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Research Article

Some Bayes Estimators for Maxwell Distribution by Using New Loss Function

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Abstract
In the current study, we have been derived some Bayesian estimations of the scale parameter of Maxwell distribution using the New loss function (NLF) which it called Generalized weighted loss function, assuming non-informative prior, namely, Jefferys prior and information prior, represented by Inverted Levy prior. Based on Monte Carlo simulation method, those estimations are compared depending on the mean squared errors (MSE's). The results show that, the behavior of Bayesian estimation under New loss function using Inverted Levy prior when (k=0, c=3) is the better behavior than other estimates for all cases.

Keywords: Maxwell distribution, Bayesian estimations, New loss Function, Jefferys prior and Inverted Levy prior, Mean squared error.

الخلاصة
في الدراسة الحالية، تم اشتقاق بعض مقدرات بيز لمعلمة القياس لتوزيع ماكسويل تحت دالة خسارة جديدة والتي سميت دالة الخسارة الموزونه المعممة، مع افتراض دالة أسبقية غير معلوماتية وهي دالة أسبقية جفري و دالة الأسبقية المعلوماتية تمثلت بدالة أسبقية معكوس ليفي. على أساس طريقة مونت كارلو للمحاكاة فإن هذه المقدرات، تمت مقارنتها بالاعتماد على متوسط مربعات الخطأ (MSE's).
أظهرت النتائج أن أداء مقدر بيز تحت دالة الخسارة الجديدة باستخدام دالة أسبقية معكوس ليفي عندما (k=0, c=3) أكثر دقة من التقديرات الأخرى في جميع الحالات.

Introduction

Maxwell [1] derived a mathematical formulation in three dimensional spaces to describe the distributions of speeds of molecules in thermal equilibrium and it came to be known as Maxwell distribution. Not all molecules move at the same speed and a few molecules move at faster speeds resulting in a leptokurtic distribution, i.e., unimodal with a longer right tail. A major characteristic of the Maxwell distribution is that it has a smooth increasing failure rate, because of which it is useful in those life-testing and reliability studies in which the assumption of constant failure rate, such as in case of exponential distribution, is not realistic. Tyagi and Bhattacharya [2] for the first time consider one parameter (scale) Maxwell distribution as a model for the distribution of life times. They obtained the minimum variance unbiased and the Bayes estimators of the scale parameter, and the reliability function of this distribution. Chaturvedi and Rani [3] generalized the Maxwell distribution through some trans-formation on a gamma distributed random variable. They also obtained the classical and Bayes estimators of the parameters. Howlader and Hossain [4] derived the highest posterior density (HPD) intervals for the unknown scale parameter, as well as for a future observation considering an asymptotically locally invariant prior proposed by Hartigan [5]. Podder and Roy [6] estimated the parameter of this distribution under Modified Linear Exponential Loss Function (MLINEX). Bekker and Roux [7] studied the maximum likelihood estimator (MLE), as well as the Bayes estimators of the truncated first moment and hazard function of the Maxwell distribution. Krishna and Malik [8] obtained themlE and the Bayes estimator of the reliability characteristics under type-II censoring scheme. Dey and Maiti [9] considered one parameter Maxwell distribution with scale parameter and obtained Bayes estimators using non-informative and conjugate priors under symmetric as well as asymmetric loss functions, namely quadratic loss function, squared-log error loss function, and modified linear exponential loss function. Performances of all these estimators were compared on the basis of their estimated risk. Krishna and Malik [10] compared themlE and the Bayes estimators of the scale parameter and the reliability function under progressive Type II censoring scheme. Recently, Dey *et al.* [11] studied one parameter Maxwell distribution under different loss functions, namely squared error loss function and precautionary loss function, and compared the performances of these estimators. They also obtained predictive density and HPD prediction interval for a future observation. Rasheed [12] derived minimax estimation of the parameter of the Maxwell distribution under Quadratic loss function.

Model Description [13]

The Maxwell (or Maxwell – Boltzmann) distribution gives the distribution of speeds of molecules in thermal equilibrium as given by statistical mechanics.

Defining $\theta = \frac{2KT}{m}$, where K is the Maxwell constant, T is temperature, m is the mass of a molecule. The probability density function and the cumulative distribution function of Maxwell distribution over the rang $x \in [0, \infty)$ are given by:

$$f(x,\theta)=\frac{4}{\sqrt{\pi}}\frac{1}{\theta^{\frac{3}{2}}}x^2 e^{-\frac{x^2}{\theta}} \quad 0<X,\ \theta \qquad (1)$$

$$F(x)=\frac{1}{\Gamma(\frac{3}{2})}\Gamma(\frac{x^2}{\theta},\frac{3}{2}) \qquad (2)$$

Where $\Gamma(x, \alpha) = \int_0^x e^{-u} u^{\alpha-1}\, du$ is the incomplete gamma function.

It can also be expressed as follows

$$F(x;\theta) = 2erf\left(\frac{x}{\sqrt{\theta}}\right) - \frac{2}{\sqrt{\pi}}\frac{x}{\theta}e^{-\frac{x^2}{\theta}}$$

Where $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-w^2}$ dw, is the error function.

Bayesian Estimations

Let $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$ be a random sample of size n with probability density function given in (1) and likelihood function is given by:

$$L\left(x_{i;}\theta\right) = \pi_{i=1}^{n} f(x_i;\theta) = \qquad (3)$$

$$\left(\frac{4}{\sqrt{\pi}}\right)^n \frac{1}{\theta^{\frac{3n}{2}}}(\pi_{i=1}^{n} x_i^2) e^{-\frac{\sum_{i=1}^{n} x_i^2}{\theta}}$$

Bayes Estimator under New Loss Function

Rasheed and Al-Shareefi [14] offered a new loss function, in estimating the scale parameter of Laplace distribution which is called Generalized weighted loss function and introduced as follows:

$$L(\hat{\theta},\theta) = \frac{\left(\sum_{j=0}^{k} a_j \theta^j\right)\left(\hat{\theta}-\theta\right)^2}{\theta^c} \qquad (4)$$

Where, $\theta > 0$; a_j is constant, j =0, 1, 2, … , k, c is constant. Hence, Bayesian estimation using the new loss function will be; [15]

$$\hat{\theta}_{NLF} = \frac{a_0 E\left(\frac{1}{\theta^{c-1}}|\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{c-2}}|\underline{x}\right) + \cdots + a_K E\left(\frac{1}{\theta^{c-(K+1)}}|\underline{x}\right)}{a_0 E\left(\frac{1}{\theta^{c}}|\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{c-1}}|\underline{x}\right) + \cdots + a_K E\left(\frac{1}{\theta^{c-K}}|\underline{x}\right)} \qquad (5)$$

Where $a_0, a_1, \dots, a_K$ are constant.

The Bayesian estimators of the parameter θ under different two prior distributions which are mentioned below can be obtained as follows:

(i) Bayesian estimation Using Jefferys Prior Information

Suppose that, the unknown scale parameter θ has non-information prior density defined as using Jefferys prior information $g(\theta)$ which is derived to be: [13-16]

$$g_1 \propto \sqrt{I(\theta)} \qquad (6)$$

Where $I(\theta) = -nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right)$, represents the Fisher's information matrix.

Hence,

$$g_1(\theta) = k\sqrt{-nE\left(\frac{\partial^2 \ln f(x;a,\theta)}{\partial\theta^2}\right)} \qquad (7)$$

k is a constant.

By taking the logarithm for distribution and taking the second partial derivative according to θ, yields:

$$\frac{\partial^2 \ln f(x;a,\theta)}{\partial\theta^2} = \frac{3}{2\theta^2} - \frac{2x^2}{\theta^3} \qquad (8)$$

$$E\left(\frac{\partial^2 \ln f(x;a,\theta)}{\partial\theta^2}\right) = \frac{3}{2\theta^2} - \frac{2}{\theta^3}E(x^2)$$

$E(x)= 2\sqrt{\frac{\theta}{\pi}}$ and $Var(x)=\frac{\theta}{2\pi}(3\pi-8)$ [7]

$$E(x^2) = \left(E(x)\right)^2 + Var(x)$$

$$= \left(2\sqrt{\frac{\theta}{\pi}}\right)^2 + \frac{\theta}{2\pi}(3\pi-8) = \frac{3}{\theta^2}$$

$$E\left(\frac{\partial^2 \ln f(x|\theta)}{\partial\theta^2}\right) = \frac{3}{2\theta^2} - \frac{3}{\theta^2} = \frac{-3}{2\theta^2} \quad (9)$$

After Substitution (9) into (7), yields:

$$g_1(\theta) = k\sqrt{-n\left(-\frac{3}{2\theta^2}\right)} = \frac{k}{\theta}\sqrt{\frac{3n}{2}} \quad (10)$$

Now, combining the prior (10) with the likelihood function (3), we have the posterior distribution of θ according to Jefferys prior information which is given by:

$$h_1(\theta|\underline{x}) = \frac{g(\theta)L(\theta;x_1x_2\ldots\ldots x_n)}{\int_0^\infty g(\theta)L(\theta;x_1x_2\ldots\ldots x_n)d\theta}$$

$$h_1(\theta|\underline{x}) = \frac{\frac{1}{\theta^{(\frac{3n}{2}+1)}}exp\left[-\frac{\sum_{i=1}^n x_i^2}{\theta}\right]}{\int_0^\infty \frac{1}{\theta^{(\frac{3n}{2}+1)}}exp\left[-\frac{\sum_{i=1}^n x_i^2}{\theta}\right]d\theta} \quad (11)$$

On Simplification, yields:

$$h_1(\theta|\underline{x}) = \frac{(\sum_{i=1}^n x_i^2)^{\frac{3n}{2}}e^{-\sum_{i=1}^n x_i^2/\theta}}{\theta^{(\frac{3n}{2}+1)}\Gamma(\frac{3n}{2})} = \frac{T^{\frac{3n}{2}}e^{-\frac{T}{\theta}}}{\theta^{(\frac{3n}{2}+1)}\Gamma(\frac{3n}{2})} \quad (12)$$

This posterior density is recognized as the density of the Inverse Gamma distribution.
Now, Recall that, according to the posterior density function (12), we derived $E(\theta^m|\underline{x})$, $E\left(\frac{1}{\theta^m}|\underline{x}\right)$ and get:

$$E(\theta^m|\underline{x}) = \frac{(\sum_{i=1}^n x_i^2)^m\Gamma(\frac{3n}{2}-m)}{\Gamma(\frac{3n}{2})}, m = 1,2,\ldots$$

And

$$E\left(\frac{1}{\theta^m}|\underline{x}\right) = \frac{\Gamma(\frac{3n}{2}+m)}{\Gamma(\frac{3n}{2})(\sum_{i=1}^n x_i^2)^m},\ m = 1,2,\ldots$$

Which can be substituted to obtain $\hat{\theta}_{NLJ}$, that is denoted to the Bayesian estimation based on Jeffery's prior using New loss function. Now, putting k = 0 in (5), yields:

$$\hat{\theta}_{NLJ} = \frac{a_0E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right)}{a_0E\left(\frac{1}{\theta^{C}}|\underline{x}\right)}$$

$$\hat{\theta}_{NLJ} = \frac{\sum_{i=1}^n x_i^2}{(\frac{3n}{2}+c-1)} \quad (13)$$

Let c=1, then the Bayesian estimation using New loss function based on Jefferys prior which is denoted by $\hat{\theta}_{NJ01}$ will be**:**

$$\hat{\theta}_{NJ01} = \frac{\sum_{i=1}^n x_i^2}{\frac{3n}{2}} \quad (14)$$

Putting c=2, then the Bayesian estimation using New loss function based on Jefferys prior that is denoted by $\hat{\theta}_{NJ02}$ will be:

$$\hat{\theta}_{NJ02} = \frac{\sum_{i=1}^n x_i^2}{\frac{3n}{2}+1} \quad (15)$$

With c=3, then the Bayesian estimation using New loss function based on Jefferys prior which is denoted by $\hat{\theta}_{NJ03}$ will be**:**

$$\hat{\theta}_{NJ03} = \frac{\sum_{i=1}^n x_i^2}{\frac{3n}{2}+2} \quad (16)$$

Now, putting k = 1 in (5), we get:

$$\hat{\theta}_{NLJ} = \frac{a_0 E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right)}{a_0 E\left(\frac{1}{\theta^{C}}|\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right)}$$

With c=1,then the Bayesian estimation using New loss function based on Jefferys prior which is denoted by $\hat{\theta}_{NJ11}$will be:

$$\hat{\theta}_{NJ11} = \frac{a_0\left(\frac{3n}{2}-1\right)\sum_{i=1}^{n} x_i^2 + a_1\left(\sum_{i=1}^{n} x_i^2\right)^2}{a_0\frac{3n}{2}\left(\frac{3n}{2}-1\right) + a_1\left(\frac{3n}{2}-1\right)\sum_{i=1}^{n} x_i^2}, \quad m = 1,2,... \qquad (17)$$

With c=2, then the Bayesian estimation using New loss function based on Jefferys prior which is denoted by $\hat{\theta}_{NJ12}$ become

$$\hat{\theta}_{NJ12} = \frac{a_0\frac{3n}{2}\left(\sum_{i=1}^{n} x_i^2\right) + a_1\left(\sum_{i=1}^{n} x_i^2\right)^2}{a_0\frac{3n}{2}\left(\frac{3n}{2}+1\right) + a_1\frac{3n}{2}\left(\sum_{i=1}^{n} x_i^2\right)} \qquad (18)$$

With c=3,then the Bayesian estimation using New loss function based on Jefferys prior that is denoted by $\hat{\theta}_{NJ13}$ is**:**

$$\hat{\theta}_{NJ13} = \frac{a_0\frac{3n}{2}\left(\sum_{i=1}^{n} x_i^2\right) + a_1\left(\sum_{i=1}^{n} x_i^2\right)^2}{a_0\frac{3n}{2}\left(\frac{3n}{2}+1\right) + a_1\frac{3n}{2}\left(\sum_{i=1}^{n} x_i^2\right)} \qquad (19)$$

(ii) Bayesian estimation Using Inverted Levy Prior Information

The inverted Levy prior is assumed to be [17]**:**

$$g_2(\theta) = \sqrt{\frac{\lambda}{2\pi}}\frac{1}{\sqrt{\theta}} e^{-\frac{\lambda}{2\theta}}, \qquad \theta, \lambda > 0 \qquad (20)$$

Where λ is the hyper-parameter. Now, the posterior density function is:

$$h_2(\theta|\underline{x}) = \frac{g_4(\theta)L(\theta;\ x_1, x_2, \dots, x_n)}{\int_0^{\infty} g_4(\theta)L(\theta;\ x_1, x_2, \dots, x_n)d\theta} \qquad (21)$$

$$h_2(\theta|\underline{x}) = \frac{\frac{1}{\theta^{\frac{3n}{2}+\frac{1}{2}}} exp\left[-\frac{\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}}{\theta}\right]}{\int_0^{\infty}\frac{1}{\theta^{\frac{3n}{2}+\frac{1}{2}}} exp\left[-\frac{\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}}{\theta}\right]d\theta}$$

Hence, the posterior density functions of (θ) based on Inverted Levy prior is given by: [14]

$$h_2(\theta|\underline{x}) = \frac{\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right)^{\frac{3n-1}{2}} e^{-(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2})/\theta}}{\theta^{\left(\frac{3n+1}{2}\right)}\,\Gamma(\frac{3n-1}{2})} \qquad (22)$$

$$= \frac{(T+\frac{\lambda}{2})^{\frac{3n-1}{2}} e^{-\frac{(T+\frac{\lambda}{2})}{\theta}}}{\theta^{\frac{3n+1}{2}}\,\Gamma(\frac{3n-1}{2})}$$

It is clear that, $h_2(\theta|\underline{x})$ is recognized as the density of the Inverse Gamma distribution.

Now, based on Inverted Levy prior, we get:

$$E(\theta^m|\underline{x}) = \frac{\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right)^m \Gamma\left(\frac{3n-1}{2} - m\right)}{\Gamma\left(\frac{3n-1}{2}\right)} \qquad (23)$$

And

$$E\left(\frac{1}{\theta^m}|\underline{x}\right) = \frac{\Gamma\left(\frac{3n-1}{2}-m\right)}{\Gamma\left(\frac{3n-1}{2}\right)\left(\sum_{i=1}^{n} x_i^2+\frac{\lambda}{2}\right)^m}, m = 1, 2, ... \qquad (24)$$

Which can be substituted to obtain $\hat{\theta}_{NLIL}$, where, $\hat{\theta}_{NLIL}$ denoted to the Bayesian estimation based on Inverted Levy prior using new loss function. Now, putting k = 0 in (5), we get:

$$\hat{\theta}_{NIL} = \frac{a_0 E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right)}{a_0 E\left(\frac{1}{\theta^{C}}|\underline{x}\right)} \qquad (25)$$

$$\hat{\theta}_{NIL} = \frac{\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right)}{\left(\frac{3n-3}{2} + c\right)}$$

With c=1, then the Bayesian estimation using New loss function based on Inverted Levy prior that is denoted by $\hat{\theta}_{NIL01}$ will be:

$$\hat{\theta}_{NIL01} = \frac{T + \frac{\lambda}{2}}{\left(\frac{3n-1}{2}\right)} \qquad (26)$$

With, c = 2 then the Bayesian estimation using New loss function based on Inverted Levy prior that is denoted by $\hat{\theta}_{NIL02}$ become:

$$\hat{\theta}_{NIL02} = \frac{T + \frac{\lambda}{2}}{\left(\frac{3n+1}{2}\right)} \qquad (27)$$

With, c = 3, then the Bayesian estimation using New loss function based on Inverted Levy prior which is denoted by $\hat{\theta}_{NIL03}$ is:

$$\hat{\theta}_{NIL03} = \frac{T + \frac{\lambda}{2}}{\left(\frac{3n+3}{2}\right)} \qquad (28)$$

Now, putting k = 1 in (5), yields:

$$\hat{\theta}_{NIL} = \frac{a_0 E\left(\frac{1}{\theta^{C-1}} |\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{C-2}} |\underline{x}\right)}{a_0 E\left(\frac{1}{\theta^{C}} |\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{C-1}} |\underline{x}\right)}$$

With, c=1, then the Bayesian estimation using New loss function based on Inverted Levy prior that is denoted by $\hat{\theta}_{NIL11}$ become:

$$\hat{\theta}_{NIL11} = \frac{a_0\left(\frac{3n-3}{2}\right)\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right) + a_1\left(\sum_{i=1}^{n} \ldots\right.}{a_0\left(\frac{3n-3}{2}\right)\left(\frac{3n-1}{2}\right) + a_1\left(\frac{3n-3}{2}\right)\left(\sum \ldots\right.} \qquad (29)$$

With, c=2, then the Bayesian estimation using New loss function based on Inverted Levy prior that is denoted by $\hat{\theta}_{NIL12}$ is:

$$\hat{\theta}_{NIL12} = \frac{a_0\left(\frac{3n-1}{2}\right)\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right) + a_1\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right)^2}{a_0\left(\frac{3n+1}{2}\right)\left(\frac{3n-1}{2}\right) + a_1\left(\frac{3n-1}{2}\right)\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right)} \qquad (30)$$

With, c = 3, then the Bayesian estimation using New loss function based on Inverted Levy prior which is denoted by $\hat{\theta}_{NIL13}$ will be:

$$\hat{\theta}_{NIL13} = \frac{a_0\left(\frac{3n+1}{2}\right)\left(\frac{3n-1}{2}\right)\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right) + a_1\left(\frac{3n-1}{2}\right)\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right)^2}{a_0\left(\frac{3n+3}{2}\right)\left(\frac{3n+1}{2}\right)\left(\frac{3n-1}{2}\right) + a_1\left(\frac{3n}{2} + \frac{1}{2}\right)\left(\frac{3n-1}{2}\right)\left(\sum_{i=1}^{n} x_i^2 + \frac{\lambda}{2}\right)} \qquad (31)$$

Simulation Results

Mean Squared Errors, are considered to compare the different estimations of the parameter θ that obtained by the method of Bayes Estimators for New loss function that derived previously. In this simulation results study, the number of recurrence used was I = 5000 samples of sizes n = 5, 10, 20, 50, 100 from the Maxwell distribution with different values of (θ = 0.5, 1.5, 3), and hyper parameter of Inverted Levy prior λ=0.8, and constant of New loss function (a_0=5000 and $a_1$0.5, 50).

The random samples from Maxwell distribution have been generated by applying an algorithm offered by [8].

In this section, Monte-Carlo simulation study is performed to compare the methods of estimation by using mean squared errors as an important criteria for comparing the efficiency of each of estimators, where:

$$\text{MSE}(\theta) = \frac{\sum_{i=1}^{I}\left(\hat{\theta}_i - \theta\right)^2}{I} \qquad (32)$$

Discussion

The results are summarized study for estimating the scale parameter (θ) of Maxwell distribution and Tables (1, 2, and 3) which contain the Expected values and MSE's, we have observed that:

The behavior of Bayesian estimation using New loss function based on Inverted Levy(NIL03) when (K=0, C=3)is the best estimate, comparing to others for all sample sizes and based on all values of the scale parameter. It is observed that mean squared error of all estimations of scale parameter is increasing with the increase of the scale parameter value. Finally, for all parameter values, an obvious reduction in mean squared error is observed based on the increase in sample size

Conclusion

It is observed that mean squared error of all estimates of the scale parameter is increases based on the increase of the scale parameter value, based on all cases.

With all values of θ, the performance of Bayesian estimations using New loss function based on K=0, C=3 is better in comparing to other estimations.

Use New loss function, based on value of the constant (a_0) is much greater than, nearly, a_0=5000, a_1=0.5.

Table 1: Estimates and MSE's for different sample sizes with $\boldsymbol{\theta}$=0.5,$\boldsymbol{\lambda}$=0.8 and $\boldsymbol{a_0}$=5000

Estimators		**Criteria** \ **n**	**5**	**10**	**20**	**50**	**100**
NJ01		**EXP.**	0.499823	0.498925	0.499653	0.499709	0.499559
		MSE	0.034281	0.017208	0.008368	0.003295	0.001646
NJ02		**EXP.**	0.441021	0.467742	0.483536	0.493134	0.496250
		MSE	0.030168	0.016164	0.008108	0.003256	0.001638
NJ03		**EXP.**	0.394597	0.440227	0.468425	0.486729	0.492986
		MSE	0.032476	0.016969	0.008351	0.003302	0.001652
$\boldsymbol{a_1}$**=0.5**	**NJ11**	**EXP.**	0.499828	0.498927	0.499654	0.499709	0.499559
		MSE	0.034282	0.017208	0.008368	0.003295	0.001646
	NJ12	**EXP.**	0.441024	0.467744	0.483536	0.493135	0.496250
		MSE	0.030169	0.016164	0.008108	0.003256	0.001638
	NJ13	**EXP.**	0.394599	0.440228	0.468426	0.486729	0.492986
		MSE	0.032476	0.016969	0.008351	0.003302	0.001652
$\boldsymbol{a_1}$**=50**	**NJ11**	**EXP.**	0.500258	0.499114	0.499741	0.499743	0.499576
		MSE	0.034400	0.017234	0.008374	0.003296	0.001646
	NJ12	**EXP.**	0.441314	0.467898	0.483616	0.493166	0.496266
		MSE	0.030205	0.016174	0.008110	0.003256	0.001638
	NJ13	**EXP.**	0.394805	0.440356	0.468497	0.486761	0.493002
		MSE	0.032477	0.016970	0.008351	0.003302	0.001652
NIL01		**EXP.**	0.592668	0.543715	0.521682	0.508431	0.503905
		MSE	0.047941	0.020325	0.009124	0.003411	0.001672
NIL02		**EXP.**	0.518584	0.508637	0.504576	0.501698	0.500558
		MSE	0.030475	0.016189	0.008117	0.003254	0.001635
NIL03		**EXP.**	0.460964	0.477810	0.488559	0.495139	0.497252
		MSE	**0.025330**	**0.014713**	**0.007721**	**0.003191**	**0.001621**
$\boldsymbol{a_1}$**=0.5**	**NIL11**	**EXP.**	0.592674	0.543717	0.521683	0.508432	0.503905
		MSE	0.047944	0.020326	0.009124	0.003411	0.001672
	NIL12	**EXP.**	0.518588	0.508639	0.504577	0.501698	0.500558
		MSE	0.030477	0.016190	0.008117	0.003254	0.001635
	NIL13	**EXP.**	0.460967	0.477811	0.488560	0.495139	0.495139
		MSE	0.025331	0.014713	0.007721	0.003191	0.003191
$\boldsymbol{a_1}$**=50**	**NIL11**	**EXP.**	0.593314	0.543947	0.521780	0.508467	0.503922
		MSE	0.048234	0.020377	0.009135	0.003412	0.001672
	NIL12	**EXP.**	0.519009	0.508826	0.504666	0.501732	0.500574
		MSE	0.030591	0.016217	0.008123	0.003255	0.001635
	NIL13	**EXP.**	0.461258	0.477967	0.488639	0.495172	0.497269
		MSE	0.025369	0.014725	0.007724	0.003191	0.001621

Table 2: Estimates and MSE's for different sample sizes with $\boldsymbol{\theta}$=1.5,$\boldsymbol{\lambda}$=0.8 and $\boldsymbol{a_0}$=5000

Estimators		Criteria \ n	5	10	20	50	100
NJ01		**EXP.**	1.499469	1.496774	1.498959	1.499125	1.498673
		MSE	0.308532	0.154873	0.075310	0.029657	0.014811
NJ02		**EXP.**	1.323063	1.403226	1.450607	1.479400	1.488749
		MSE	0.271514	0.145474	0.072968	0.029305	0.014740
NJ03		**EXP.**	1.183792	1.403226	1.405273	1.460188	1.478957
		MSE	0.292286	0.145474	0.075162	0.029720	0.014865
a_1=0.5	**NJ11**	**EXP.**	1.499510	1.496791	1.498968	1.499128	1.498675
		MSE	0.308564	0.154879	0.075312	0.029657	0.014811
	NJ12	**EXP.**	1.323089	1.403241	1.450613	1.479403	1.488750
		MSE	0.271524	0.145477	0.072969	0.029305	0.014740
	NJ13	**EXP.**	1.183810	1.320694	1.405280	1.460191	1.478958
		MSE	0.292286	0.152722	0.075162	0.029720	0.014865
a_1=50	**NJ11**	**EXP.**	1.503330	1.498458	1.499747	1.499428	1.498825
		MSE	0.311705	0.155564	0.075467	0.029680	0.014816
	NJ12	**EXP.**	1.325672	1.404608	1.451317	1.479692	1.488897
		MSE	0.272484	0.145748	0.073037	0.029315	0.014742
	NJ13	**EXP.**	1.185641	1.321831	1.405922	1.460468	1.479101
		MSE	0.292317	0.152733	0.075162	0.029719	0.014864
NIL01		**EXP.**	1.663719	1.575978	1.575978	1.514560	1.506367
		MSE	0.380984	0.171499	0.171499	0.030267	0.014949
NIL02		**EXP.**	1.455753	1.474298	1.487503	1.494497	1.496355
		MSE	0.273129	0.145693	0.073016	0.029295	0.014724
NIL03		**EXP.**	1.294002	1.384948	1.440279	1.474961	1.486478
		MSE	**0.256693**	**0.141223**	**0.071874**	**0.029131**	**0.014700**
a_1=0.5	**NIL11**	**EXP.**	1.663772	1.575996	1.537932	1.514563	1.506369
		MSE	0.381046	0.171510	0.079324	0.030267	0.014949
	NIL12	**EXP.**	1.455785	1.474313	1.487511	1.494500	1.496357
		MSE	0.273150	0.145699	0.073018	0.029295	0.014724
	NIL13	**EXP.**	1.294025	1.384960	1.440286	1.474964	1.486479
		MSE	0.256699	0.141225	0.071875	0.029131	0.014700
a_1=50	**NIL11**	**EXP.**	1.668816	1.577902	1.538769	1.514867	1.506514
		MSE	0.387016	0.172610	0.079557	0.030301	0.014957
	NIL12	**EXP.**	1.459105	1.475869	1.488263	1.494795	1.496502
		MSE	0.275340	0.146237	0.073147	0.029315	0.014729
	NIL13	**EXP.**	1.296324	1.386246	1.440969	1.475247	1.486621
		MSE	0.257280	0.141408	0.071923	0.029139	0.014702

Table 3: Estimates and MSE's for different sample sizes with $\boldsymbol{\theta}$=3,$\boldsymbol{\lambda}$=0.8 and $\boldsymbol{a_0}$=5000

Estimators		Criteria \ n	5	10	20	50	100
NJ01		**EXP.**	2.998938	2.993549	2.997918	2.998251	2.997346
		MSE	1.234127	0.619490	0.301240	0.118626	0.059243
NJ02		**EXP.**	2.646127	2.806452	2.901214	2.958800	2.977498
		MSE	1.086054	0.581897	0.291874	0.117220	0.058960
NJ03		**EXP.**	2.367584	2.641365	2.810546	2.920375	2.957915
		MSE	1.169143	0.610886	0.300649	0.118881	0.059459
a_1=0.5	**NJ11**	**EXP.**	2.999096	2.993617	2.997952	2.998263	2.997352
		MSE	1.234384	0.619547	0.301253	0.118628	0.059243
	NJ12	**EXP.**	2.646229	2.806511	2.901238	2.958811	2.977504

		MSE	1.086135	0.581920	0.291879	0.117220	0.058960
	NJ13	**EXP.**	2.367661	2.641409	2.810574	2.920387	2.957920
		MSE	1.169145	0.610890	0.300650	0.118881	0.059459
	NJ11	**EXP.**	3.014093	3.000170	3.001023	2.999444	2.997946
		MSE	1.259003	0.624908	0.302472	0.118809	0.059286
a_1=50	**NJ12**	**EXP.**	2.656391	2.811893	2.904030	2.959946	2.978077
		MSE	1.093657	0.584031	0.292409	0.117303	0.058979
	NJ13	**EXP.**	2.374866	2.645895	2.813105	2.921480	2.958474
		MSE	1.169361	0.610959	0.300642	0.118874	0.059455
NIL01		**EXP.**	3.270289	3.124364	3.062290	3.023734	3.010049
		MSE	1.489784	0.678371	0.315414	0.120784	0.059734
NIL02		**EXP.**	2.861506	2.922787	2.961891	2.983689	2.990048
		MSE	1.103861	0.586091	0.292893	0.117323	**0.058942**
NIL03		**EXP.**	2.543555	2.745653	2.867859	2.944689	2.970314
		MSE	**1.065371**	**0.576635**	**0.290690**	**0.117076**	0.058950
	NIL11	**EXP.**	3.270493	3.124442	3.062323	3.023746	3.010054
		MSE	1.490244	0.678458	0.315433	0.120787	0.059734
a_1=0.5	**NIL12**	**EXP.**	2.861637	2.922853	2.961920	2.983703	2.990054
		MSE	1.104028	0.586131	0.292903	0.117325	0.058942
	NIL13	**EXP.**	2.543652	2.745707	2.867886	2.944701	2.970321
		MSE	1.065408	0.576648	0.290694	0.117077	0.058950
	NIL11	**EXP.**	3.289675	3.131821	3.065582	3.024968	3.010652
		MSE	1.533912	0.686589	0.317161	0.121035	0.059793
a_1=50	**NIL12**	**EXP.**	2.874289	2.928884	2.964863	2.984870	2.990632
		MSE	1.119701	0.590010	0.293836	0.117468	0.058976
	NIL13	**EXP.**	2.552429	2.750689	2.870559	2.945818	2.970893
		MSE	1.069214	0.577846	0.291005	0.117125	0.058960

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