

$$+ \|x(s+\tau) - x(s)\| \|u_2(s)\|] \|(C + \Delta C)\|$$

$$\le \frac{(t-s)^\alpha}{\alpha} \Big]_0^t \frac{c_1 m}{\Gamma(\alpha)} \left[L_0\ \frac{k_2}{n} + \frac{1}{n}\ \frac{k_1}{n} \right] F_2$$

$$\left\| \sum_{t_i < t} T_\alpha(t - t_i)\ E\ \left[I_{i+q}(x(t_{i+q})) - I_i(x(t_i)) \right] \right\|$$

$$\leq \sum_{t_i<t} \left\| T_\alpha(t-t_i) \right\| \left\| E \right\| \left\| I_{i+q}(x(t_{i+q})) - I_i(x(t_i)) \right\|$$

$$\leq \frac{c_1 m}{\Gamma(\alpha)(1-e^{-(w+M\|\Delta A(t)\|)\alpha})} \left\| E \right\| \frac{k_2}{n}$$

Then $\left\| \Psi x(t+\tau) - \Psi x(t) \right\| \leq \frac{1}{n}$.

To introduce the following main theorem

Theorem (3.11):

If assumptions condition (a)-(c) are satisfied, then the system (3.1)-(3.2) has the mild almost periodic solution on [0,T].

proof:

To prove the existence of a fixed point of the operator Ψ which is defined in Equation (3.8) by using the contraction mapping principle. Let $x, y \in C([0,T]; L^2([0,T], X))$. From (3.8) for any $t \in [0,T]$, we have

$$\left\| (\Psi x)(t) - (\Psi y)(t) \right\| = \left\| \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \right.$$

$$h(x(s))ds - \breve{T}_\alpha(t)\; g(0,x(0)) + E^{-1}\; g(t,x(t)) +$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds +$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} F(s,x(s))ds +$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}$$

$$(C+\Delta C)\; x(s)\; u_2(s)ds + \sum_{t_i<t} T_\alpha(t-t_i)$$

$$E\; I_i(x(t_i)) - \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} h(y(s))\; ds +$$

$$\breve{T}_\alpha(t)\; g(0,y(0)) - E^{-1}\; g(t,y(t)) -$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(B+\Delta B)$$

$$u_1(s)ds - \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}$$

$$F(s,y(s))ds - \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C+\Delta C)$$

$$\left. y(s)\; u_2(s)ds - \sum_{t_i<t} T_\alpha(t-t_i)\; E\; I_i(y(t_i)) \right\|$$

$$\leq \left\| \breve{T}_\alpha(t)\; E\; \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \right.$$

$$\left. \left[h(x(s)) - h(y(s)) \right] ds \right\|$$

$$+ \left\| \breve{T}_\alpha(t)\; \left[g(0,x(0)) - g(0,y(0)) \right] \right\| +$$

$$\left\| E^{-1} \left[g(t,x(t)) - g(t,y(t)) \right] \right\| +$$

$$\left\| \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} \left[F(s,x(s)) - F(s,y(s)) \right] \right\| +$$

$$\left\| \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C+\Delta C)\left[x(s) - y(s) \right] u_2(s) \right\|$$

$$+ \left\| \sum_{t_i<t} T_\alpha(t-t_i)\; E \left[I_i(x(t_i)) - I_i(y(t_i)) \right] \right\|$$

By using lemma (3.4) we obtain

$$\left\| (\Psi x)(t) - (\Psi y)(t) \right\| \leq \frac{c_1 m \left\| E \right\| T^{1-\alpha}}{(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}$$

$$\sup_{s\in[0,t]} \left\| h(x(s)) - h(y(s)) \right\| + \frac{c_1 m}{\Gamma(\alpha)}$$

$$\left\| g(0,x(0)) - g(0,y(0)) \right\| +$$

$$\left\| C_1 \left[g(t,x(t)) - g(t,y(t)) \right] \right\| + \frac{c_1 m\; L_B T^\alpha}{|2\alpha-1|}$$

$$\sup_{s\in[0,t]} \left\| F(s,x(s)) - F(s,y(s)) \right\| +$$

$$\frac{c_1 m\; T^{2\alpha-1}}{|2\alpha-1|}\; F_2\; K_2 \sup_{s\in[0,t]} \left\| x(s) - y(s) \right\| +$$

$$\sum_{t_i<t} \left\| T_\alpha(t-t_i) \right\| \left\| E \right\| \left\| I_i(x(t_i)) - I_i(y(t_i)) \right\|$$

$$\|(\Psi x)(t)-(\Psi y)(t)\| \le \frac{c_1 m\|E\|T^{1-\alpha}\ k_5}{(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}$$

$$\sup_{s\in[0,t]}\|x(s)-y(s)\|+\frac{c_1 m\ T^{1-\alpha}\ k_5\ k_7}{(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}$$

$$\sup_{s\in[0,t]}\|x(s)-y(s)\| c_1\ k_7\|x(t)-y(t)\|+$$

$$\frac{c_1 m\ T^{\alpha}k_1}{|2\alpha-1|}\frac{c_1 m\ T^{\alpha-1}\ F_2\ K_2\ K_3}{|2\alpha-1|}$$

$$\sup_{s\in[0,t]}\|x(s)-y(s)\|+\sup_{s\in[0,t]}\|x(s)-y(s)\|$$

$$+\frac{c_1 m}{\Gamma(\alpha)\ n\ (1-e^{-(w+M\|\Delta A(t)\|)\alpha})}$$

By taking the supermom over $t\in[0,T]$ for both sides, we get

$$\|\Psi x-\Psi y\|\le Y\ (T)\|x-y\|$$

where

$$Y\ (T)=\frac{c_1 m\|E\|T^{1-\alpha}\ k_5}{(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}+$$

$$\frac{c_1 m\ T^{1-\alpha}\ k_5\ k_7}{(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}+c_1\ k_7+\frac{c_1 m\ T^{\alpha}k_1}{|2\alpha-1|}$$

$$+\frac{c_1 m\ T^{\alpha-1}\ F_2\ K_2\ K_3}{|2\alpha-1|}$$

then, there exists $T_1\in(0,T)$ such that $0<X(T_1)<1$ and Ψ is a contraction mapping on $C\left([0,T_1];L^2([0,T],X)\right)$ and therefore has a unique fixed point which is a mild solution of Equation (3.1)-(3.2) on $[0,T_1]$.

This procedure can be repeated in order to extend the solution to the entire intere interval $[0,T]$ infinitely many steps complete the proof.

Asymptotically Stable Almost Periodic Solution of Problem Formulation.

Lemma(3.12):

Let a nonnegative piecewise continuous function x(t) satisfy for $t>t_0$ the inequality

$$x(t)\le C+\int_{t_0}^{t}(t-s)^{\alpha-1}v(\tau)u(\tau)d\tau+\sum_{t_0<\tau_i<t}B_i x(\tau_i).$$

...(3.9)

Where $c>0,\ B_i\ge0,\ v(\tau)\ge0,\ \tau_i$ are the first kind discontinuity points of the function x(t). Then,

$$x(t)\le C\prod_{t_0<\tau_i<t}(1+B_i)\ E_\alpha\left(v(\tau)\ \Gamma(\alpha)\ \tau^{\alpha}\right)\cdot$$

...(3.10)

Proof:

By induction on the interval $[t_0,t_i]$, then (3.9) has the form.

$$x(t)\le C+v(t)\int_0^t(t-s)^{\alpha-1}x(s)ds$$

and so by the lemma (2.6), we have that

$$x(t)\le C\prod_{t_0<\tau_i<t}(1+B_i)\ E_\alpha\left(v(\tau)\ \Gamma(\alpha)\ \tau^{\alpha}\right)\cdot$$

for $t\in[t_0,t_i]$, from (3.10) suppose it holds for $t\in[\tau_i,\tau_{i+1}],\ i=1,2,\dots,K-1$, then for $t\in[\tau_K,\tau_{K+1}]$, we have

$$x(t)\le C+\sum_{i=1}^{K}B_i C\prod_{j=1}^{i-1}(1+B_i)E_\alpha\left(v(\tau_i)\Gamma(\alpha)\tau_i^{\alpha}\right)$$

$$+\sum_{i=1}^{K}\int_{T_{i-1}}^{T_i}v(\tau)C$$

$$\prod_{j=1}^{i-1}(1+B_i)E_\alpha\left(v(\tau_i-(\tau_i-1))\Gamma(\alpha)(\tau_i-(\tau_i-1)^{\alpha}\right)d(\tau)+$$

$$\int_{\tau_k}^{t}(t-s)^{\alpha-1}v(\tau)x(\tau)d(\tau)$$

$$=C\left[1+\sum_{i=1}^{K}\prod_{j=1}^{i-1}(1+B_j)E_\alpha\left(v(\tau-\tau_i)\Gamma(\alpha)(\tau-\tau_i)^{\alpha}\right)(1+B_i)-\right.$$

$$\sum_{i=1}^{K}\prod_{j=1}^{i-1}(1+B_j)E_\alpha\left(v(\tau-\tau_{i-1})\Gamma(\alpha)(\tau-\tau_{i-1})^\alpha\right)+\int_{\tau_k}^{t}(t-s)^{\alpha-1}v(\tau)x(\tau)d(\tau)$$

$$=C\prod_{j=1}^{K}(1+B_j)E_\alpha\left(v(\tau-\tau_k)\Gamma(\alpha)(\tau-\tau_k)^\alpha\right)+\int_{\tau_k}^{t}(t-s)^{\alpha-1}v(\tau)x(\tau)d(\tau)$$

for $t\in[\tau_k,\tau_{k+1}]$, the function $x(t)$ satisfies $x(t)\le C_1+\int_{\tau_k}^{\tau}(t-s)^{\alpha-1}v(\tau)x(\tau)d(\tau)$ *where*

$$C_1=C\prod_{j=1}^{K}(1+B_j)E_\alpha\left(v(\tau-\tau_k)\Gamma(\alpha)(\tau-\tau_k)^\alpha\right)$$

and by lemma (2.6), for $t\in[\tau_k,\tau_{k+1}]$, $x(t)\le C_1E_\alpha\left(v(\tau-\tau_k)\Gamma(\alpha)(\tau-\tau_k)^\alpha\right)$ and

$$x(t)\le C\prod_{t_0<\tau_i<t}(1+B_i)\ E_\alpha\left(v(\tau)\ \Gamma(\alpha)\ \tau^\alpha\right)$$

The following theorem is important theorem for investigate the stability of system (3.1) by using Gronwal Bellman of fractional integral Equation which introdued above.

Theorem (3.13):
If assumptions conditions (a)-(c) are satisfied, then the system (3.1)-(3.2) has a mild almost periodic solution on [0,T] with the following sufficient condition:

$$\omega'=\Bigg[L_\alpha\Bigg\|\Bigg(\frac{1}{\|(\Phi-\Psi)\|(t-s)}+\frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)}+\frac{\alpha\ k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}+\frac{\alpha\|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}\Bigg)\frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7}\Bigg\|\Bigg]$$

$$+\varepsilon\Bigg[\Bigg\|\Bigg(\frac{1}{\|(\Phi-\Psi)\|(t-s)}+\frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)}+\frac{\alpha\ k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}+\frac{\alpha\|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}\Bigg)\frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7}\Bigg\|\Bigg]^{\frac{1}{\alpha}}$$

$$<0$$

proof :

$$x(t)=\breve{T}_\alpha(t)E\left[X_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}h(x(s))ds\right]-\breve{T}_\alpha(t)\ g(0,x(0))+E^{-1}\ g(t,x(t))+$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ F(s,x(s))ds+$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C+\Delta C)x(s)\ u_2(s)ds+\sum_{t_i<t}T_\alpha(t-t_i)\ E\ I_i(x(t_i))$$

Let $X(t)=X(t,0,\Phi)$ and $Y(t)=Y(t,0,\Phi)$ be two solution of Equation (3.1)-(3.2)

$$X(t)=\breve{T}_\alpha(t)\ E\left[\Phi+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\ \ h(x(s))\ ds\right]-\breve{T}_\alpha(t)\ g(0,x(0))+E^{-1}\ g(t,x(t))+$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ F(s,x(s))ds+$$

$$\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C+\Delta C)\ x(s)\ u_2(s)ds + \sum_{t_i<t} T_\alpha(t-t_i)\ E\ I_i(x(t_i)).$$

$$Y(t) = \breve{T}_\alpha(t)\ E\left[\Psi + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\ \ h(y(s))\ ds\right]$$

$$-\breve{T}_\alpha(t)\ g(0,y(0)) + E^{-1}\ g(t,y(t)) + \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(B+\Delta B)u_1(s)ds + \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\ F(s,y(s))ds +$$

$$\|x(t)-y(t)\| = \|\breve{T}_\alpha(t)\ E(\Phi-\Psi) + \breve{T}_\alpha(t)\ E\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\ \left[h(x(s))-h(y(s))\right]\ ds + \breve{T}_\alpha(t)\left[g(0,x(0))-g(0,y(0))\right] + E^{-1}\left[g(t,x(t))-g(t,y(t))\right] + \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\left[F(s,x(s))-F(s,y(s))\right] + \int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}(C+\Delta C)\ \left[x(s)-y(s)\right]\ u_2(s)ds + \sum_{t_i<t} T_\alpha(t-t_i)\ E\ \left[I_i(x(t_i))-I_i(y(t_i))\right]\Bigg\|$$

$$\leq \|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\| + \|\breve{T}_\alpha(t)\|\|E\|\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\|h(x(s))-h(y(s))ds\| + \|\breve{T}_\alpha(t)\|\|g(0,x(0))-g(0,y(0))\| + \|E^{-1}\|\|g(t,x(t))-g(t,y(t))\| + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\|F(s,x(s))-F(s,y(s))\| + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|\|(C+\Delta C)\|\|x(s)-y(s)\|\|u_2(s)\|ds + \sum_{t_i<t}\|T_\alpha(t-t_i)\|\|E\|\|I_i(x(t_i))-I_i(y(t_i))\|$$

$$\frac{\|x(t)-y(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} \leq 1 + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\|h(x(s))-h(y(s))\|ds}{\|(\Phi-\Psi)\|} + \frac{\|g(0,x(0))-g(0,y(0))\|}{\|E\|\|(\Phi-\Psi)\|} + \frac{\|E^{-1}\|\|g(t,x(t))-g(t,y(t))\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} + \int_0^t (t-s)^{\alpha-1}\frac{\|T_\alpha(t-s)\|\|F(s,x(s))-F(s,y(s))\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} + \int_0^t (t-s)^{\alpha-1}\frac{\|T_\alpha(t-s)\|\|(C+\Delta C)\|\|x(s)-y(s)\|\|u_2(s)\|ds}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} + \sum_{t_i<t}\frac{\|T_\alpha(t-t_i)\|\|E\|\|I_i(x(t_i))-I_i(y(t_i))\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}$$

$$u_2(t) = F(x(t)-y(t))\|x(t)-y(t)\|^{-1}$$

$$\frac{\|x(t)-y(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} \leq 1 + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{k_5\|x-y\|}{\|(\Phi-\Psi)\|}ds + \frac{1}{\|E\|} + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}$$

$$\frac{\|h(x(s))-h(y(s))\|}{\|E\|\|(\Phi-\Psi)\|}ds+\frac{\|E^{-1}\|k_7\|x-y\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}$$

$$+\int_0^t (t-s)^{\alpha-1}\frac{\alpha\, k_1\|x-y\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}ds+$$

$$\int_0^t (t-s)^{\alpha-1}\frac{\alpha\|(C+\Delta C)\|\|x(s)-y(s)\|F(x(s)-y(s))}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}$$

$$\|x(s)-y(s)\|^{-1}ds+\sum_{t_i<t}\frac{\alpha\, k_0\|x-y\|}{\|\breve{T}_\alpha(t_i)\|\|(\Phi-\Psi)\|}$$

Substitute

$$\|F(x(t)-y(t))\|\le f(t)\|x(t)-y(t)\|$$

$$\frac{\|x(t)-y(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}\le 1+\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}$$

$$\frac{\|x-y\|}{\|(\Phi-\Psi)\|}ds+\frac{1}{\|E\|}+\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}$$

$$\frac{k_5\|x(s)-y(s)\|}{\|E\|\|(\Phi-\Psi)\|}ds+\frac{\|E^{-1}\|k_7\|x-y\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}+$$

$$\int_0^t (t-s)^{\alpha-1}\frac{\alpha\, k_1\|x-y\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}ds+$$

$$\int_0^t (t-s)^{\alpha-1}\frac{\alpha\|(C+\Delta C)\|\|f(t)\|\|x(s)-y(s)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}ds$$

$$+\sum_{t_i<t}\frac{\alpha\, k_0\|x-y\|}{\|\breve{T}_\alpha(t_i)\|\|(\Phi-\Psi)\|}$$

Where $\|T_\alpha(t)\|\neq 0, \|E\|\neq 0 \quad \|\phi-\Psi\|\neq 0.$

$$\le 1+\frac{1}{\|E\|}+\frac{\|E^{-1}\|k_7\|x(t)-y(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}+$$

$$\int_0^t (t-s)^{-\alpha}\left[\frac{1}{\|(\Phi-\Psi)\|}+\frac{k_5}{\|E\|\|(\Phi-\Psi)\|}+\right.$$

$$\left.\frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}+\frac{\alpha\|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}\right]$$

$$\|x(s)-y(s)\|+$$

$$\sum_{t_i<t}\frac{\alpha\, k_0}{\|\breve{T}_\alpha(t_i)\|\|(\Phi-\Psi)\|}\|x(t_i)-y(t_i)\|$$

Then $\left(\frac{1-\|E^{-1}\|k_7}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}\right)\|x(t)-y(t)\|\le 1+\frac{1}{\|E\|}$

$$+\int_0^t (t-s)^{-\alpha}\left[\frac{1}{\|(\Phi-\Psi)\|}+\frac{k_5}{\|E\|\|(\Phi-\Psi)\|}+\right.$$

$$\frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}+\frac{\alpha\|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}$$

$$\left.+\sum_{t_i<t}\frac{\alpha\, k_0}{\|\breve{T}_\alpha(t_i)\|\|(\Phi-\Psi)\|}\right]\|x(s)-y(s)\|$$

$$\|x(t)-y(t)\|\le\frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7}+$$

$$\frac{\|\breve{T}_\alpha(t)\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7}+\int_0^t (t-s)^{-\alpha}$$

$$\left[\frac{1}{\|(\Phi-\Psi)\|(t-s)}+\frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)}\right.$$

$$+\frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}+\frac{\alpha\|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}$$

$$\left.+\sum_{t_i<t}\frac{\alpha\, k_0}{\|\breve{T}_\alpha(t_i)\|\|(\Phi-\Psi)\|}\right]$$

$$\|x(s)-y(s)\|\quad\frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7}$$

$$\frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)}+\frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}+$$

$$\left.\frac{\alpha\|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}\right)\frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7}\Gamma(\alpha)t^{\alpha}\Big]$$

$$\Big| E_\alpha \Big(\frac{1}{\|(\Phi-\Psi)\|(t-s)} + \frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)} + \frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} + \frac{\alpha \|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} \Big) \frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7} \quad t^\alpha \Big] <$$

$$C_\varepsilon \exp\Big[L_\alpha \Big\| \Big(\frac{1}{\|(\Phi-\Psi)\|(t-s)} + \frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)} + \frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} + \frac{\alpha \|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} \Big) \frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7} \Big\| \; t^\alpha \Big] + \varepsilon \Big[\Big\| \Big(\frac{1}{\|(\Phi-\Psi)\|(t-s)} + \frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)} + \frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} + \frac{\alpha \|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} \Big) \frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7} \Big\| \; t^\alpha \Big]^{\frac{1}{\alpha}}$$

Let

$$\omega' = \Big[L_\alpha \Big\| \Big(\frac{1}{\|(\Phi-\Psi)\|(t-s)} + \frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)} + \frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} + \frac{\alpha \|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} \Big) \frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7} \Big\| \Big] + \varepsilon \Big[\Big\| \Big(\frac{1}{\|(\Phi-\Psi)\|(t-s)} + \frac{k_5}{\|E\|\|(\Phi-\Psi)\|(t-s)} + \frac{\alpha\, k_1}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} + \frac{\alpha \|(C+\Delta C)\|\|f(t)\|}{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|} \Big) \frac{\|\breve{T}_\alpha(t)\|\|E\|\|(\Phi-\Psi)\|}{1-\|E^{-1}\|k_7} \Big\| \Big]^{\frac{1}{\alpha}}$$

$$\|x(t)-y(t)\| \le \frac{\|\breve{T}_\alpha(t)\|.\|E\|.\|\Phi-\Psi\|}{1-\|E^{-1}\|k_7} + \frac{\|\breve{T}_\alpha(t)\|\|\Phi-\Psi\|}{1-\|E^{-1}\|k_7} \prod_{0<t_1<t} \Big(1 + \frac{\alpha k_0}{\|\breve{T}_\alpha(t_i)\|\|\Phi-\Psi\|} . \frac{\|\breve{T}_\alpha(t)\|.\|E\|.\|\Phi-\Psi\|}{1-\|E^{-1}\|k_7}\Big).e^{w' t}.$$

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