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Conic Parameterization in $PG(2,25)$

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Abstract
The main aim of this paper is to parameterize the conics form through the inequivalent 5-arcs in PG (2,25) using one-one correspondence property between line and conic. The inequivalent 6-arcs in PG (2,25), also have been computed with some examples.

Keywords: Projective space, arc, conic.

الخلاصة
الهدف من هذا البحث هو اعادة تمثيل صيغ القطوع المارة خلال الاقواس المؤلفة من خمسة عناصرالغير متكافئة في $PG(2,25)$ مستخدما خاصية التقابل بين الخط والقطع. الاقواس المؤلفة من 6 عناصر الغير متكافئة في $PG(2,25)$ كذلك تم حسابها مع بعض الامثلة.

Introduction

In $PG\ (2,q)$, the projective plane of order q, there have been many characterizations of the classical curves given by the zeros of quadratic forms called conics. For example, Al-Zangana studied the group effect on the conic in $PG\ (2,q), q = 19,29,31$ [1] [2]. Also, Al-Zangana started to parameterized the conics through the inequivalent 5-arc in PG (2,19), PG (2,23) [1][3]. It is worth mentioning that, the projective plane $PG(2,25)$ has been studied by calculated the complete arcs only as in [4] [5].

The purpose of the research is to compute the 5-arc and then parameterized the conics through these 5-arc in PG (2,15). Also, in this paper, the inequivalent 6-arcs have been computed and then show that, there is a unique 6-arc with ten B-points but does not form a 10-arc.

Preliminary

Definition 1[6]**.** A k-arc, K in projective plane $PG(2,q)$ is a set of k points no three of them are collinear, but there is some two collinear. A k-set, S in projective line $\mathrm{PG}(1,\mathrm{q})$ is a set of k distinct points.

Definition 2 [6]**.** A line ℓ of $PG(2,q)$ is an i-secant of a k-arc K if $|\ell \cap K| = i$. A 2-secant is called a bisecant, a 1-secant is a unisecant and a 0-secant is an external line. The number of bisecants through a point Q out of K is called the index of Q with respect to K.

Definition 3 [6]**.** Let K be an arc and c_i be the number of points of $PG(2,q)\backslash K$ with index exactly i. A point of index three is called a Brianchon point or B-point for short.

During this research, write $ij \cdot kl \cdot mn = P_iP_j \cap P_kP_l \cap P_mP_n$ for B-point, where $P_⅄P_ב$ represent the line through the points $P_⅄$ and $P_ב$.

Definition 4 [6]**.** The zero set of the form F of degree two

$$V(F) = V(aX_0^2 + bX_1^2 + cX_2^2 + dX_0X_1 + eX_0X_2 + fX_1X_2)$$

is called plane quadric. A non-singular plane quadric is called conic.

For details about groups that appear in this paper like, $Z_n \rtimes Z_m =$ semi direct product group, $S_n =$ symmetric group of degree n, $V_4 =$ Klein 4-group and $A_n =$alternating group of degree n, see [7].

To start with this research, the points and lines of $PG(2,25)$ are needed to construct.

The projective plane of order twenty five, $PG(2,25)$, has 651 points and lines, 26 points on each line and 26 lines passing through each point.

Let $(X) = X^3 - \beta^{16}X - \beta \in F_{25}[X]$, where β is the primitive element of F_{25}. Then f is primitive polynomial over F_{25} since

$f(0) = \beta^{13}$,	$f(1) = \beta^2$,
$f(\beta) = \beta^{21}$,	$f(\beta^2) = \beta^{10}$
$f(\beta^3) = \beta^5$,,	$f(\beta^4) = \beta$,
$f(\beta^5) = \beta^{16}$,	$f(\beta^6) = \beta^{11}$,
$f(\beta^7) = \beta^5$,	$f(\beta^8) = \beta^{13}$,
$f(\beta^9) = \beta^{23}$,	$f(\beta^{10}) = \beta^{20}$,
$f(\beta^{11}) = \beta^{16}$,	$f(\beta^{12}) = \beta^4$
$f(\beta^{13}) = \beta^{22}$	$f(\beta^{14}) = \beta^5$,
$f(\beta^{15}) = \beta^{10}$,	$f(\beta^{16}) = \beta^7$,
$f(\beta^{17}) = \beta^6$,	$f(\beta^{18}) = \beta^{18}$,
$f(\beta^{19}) = \beta^{10}$,	$f(\beta^{20}) = \beta^{13}$,
$f(\beta^{21}) = \beta^{15}$,	$f(\beta^{22}) = 1$,
$f(\beta^{23}) = \beta^6$.	

That is, f irreducible over F_{25}, but f has three zeros $\gamma, \gamma^{25}, \gamma^{625}$ in F_{25^3}, where γ is the primitive element of F_{25^3}. Therefore, the companion matrix of f

$$C(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta & \beta^{16} & 0 \end{pmatrix}$$

cycle is projectivity, and then the points of $PG(2,25)$ are

$$P(i) = (1,0,0)C(f)^i.$$

Dully, the lines of $PG(2,25)$ are

$$P(i) = \ell_1 C(f)^i,$$

where $i = 0,1,\dots,650$ and $\ell_1 = V(X_2)$. The line ℓ_1 in numeral form is

$1, 2, 4, 44, 65, 74, 93, 162, 170, 176, 215,$
$252, 269, 310, 397, 422, 454, 472, 501,$
$506, 516, 528, 532, 539, 552, 587.$

For a comprehensive bibliography and more theoretical details about the lines and points structure see [6], and about field theory see [8].

Inequivalent 5*-Arcs*

From the fundamental theorem of projective geometry, there is projectively a unique 4-arc called frame. The stabilizer group of any 4-arc is S_4. Let $\Gamma_{25} = \{U_0, U_1, U_2, U\}$ be the representative 4-arc (standard frame) where $U_0 = [1,0,0] = P(0)$, $U_1 = [0,1,0] = P(1)$, $U_2 = [0,0,1] = P(2)$, $U = [1,1,1] = P(603)$. The 5-arcs are formed by adding points of index zero and the inequivalent one are computed using mathematical program language Gap as summarized in the following theorem.

Theorem 5. In $PG(2,25)$, there are eight inequivalent 5-arcs through Γ_{25}. The values of the constants c_i for any 5-arc are $c_0 = 421$; $c_1 = 210$; $c_2 = 15$. These arcs with their stabilizer group types are given in Table 1.

Table 1: inequivalent 5-arcs

$\mathcal{A}_i$	The 5-arc	SG- type
$\mathcal{A}_1$	$\Gamma_{25} \cup P(\beta^{16}, \beta^6, 1)$	Z_2
$\mathcal{A}_2$	$\Gamma_{25} \cup P(\beta^{18}, \beta^{15}, 1)$	I
$\mathcal{A}_3$	$\Gamma_{25} \cup P(\beta^7, \beta^{10}, 1)$	Z_2
$\mathcal{A}_4$	$\Gamma_{25} \cup P(\beta^{20}, \beta^9, 1)$	Z_2
$\mathcal{A}_5$	$\Gamma_{25} \cup P(\beta^{18}, \beta^6, 1)$	$Z_5 \rtimes Z_4$
$\mathcal{A}_6$	$\Gamma_{25} \cup P(\beta^{22}, \beta^{23}, 1)$	Z_2
$\mathcal{A}_7$	$\Gamma_{25} \cup P(\beta^{14}, \beta^{18}, 1)$	I
$\mathcal{A}_8$	$\Gamma_{25} \cup P(\beta^{20}, \beta ,1)$	S_3

Conic Representation through 5- Arc

It is well known that, through any 5-arc there is a unique conic and the rational points X of the conic $C^* = V(X_1 - X_0X_2)$ parameterized as $(t^2, t, 1)$ [6]. So, There is a unique conic through each 5-arc, $\mathcal{A}_i$ and since each of this arcs passes through Γ_{25}, therefore, each conic $C_{\mathcal{A}_i}$, take the form

$$C_{\mathcal{A}_i} = V(F_{\mathcal{A}_i}) = X_0X_1 + aX_0X_2 - (a+1)X_1X_2.$$

After substituted the fifth point of the arcs $\mathcal{A}_i$ into $F_{\mathcal{A}_i}$ the following are deduced.

$$\begin{aligned} C_{\mathcal{A}_1} &= V(X_0X_1 + \beta^4X_0X_2 + \beta^{11}X_1X_2). \\ &= \{1, 2, 3, 6, 7, 8, 107, 119, 137, 139, 279, 319, 342, 361, 431, 434, 452, \\ &\quad 466, 555, 562, 584, 594, 601, 603, 613, 620\}. \\ C_{\mathcal{A}_2} &= V(X_0X_1 + \beta^4X_0X_2 + \beta^{11}X_1X_2). \\ &= \{1, 2, 3, 6, 7, 8, 107, 119, 137, 139, 279, 319, 342, 361, 431, 434, 452, \end{aligned}$$

$466, 555, 562, 584, 594, 601, 603, 613, 620\}.$

$C_{\mathcal{A}_3} = V(X_0X_1+\beta^{10}X_0X_2+\beta X_1X_2).$
$= \{1, 2, 3, 9, 86, 120, 151, 178, 180, 209, 222, 239, 244, 273, 281, 284, 294, 406, 487, 525, 526, 579, 592, 603, 606, 607\}.$

$C_{\mathcal{A}_4} = V(X_0X_1+\beta^{19}X_0X_2+\beta^{21}X_1X_2).$
$= \{1, 2, 3, 12, 26, 72, 81, 187, 194, 208, 227, 243, 260, 331, 352, 379, 467, 484, 494, 549, 570, 582, 589, 600, 603, 627\}.$

$C_{\mathcal{A}_5} = V(X_0X_1+X_0X_2+\beta^{18}X_1X_2).$
$= \{1, 2, 3, 17, 34, 49, 116, 141, 166, 168, 212, 223, 256, 265, 287, 333, 345, 381, 427, 429, 430, 508, 593, 603, 605, 632\}.$

$C_{\mathcal{A}_6} = V(X_0X_1+\beta^{9}X_0X_2+\beta^{23}X_1X_2).$
$= \{1, 2, 3, 19, 23, 57, 127, 153, 230, 232, 241, 258, 285, 290, 306, 358, 369, 376, 387, 399, 465, 550, 565, 583, 603, 645\}.$

$C_{\mathcal{A}_7} = V(X_0X_1+X_0X_2+\beta^{10}X_1X_2).$
$= \{1, 2, 3, 21, 61, 112, 149, 156, 220, 242, 247, 249, 298, 315, 336, 383, 392, 448, 530, 548, 566, 596, 603, 611, 629, 643\}.$

$C_{\mathcal{A}_8} = V(X_0X_1+\beta^{8}X_0X_2+\beta^{16}X_1X_2).$
$= \{1, 2, 3, 35, 37, 39, 76, 99, 124, 125, 131, 157, 173, 322, 324, 346, 347, 378, 384, 444, 475, 522, 599, 603, 609, 646\}.$

Lemma 6 [9]**.**
On $PG(1,25)$, there are precisely eight distinct pentads given with their stabilizer groups in Table 2 and Table 3

Table 2: Inequivalent pentads

Type	The pentads
$\mathcal{P}_1$	$\{\infty, 0, 1, \beta^{12}, \beta^{6}\}$
$\mathcal{P}_2$	$\{\infty, 0, 1, \beta^{12}, \beta\}$
$\mathcal{P}_3$	$\{\infty, 0, 1, \beta^{12}, \beta^{2}\}$
$\mathcal{P}_4$	$\{\infty, 0, 1, \beta^{12}, \beta^{3}\}$
$\mathcal{P}_5$	$\{\infty, 0, 1, \beta^{4}, \beta^{2}\}$
$\mathcal{P}_6$	$\{\infty, 0, 1, \beta^{4}, \beta^{5}\}$
$\mathcal{P}_7$	$\{\infty, 0, 1, \beta, \beta^{2}\}$
$\mathcal{P}_8$	$\{\infty, 0, 1, \beta, \beta^{8}\}$

Table 3: Stabilizer of inequivalent pentads

Type	SG-type
$\mathcal{P}_1$	$Z_5 \rtimes Z_4 = \langle 1/(t+\beta^{12}), (t\beta^{18}+\beta^{12})\rangle$
$\mathcal{P}_2$	I
$\mathcal{P}_3$	$Z_2 = \langle (t+1)/(t+\beta^{12})\rangle$
$\mathcal{P}_4$	I
$\mathcal{P}_5$	$Z_2 = \langle \beta^4/t\rangle$
$\mathcal{P}_6$	$S_3 = \langle (\beta^8 t+1), \beta^5 t/(t+\beta^{17})\rangle$
$\mathcal{P}_7$	$Z_2 = \langle \beta^2/t\rangle$
$\mathcal{P}_8$	$Z_2 = \langle t/(t+\beta^{12})\rangle$

Using the corresponding properties between $PG(1,25)$ and the conic C^*, the eight 5-sets, $\mathcal{P}_i$ in Table 2 are transformed by $t \longmapsto (t^2, t, 1)$ into 5-arcs, ${\mathcal{P}_i}^*$ in C^* but not through the frame Γ_{25}, where

$C^* = \{1, 3, 19, 42, 47, 111, 149, 157, 174, 210, 217, 273, 288, 303, 325, 348, 357, 416, 430, 466, 509, 549, 597, 603, 623, 631\}.$

Each ${\mathcal{P}_i}^*$ is projectively equivalent to 5-arc, $\mathcal{A}_i$ as given below.

$${\mathcal{P}_1}^* = \{1,3,603,357,210\} \xrightarrow{\begin{pmatrix}\beta^{18} & 0 & 0\\ 0 & 0 & \beta^{14}\\ \beta^{15} & \beta^{13} & \beta^{14}\end{pmatrix}} \mathcal{A}_5$$

$${\mathcal{P}_2}^* = \{1,3,603,357,273\} \xrightarrow{\begin{pmatrix}\beta & 0 & 0\\ 0 & 0 & \beta^{10}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}} \mathcal{A}_2$$

$${\mathcal{P}_3}^* = \{1,3,603,357,42\} \xrightarrow{\begin{pmatrix}\beta^{11} & 0 & 0\\ 0 & 0 & \beta^{4}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}} \mathcal{A}_3$$

$${\mathcal{P}_4}^* = \{1,3,603,357,228\} \xrightarrow{\begin{pmatrix}\beta^{2} & 0 & 0\\ 0 & 0 & \beta^{13}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}} \mathcal{A}_7$$

$${\mathcal{P}_5}^* = \{1,3,603,111,42\} \xrightarrow{\begin{pmatrix}\beta^{15} & 0 & 0\\ 0 & 0 & \beta^{19}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}} \mathcal{A}_4$$

$$\mathcal{P}_6{}^* = \{1,3,603,111,47\} \xrightarrow{\begin{pmatrix}\beta^{20} & 0 & 0\\ 0 & 0 & \beta^{6}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}} \mathcal{A}_8$$

$$\mathcal{P}_7{}^* = \{1,3,603,111,47\} \xrightarrow{\begin{pmatrix}\beta^{13} & 0 & 0\\ 0 & 0 & \beta^{5}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}} \mathcal{A}_6$$

$$\mathcal{P}_8{}^* = \{1,3,603,273,631\} \xrightarrow{\begin{pmatrix}\beta & 0 & 0\\ 0 & 0 & \beta^{10}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}} \mathcal{A}_1.$$

Theorem 7. By uniqueness properties of conics, the parameterization of each conic $C_{\mathcal{A}_i}$ are given below using the matrix transformation between C^* and $C_{\mathcal{A}_i}$. Let $t \in F_{25} \cup \{\infty\}$,

$C_{\mathcal{A}_i}$	Matrix trans. of $C_{\mathcal{A}_i}$ to C^*	Parameterization of $C_{\mathcal{A}_i}$
$C_{\mathcal{A}_1}$	$\begin{pmatrix}\beta & 0 & 0\\ 0 & 0 & \beta^{10}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}$	$\{P(\beta^{-1}(t^2 - \beta^{-1}t), \beta^{-10}(1-\beta t),\ \beta^{-13}t)\}$
$C_{\mathcal{A}_2}$	$\begin{pmatrix}\beta & 0 & 0\\ 0 & 0 & \beta^{10}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}$	$\{P(\beta^{-1}(t^2 - \beta^{-1}t), \beta^{-10}(1-\beta t),\ \beta^{-13}t)\}$
$C_{\mathcal{A}_3}$	$\begin{pmatrix}\beta^{11} & 0 & 0\\ 0 & 0 & \beta^{4}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}$	$\{P(\beta^{-11}(t^2 - \beta^{-1}t), \beta^{-4}(1-\beta t), \beta^{-13}t)\}$
$C_{\mathcal{A}_4}$	$\begin{pmatrix}\beta^{15} & 0 & 0\\ 0 & 0 & \beta^{19}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}$	$\{P(\beta^{-15}(t^2 - \beta^{-1}t), \beta^{-19}(1-\beta t), \beta^{-13}t)\}$
$C_{\mathcal{A}_5}$	$\begin{pmatrix}\beta^{18} & 0 & 0\\ 0 & 0 & \beta^{14}\\ \beta^{15} & \beta^{13} & \beta^{14}\end{pmatrix}$	$\{P(\beta^{-18}(t^2 - \beta^{-1}t), \beta^{-15}(1-\beta \mathrm{t}), \beta^{-13}t)\}$
$C_{\mathcal{A}_6}$	$\begin{pmatrix}\beta^{13} & 0 & 0\\ 0 & 0 & \beta^{5}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}$	$\{P(\beta^{-13}(t^2 - \beta^{-1}t), \beta^{-5}(1-\beta t), \beta^{-13}t)\}$
$C_{\mathcal{A}_7}$	$\begin{pmatrix}\beta^{2} & 0 & 0\\ 0 & 0 & \beta^{13}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}$	$\{P(\beta^{-2}(t^2 - \beta^{-1}t), \beta^{-13}(1-\beta t), \beta^{-13}t)\}$
$C_{\mathcal{A}_8}$	$\begin{pmatrix}\beta^{20} & 0 & 0\\ 0 & 0 & \beta^{6}\\ \beta^{12} & \beta^{13} & \beta^{14}\end{pmatrix}$	$\{P(\beta^{-20}(t^2 - \beta^{-1}t), \beta^{-6}(1-\beta \mathrm{t}), \beta^{-13}t)\}$

Inequivalent 6-Arcs

After calculating the orbit of each 5-arc $\mathcal{A}_i$ and adding one point from each orbit to $\mathcal{A}_i$, the 6-arcs are constructed. In the following theorem the details of inequivalents 6-arcs are given.

Theorem 8: In $PG(2,25)$, there are 365 inequivalent 6-arcs through the standard frame. These arcs partitioned according to stabilizer group types and the parameters $[c_0, c_1, c_2, c_3]$ as given below.

SG-type:*No.*
I: 255
Z_2: 53
Z_3: 29
V_4: 5, Z_4: 4
S_3: 12
A_4: 5
G_{36}: 1
S_5: 1

The elements of the group G_{36} have order as follows.

G_{36}

Ord(g):*No.*
2: 9
3: 8
4: 18

$[c_0, c_1, c_2, c_3]$	:*No.*
[320, 300, 15, 10]	:1
[324, 288, 27, 6]	:6
[326, 282, 33, 4]	:9
[327, 279, 36, 3]	:32
[328, 276, 39, 2]	:50
[329, 273, 42, 1]	:133
[330, 270, 45,0]	:134

Example 9:
The unique 6-arc with stabilizer group of order 120 and ten B-points is

$$\mathcal{H} = \mathcal{A}_5 \cup P(\beta^{12}, \beta^{18}, 1).$$

The arc $\mathcal{H}$ in numeral form is

$$\{1, 2, 3, 603, 17, 430\}.$$

The ten B-points of $\mathcal{H}$ in numeral form is

$$\mathcal{K}_{10} = \{176, 93, 396, 268, 624, 380, 533, 351, 517, 574\},$$

where

$ij \cdot kl \cdot mn$	Point in coordinate form	Point in numeral form
$12 \cdot 34 \cdot 56$	$P(1,1,0)$	176
$12 \cdot 35 \cdot 46$	$P(\beta^{12}, 1,0)$	93
$13 \cdot 24 \cdot 56$	$P(1,0,1)$	396
$13 \cdot 26 \cdot 45$	$P(\beta^{12}, 0,1)$	268
$14 \cdot 25 \cdot 36$	$P(\beta^{18}, 1,1)$	624
$14 \cdot 26 \cdot 35$	$P(\beta^{12}, 1,1)$	380
$15 \cdot 23 \cdot 46$	$P(0, \beta^{6}, 1)$	533
$15 \cdot 24 \cdot 36$	$P(1, \beta^{6}, 1)$	351
$16 \cdot 23 \cdot 45$	$P(0, \beta^{18}, 1)$	517
$16 \cdot 25 \cdot 34$	$P(\beta^{18}, \beta^{18}, 1)$	574

The set $\mathcal{K}_{10}$ does not form 10-arc since it has ten 3-secants as given below.

$\mathcal{K}_{10} \cap \ell_{93}$	$= 93, 268, 624$
$\mathcal{K}_{10} \cap \ell_{112}$	$= 176, 380, 533$
$\mathcal{K}_{10} \cap \ell_{176}$	$= 176, 268, 351$
$\mathcal{K}_{10} \cap \ell_{265}$	$= 268, 533, 574$
$\mathcal{K}_{10} \cap \ell_{323}$	$= 93, 396, 574$
$\mathcal{K}_{10} \cap \ell_{348}$	$= 93, 351, 517$
$\mathcal{K}_{10} \cap \ell_{356}$	$= 176, 517, 624$
$\mathcal{K}_{10} \cap \ell_{516}$	$= 380, 396, 517$
$\mathcal{K}_{10} \cap \ell_{531}$	$= 351, 380, 574$
$\mathcal{K}_{10} \cap \ell_{532}$	$= 396, 533, 624$

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