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Research Article

The Continuous Classical Boundary Optimal Control of a Couple Linear Of Parabolic Partial Differential Equations

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Abstract
In this paper the continuous classical boundary optimal problem of a couple linear partial differential equations of parabolic type is studied, The Galerkin method is used to prove the existence and uniqueness theorem of the state vector solution of a couple linear parabolic partial differential equations for given (fixed) continuous classical boundary control vector. The proof of the existence theorem of a continuous classical optimal boundary control vector associated with the couple linear parabolic is given. The Frechet derivative is derived; finally we give a proof of the necessary conditions for optimality (boundary control) of the above problem.

Keywords: boundary optimal control, couple linear parabolic partial differential equations.

الخلاصة
في هذا البحث تمت دراسة المسالة التقليدية المباشرة للسيطرة الامثلية الحدودية لزوج من المعادلات التفاضلية الجزئية الخطية من النوع المكافئ .استخدمت طريقة كاليركن لبرهان مبرهنة وجود ووحدانية الحل لمتجه الحالة لزوج من المعادلات التفاضلية الجزئية الخطية من النوع المكافئ عندما يكون متجه السيطرة الحدودية ثابتا. كذالك تم برهان مبرهنة وجود متجه السيطرة امثلية الحدودية المصاحبة الحدودية لزوج من المعادلات التفاضليةالجزئية الخطية من النوع المكافئ .تم اشتقاق مشتقة فرشيه وكذالك تم برهان مبرهنة الشروط الضرورية لوجود سيطرة امثلية حدودية.

Introduction

Control theory is a mathematical study of influence the behavior of dynamical system to achieve a desired goal. The subject of the optimal control theory developed in the latter half of 20th century in response to diverse applied problem [1].
Control theory is an application-oriented mathematics that deals with the basic principles underlying the analysis and design of (control) system. Systems can be engineering (air conditioner, air craft, and CD player etcetera), economic, and biological [2].
In general there are many optimal control problems, usually are governed either by ODEs as in [3] or by different type PDEs and are subject to control and state constraints, in 2010 [4], and in 2015 [5] studied an optimal control of parabolic partial differential equations,. in 2014[6] studied an optimal control of hyperbolic partial differential equations, in 2012 [7] studied an optimal control of elliptic partial differential equations, in 2014 [8] studied an optimal control of a coupled of nonlinear elliptic partial differential equations and in 2016 [9] studied optimal control of a coupled of nonlinear parabolic partial differential equations while, in 2016 [10] studied an optimal control of a coupled of nonlinear hyperbolic partial differential equations.
In this paper, the existence and uniqueness theorem of the state vector solution of couple linear parabolic partial differential equations for given (fixed) continuous classical boundary control vector is studied, the existence theorem of a continuous classical boundary optimal control vector associated with a couple linear partial differential equations of parabolic type is developed and proved, also the derivation of the Frechet derivative is done, the existence and uniqueness of the solution of the adjoint

equations which corresponds to the state vector is studied. Finally the necessary conditions for optimality of the above considered problem is developed and proved.

Description of the problem:

Let $I=(0,T), T<\infty$ and $\Omega\subset\mathbb{R}^2$ be an open and bounded region with Lipschitz boundary $\Gamma=\partial\Omega, Q=\Omega\times I, \Sigma=\Gamma\times I$. Consider the following continuous classical boundary optimal control problem:

The state equation is given by the following linear parabolic equations with the initial and boundary conditions:

$y_{1t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial y_1}{\partial x_j})+b_1(x,t)y_1-b(x,t)y_2=f_1(x,t)$ (1)

$y_{2t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(b_{ij}(x,t)\frac{\partial y_2}{\partial x_j})+b_2(x,t)y_2+b(x,t)y_1=f_2(x,t)$ (2)

$\sum_{ij}^{n}a_{ij}\frac{\partial y_1}{\partial n}=u_1(x,t)$,on$\Sigma$ (3)

$y_1(x,0)=y_1^0(x)$,onΩ (4)

$\sum_{ij=1}^{n}b_{ij}\frac{\partial y_2}{\partial n}=u_2(x,t)$,on$\Sigma$ (5)

$y_2(x,0)=y_2^0(x)$,onΩ (6)

Where $(y_1,y_2)=(y_1(\mathrm{x}),y_2(\mathrm{x}))\in(H^1(\Omega))^2$ is the state vector, $(u_1,u_2)=(u_1(\mathrm{x}),u_2(\mathrm{x}))\in(L^2(\Sigma))^2$ is the classical boundary control vector and $(f_1,f_2)=(f_1(\mathrm{x}),f_2(\mathrm{x}))\in(L^2(\mathrm{Q}))^2$ is a vector of a given functions, for all $x=(x_1,x_2)\in\Omega$ .since a_{ij} ,b_{ij} ,$b(x,t)$ and $b_i(x,t)\in C^\infty(Q)$ Let $\overrightarrow{W}_A$ be the set of admissible classical controls, where

$\overrightarrow{W}_A=\left\{\vec{u}\in(L^2(\Sigma))^2/\vec{u}\in U_1\times U_2\subset R^2\text{ a.e. in }\Omega\right\}$

While $G_0(\vec{u})$ be the cost function, such that

$Min.\,G_0(\vec{u})=\frac{1}{2}\|\vec{y}-\overrightarrow{y_d}\|_Q^2+\frac{\beta}{2}\|\vec{u}\|_\Sigma^2, \vec{u}\in\overrightarrow{W_A}$ (7)

Where $\beta$is a positive real number,$(y_{1d},y_{2d})$is the desired data and $(y_1,y_2)=(y_{u1},y_{u2})$is the solution of (1-6) corresponding to the boundary control vector $\vec{u}$.

Let $\vec{V}=V_1\times V_2=H^1(\Omega)\times H^1(\Omega)$ and $(v_1,v_2)=(v_1(\mathrm{x}),v_2(\mathrm{x}))$

We denoted $(v,v)_1$ and $\|v\|_1$ the inner product and the norm in $H^1(\Omega)$**,** by $(\vec{v},\vec{v})$ and $\|\vec{v}\|_0$ the inner product and the norm in $(L^2(\Omega))^2$ by $(\vec{v},\vec{v})_1=(v_1,v_1)_1+(v_2,v_2)_1$ and $\|\vec{v}\|_1=\|v_1\|_1+\|v_2\|_1$ the inner product and the norm in $\vec{V}$ and $\vec{V}^*$ (is the dual of $\vec{V}$).

Weak Formulation of the State Equations:

The weak forms (1-6) is obtained through multiplying both sides of equations (1 & 4) and (2 &6) by $v_1\in V$ and $v_2\in V$ respectively, then taking the integral for both sides and finally using the generalize Green theorem in Hilbert space for the term which have the 2^{nd} derivatives in the L.H.S. of the obtained equations from (1&2), to get $\forall v_1,v_2\in V$

$\langle y_{1t},v_1\rangle+a_1(t,y_1,v_1)+(b_1(t)y_1,v_1)-(b(t)y_2,v_1)=(f_1,v_1)+(u_1,v_1)_\Gamma$ (8a)

$(y_1^0,v_1)=(y_1(0),v_1)$ (8b)

and

$\langle y_{2t},v_2\rangle+a_2(t,y_2,v_2)+(b_2(t)y_2,v_2)+(b(t)y_1,v_2)=(f_2,v_2)+(u_2,v_2)_\Gamma$, (9a)

$(y_2^0,v_2)=(y_2(0),v_2)$, (9b)

Where

$a_1(t,y_1,v_1)=\int_\Omega\sum_{i,j=1}^{n}a_{ij}\frac{\partial y_1}{\partial x_i}\frac{\partial v_1}{\partial x_j}dx$, $a_2(t,y_2,v_2)=\int_\Omega\sum_{i,j=1}^{n}b_{ij}\frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j}dx$

To study the existence of unique solution of the weak form (8-9), we consider the following

Assumptions (A):

i) f_i Satisfies the following condition w.r.t. x &t, i.e.

$|f_i(x,t)|\le\lambda_i(x,t),\forall i=1,2,$ where $(x,t)\in Q,\lambda_i\in L^2(Q,R)$

ii)$|a_i(t,y_i,v_i)|\le\alpha_i\|y_i\|_1\|v_i\|_1$, $|(b_i(t)y_i,v_i)|\le\beta_i\|y_i\|_0\|v_i\|_0\ \forall\ i=1,2$

$a_i(t,y_i,y_i)\ge\bar{\alpha}_i\|y_i\|_1{}^2,\forall\ i=1,2$

$(b_i(t)y_i,y_i)\ge\zeta_i\|y_i\|_0{}^2,\forall\ i=1,2,$ where $\alpha_i,\beta_i,\bar{\alpha}_i\&\zeta_i$ are positive constants $\forall\ i=1,2$

iii) $c(t,\vec{y},\vec{y})=a_1(t,y_1,v_1)+(b_1(t)y_1,y_1)+a_2(t,y_2,y_2)+(b_1(t)y_2,y_2)$, and $|c(t,\vec{y},\vec{v})|\le\alpha\|\vec{y}\|_1\|\vec{v}\|_1$, $(t,\vec{y},\vec{y})\ge\bar{\alpha}\ \|\vec{y}\|_1{}^2$, where $\alpha,\bar{\alpha}$ are real positive constants**.**

Theorem (1) :(Existence and Uniqueness Solution of the State Equations)

With assumptions (A), for each fixed boundary control $\vec{u}\in(L^2(\Sigma))^2$, the weak form of the state equations (8-9) has a unique solution $\vec{y}=$

tion $\vec{y} = (y_1, y_2)$ s.t. $\vec{y} \in \left(L^2(I,V)\right)^2$, $\vec{y}_t = (y_{1t}, y_{2t}) \in \left(L^2(I,V^*)\right)^2$

Proof:

Let $\vec{V}_n \subset \vec{V}$ be the set of continuous and piecewise affine functions in Ω, let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be basis of $\vec{V}_n$ where $n = 2N$, then the approximate solution $\vec{y}$ of (8-9) is approximated by $\vec{y}_n = (y_{1n}, y_{2n}), \forall n$ s.t

$\vec{y}_n = \sum_{j=1}^{N} C_j\,(t)\vec{V}_j(x)$ (10)

Where $\vec{V}_j = ((2-l)v_k, (l-1)v_k)$ (11),

For $\; = 1, \dots N, l = 1,2$ $\;,C_j = c_{lj}, j = 1, \dots n$ $\;,l = 1, j = n+1, \dots, N, j = k + n(l-1), l = 1,2$ and $c_{ij}(t)$ is unknown function of t $,\forall\, j = 1,2, \dots n$

The weak forms of the state equations (8) and (9) can be approximated w.r.t the space variable, using the Galerkin's method to get:

$\langle y_{1nt}, v_1 \rangle + a_1(t, y_{1n}, v_1) + (b_1(t)y_{1n}, v_1) - (b(t)y_{2n}, v_1) = (f_1, v_1) + (u_1, v_1)_\Gamma$ (12a)

$(y_{1n}^0, v_1) = (y_1^0, v_1), \forall\, v_1 \in V_n$ (12b)

&

$\langle y_{2nt}, v_2 \rangle + a_2(t, y_{2n}, v_2) + (b_2(t)y_{2n}, v_2) + (b(t)y_{1n}, v_2) = (f_2, v_2) + (u_2, v_2)_\Gamma$ (13a)

$(y_{2n}^0, v_2) = (y_2^0, v_2), \forall\, v_2 \in V_n$ (13b)

Where $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n \subset L^2(\Omega)$ is the projection of y_i^0, w.r.t. the norm $\|\,.\,\|_0$ i.e$(y_{in}^0, v_i) = (y_i^0, v_i) \iff \left\|y_{in}^0 - y_i^0\right\|_0 \le \left\|y_i^0 - v_i\right\|_0, \forall v_i \in V_n, \forall i = 1,2$

From (10-11) and (12-13), with setting $v_l = v_{li}$ $\;,l = 1, 2$, the weak form can be written as

$\sum_{j=1}^{n} c_{1j}'(t)\left(v_{1j}, v_{1i}\right) + \sum_{j=1}^{n} c_{1j}\left[a_1\left(t, v_{1j}, v_{1i}\right) + \left(b_1(t)v_{1j}, v_{1i}\right)\right] - \sum_{j=1}^{n} c_{2j}\left(b(t)v_{2j}, v_{1i}\right) = (f_1, v_{1i}) + (u_1, v_{1i})_\Gamma$

&

$\sum_{j=1}^{n} c_{2j}'(t)\left(v_{2j}, v_{2j}\right) + \sum_{j=1}^{n} c_{2j}\left[a_2\left(t, v_{2j}, v_{2j}\right) + \left(b_2(t)v_{2j}, v_{1i}\right)\right] + \sum_{j=1}^{n} c_{2j}\left(b(t)v_{1j}, v_{2j}\right) = \left(f_2, v_{2j}\right) + (u_2, v_{2j})_\Gamma$

The above equations are equivalent to the following linear system of 1^{st} ordinary differential equations with its initial conditions, i.e.

$A_1 C_1'(t) + D_1 C_1(t) - E_1 C_2(t) = b_1$
$A_1 C_1(0) = b_1^0$
$A_2 C_2'(t) + D_2 C_2(t) + E_2 C_1(t) = b_2$
$A_2 C_2(0) = b_2^0$

From assumptions (A), we can get easily that the matrices $A_1 \& A_2$ are positive definite therefore system of 1^{st} order differential equation has unique solution [11]

The convergence of the solution:

Let $\{\vec{V}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{V}$, such that $\forall\, \vec{v} = (v_1, v_2) \in \vec{V}$, then usually there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n = (v_{1n}, v_{2n}) \in \vec{V}_n, \forall n$, and $\vec{v}_n \longrightarrow \vec{v}$ strongly in $\vec{V}$ $\Rightarrow\; \vec{v}_n \longrightarrow \vec{v}$ Strongly in $\left(L^2(\Omega)\right)^2$

The approximation problem (12-13) with substituting $\vec{v} = \vec{v}_n$ becomes.

$\langle y_{1nt}, v_{1n} \rangle + a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n}) - (b(t)y_{2n}, v_{1n}) =$
$(f_1, v_{1n}) + (u_1, v_{1n})_\Gamma, y_{1n} \in L^2(I, V_n)$ a.e. in I, (15a)

$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n}), \forall\, v_{1n} \in V_n, \forall n$ (15b)

$(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n}) + (b(t)y_{1n}, v_{2n}) =$
$(f_2, v_{2n}) + (u_2, v_{2n})_\Gamma, y_{2n} \in L^2(I, V_n)$ a.e.inI, (16a) $\quad (y_{2n}^0, v_{2n}) = (y_2^0, v_{2n}), \forall\, v_{2n} \in V_n, \forall n$ (16b)

Which has a sequence of solution $\{\vec{y}_n\}_{n=1}^{\infty}$, where $\vec{y}_n = (y_{1n}, y_{2n})$.

Also from Assumption (A) and the weak forms (12a)-(13a) one can show that the norms $\left\|\overrightarrow{\boldsymbol{y}_n^0}\right\|_{\boldsymbol{0}}$, $\|\vec{\boldsymbol{y}}_{\boldsymbol{n}}(\boldsymbol{t})\|_{L^\infty\left(I,L^2(\Omega)\right)}$, $\|\vec{\boldsymbol{y}}_{\boldsymbol{n}}(\boldsymbol{t})\|_{\boldsymbol{Q}}$ and $\|\vec{\boldsymbol{y}}_{\boldsymbol{n}}(\boldsymbol{t})\|_{L^2(I,V)}$ are bounded**.** Then by Alaoglu theorem [12], there exists a subsequence of $\{\vec{y}_n\}_{n\in N}$, say again $\{\vec{y}_n\}_{n\in N}$ such that $\vec{y}_n \longrightarrow \vec{y}$ Weakly in $\left(L^2(Q)\right)^2$ and $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(I,V)\right)^2$

$\sum_{j=1}^{n} c_{1j}(0)\left(v_{1j}, v_{1i}\right) =$

Multiplying both sides of (15a) and (16a) by $\varphi_i(t) \in C^1[0,T]$, such that $\varphi_i(t) = 0\; \forall i = 1,2$, integrating both sides w.r.t. t from 0 to T, then using integrating by parts for the 1^{st} terms in each obtained equation, to get

$\sum_{j=1}^{n} c_{2j}(t)\left(v_{2j}, v_{2i}\right) -$

$-\int_0^T \langle y_{1nt}, v_{1n} \rangle \varphi_1'(t)dt + \int_0^T [\, a_1(t, y_{1n}, v_{1n}) + \; (b_1(t)y_{1n}, v_{1n}) -$

$(b(t)y_{2n}, v_{1n})]\varphi_1(t)dt = \int_0^T (f_1, y_{1n})\varphi_1(t)dt + \int_0^T (u_1,, y_{1n})_\Gamma \varphi_1(t)dt$
$+(y_{1n}^0, v_{1n})\varphi_1(0)$ (17)
&
$-\int_0^T \langle y_{2nt}, v_{2n}\rangle \varphi_2'(t)dt + \int_0^T [\, a_2(t, y_{2n}, v_{1n}) + (b_2(t)y_{2n}, v_{2n}) - (b(t)y_{1n}, v_{2n})]\varphi_2(t)dt = \int_0^T (f_2, y_{2n})\varphi_2(t)dt + \int_0^T (u_2,, y_{2n})_\Gamma \varphi_2(t)dt + (y_{2n}^0, v_{2n})\varphi_2(0)$, (18)

Since $\forall\, i = 1,2$, $v_{in} \longrightarrow v_i$ strongly in $L^2(\Omega)$ and in V then $v_{in}\varphi_i' \longrightarrow v_i\varphi_i'$ and $v_{in}\varphi_i \longrightarrow v_i\varphi_i$ strongly in $L^2(Q)$ and $L^2(I, V)$ respectively, and since $y_{in} \longrightarrow y_i$ weakly in $L^2(Q)$, also since $y_{in}^0 \longrightarrow y_i^0$ strongly in $L^2(\Omega)$ $\forall i = 1,2$, then

$\int_0^T \langle y_{1nt}, v_{1n}\rangle \varphi_1'(t)dt + \int_0^T [\, a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n}) - (b(t)y_{2n}, v_{1n})]\varphi_1(t)dt \to \int_0^T \langle y_{1t}, v_1\rangle \varphi_1'(t)dt + \int_0^T [\, a_1(t, y_1, v_1) + (b_1(t)y_1, v_1) - (b(t)y_2, v_1)]\varphi_1(t)dt$, (19a)
$(y_{1n}^0, v_{1n})\varphi_1(0) \to (y_1^0, v_1)\varphi_1(0)$, (19b)
$\int_0^T \langle y_{2nt}, v_{2n}\rangle \varphi_2'(t)dt + \int_0^T [\, a_2(t, y_{2n}, v_{1n}) + (b_2(t)y_{2n}, v_{2n}) - (b(t)y_{1n}, v_{2n})]\varphi_2(t)d\,t \longrightarrow \int_0^T \langle y_{2t}, v_2\rangle \varphi_2'(t)dt + \int_0^T [\, a_2(t, y_2, v_1) + (\, b_2(t)y_2, v_2) - (b(t)y_1, v_2)]\varphi_2(t)dt$, (20a)
$(y_{2n}^0, v_{2n})\varphi_2(0) \to (y_2^0, v_2)\varphi_2(0)$ (20b)
$\int_0^T [(f_1, y_{1n}) + (u_1,, y_{1n})_\Gamma]\varphi_1(t)dt \to \int_0^T [(f_1, y_1) + (u_1,, y_1)_\Gamma]\varphi_1(t)dt$ (19c)
And
$\int_0^T [(f_2, y_{2n}) + (u_2,, y_{2n})_\Gamma]\varphi_2(t)dt \to \int_0^T [(f_2, y_2) + (u_2,, y_2)_\Gamma]\varphi_2(t)dt$ (20c)
From (19 a, b& c), (17) becomes
$-\int_0^T \langle y_{1t}, v_1\rangle \varphi_1'(t)dt + \int_0^T [\, a_1(t, y_1, v_1) + (b_1(t)y_1, v_1) - (b(t)y_2, v_1)]\varphi_1(t)dt = \int_0^T (f_1, y_1)\varphi_1(t)dt + \int_0^T (u_1,, y_1)_\Gamma \varphi_1(t)dt + (y_1^0, v_1)\varphi_1(0)$ (21)
And from (20 a, b & c), (18) becomes
$-\int_0^T \langle y_{2t}, v_2\rangle \varphi_2'(t)dt + \int_0^T [\, a_2(t, y_2, v_1) + (b_2(t)y_2, v_2) + (b(t)y_1, v_2)]\varphi_2(t)dt = \int_0^T (f_2, y_2)\varphi_2(t)dt + \int_0^T (u_2, y_2)_\Gamma \varphi_2\,(t)dt + (y_2^0, v_2)\varphi_2(0)$ (22)

Case1: Choose $\varphi_i \in D[0, T]$ with $\varphi_i(0) = \varphi_i(T) = 0$ in (21) and (22), and then using integrating by parts for each 1^{st} term in the L.H.S. of each equation
$\int_0^T \langle y_{1t}, v_1\rangle \varphi_1(t)dt + \int_0^T [\, a_1(t, y_1, v_1) + (b_1(t)y_1, v_1) - (b(t)y_2, v_1)]\varphi_1(t)dt = \int_0^T (f_1, y_1)\varphi_1(t)dt + \int_0^T (u_1,, y_1)_\Gamma \varphi_1(t)dt$ (23)
We get that y_1 is a solution for (8a) by the same technique y_2 is solution for (9a).
Case2: Choose $\varphi_i \in C^1[0, T]$, such that $\varphi_i(T) = 0$, and $\varphi_i(0) \neq 0, \forall i = 1,2$
The initial condition (8 b) is obtained through using integrating by part in equation (23) and subtracting the obtained equation from equation (21).
And by the same technique gives the initial condition (9 b) holds

The strong convergence for $\vec{y}_n$:
By substituting $v_1 = y_1$ and $v_1 = y_{1n}$ in (8a) and (12a) respectively and also substituting $v_2 = y_2$ and $v_2 = y_{2n}$ in (9a) and (13a) respectively, integrating these four equations from 0 to T, finally adding the equations which is obtained from (8a) with that obtained from (12a) to gather and the same happened for (9a)-(13a), to get
$\int_0^T \langle \vec{y}_t - \vec{y}_{nt}\,, \vec{y} - \vec{y}_n\rangle\, dt + \int_0^T c(t, \vec{y} - \vec{y}_n, \vec{y} - \vec{y}_n)\, dt = \int_0^T (f_1, y_1 - y_{1n})dt + \int_0^T (u_1,, y_1 - y_{1n})_\Gamma dt + \int_0^T (f_2, y_2 - y_{2n})dt + \int_0^T (u_2,, y_2 - y_{2n})_\Gamma dt$
$\Rightarrow \vec{y}_n \longrightarrow \vec{y}$ Strongly in $\left(L^2(I, V)\right)^2$.

Uniqueness of the Solution :
Let (y_1, y_2) and $(\hat{y}_1, \hat{y}_2)$ be two solutions for (8-9). Substituting once y_1 in (8a) and $\hat{y}_1$once again then subtracting each equation from the other, then substituting $v_1 = y_1 - \hat{y}_1$, to get .
$\langle (y_1 - \hat{y}_1)_t, y_1 - \hat{y}_1\rangle + a_1(t, y_1 - \hat{y}_1, y_1 - \hat{y}_1) + (b_1(t)y_1 - \hat{y}_1, y_1 - \hat{y}_1) - (b(t)y_2 - \hat{y}_2, y_1 - \hat{y}_1) = 0$ (24)
By the same way which can used in above steps, one have that
$\langle (y_2 - \hat{y}_2)_t, y_2 - \hat{y}_2\rangle + a_2(t, y_2 - \hat{y}_2, y_2 - \hat{y}_2) + (b_2(t)y_2 - \hat{y}_2, y_2 - \hat{y}_2) - (b(t)y_1 - \hat{y}_1, y_2 - \hat{y}_2) = 0$ (25)
Adding (24) and (25) and, then using Lemma 1.2 in [12]. In the 1^{st}and2^{nd} terms and using

Assumption (A-ii) for the 3^{rd}and4^{th}, one obtains that

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left\|\vec{y}-\hat{\vec{y}}\right\|_0^2+\alpha\left\|\vec{y}-\hat{\vec{y}}\right\|_1^2=0 \quad (26)$$

Using that the fact the 2^{nd} term of the L.H.S. of (26) is positive, then integrating both sides of (26) from 0 to T, using Bellman- Gronwall inequality in [11] to get $\vec{y}=\hat{\vec{y}}$

Lemma (1):

a-Consider all the hypotheses in (A) are hold, and $\vec{y}$ and $\vec{y}+\overrightarrow{\Delta y}$ are the states corresponding to the controls $\vec{u}$ and $\vec{u}+\overrightarrow{\Delta u}$ respectively where $\vec{u}$ and $\overrightarrow{\Delta u}$ are bounded in $L^2(\Sigma)\times L^2(\Sigma)$, then

$$\left\|\overrightarrow{\Delta y}\right\|_{L^\infty(I,L^2(\Omega))}\le K\left\|\overrightarrow{\Delta u}\right\|_\Sigma$$

$$\left\|\overrightarrow{\Delta y}\right\|_{L^2(Q)}\le K\left\|\overrightarrow{\Delta u}\right\|_\Sigma$$

$$\left\|\overrightarrow{\Delta y}\right\|_{L^2(I,V)}\le K\left\|\overrightarrow{\Delta u}\right\|_\Sigma$$

b-The operator $\vec{u}\mapsto\vec{y}_{\vec{u}}$ from $(L^2(\Sigma))^2$ into $(L^\infty(I,L^2(\Omega)))^2$ or in to $(L^2(I,V))^2$ or into $(L^2(Q))^2$ is continuous.

Proof:

a-Let$\vec{u}=(u_1,u_2),\hat{\vec{u}}=(\hat{u}_1,\hat{u}_2)\in L^2(\Sigma),\overrightarrow{\Delta u}=\hat{\vec{u}}-\vec{u}$ then hence by theorem(3.1), there exists a solutions $\vec{y}=(y_1,y_2)$ and $\hat{\vec{y}}=(\hat{y}_1,\hat{y}_2)$ of (8-9) (corresponding to the boundary controls $\vec{u}$ and $\hat{\vec{u}}$

$$\langle\hat{y}_{1t},v_1\rangle+a_1(t,\hat{y}_1,v_1)+(b_1(t)\hat{y}_1,v_1)-(b(t)\hat{y}_2,v_1)=(f_1,v_1)+(u_1+\Delta u_1,v_1)_\Gamma \quad (27a)$$

$$(\hat{y}_1(0),v_1)=(y_1^0,v_1) \quad (27b)$$

&

$$\langle\hat{y}_{2t},v_2\rangle+a_2(t,\hat{y}_2,v_2)+(b_2(t)\hat{y}_2,v_2)+(b(t)\hat{y}_1,v_2)=(f_2,v_2)+(u_2+\Delta u_2,v_2)_\Gamma \quad (28a)$$

$$(\hat{y}_2(0),v_2)=(y_2^0,v_2) \quad (28b)$$

By subtracting (8a&b) from (27a&b), and (9a&b) from (28a&b), then setting$\Delta y_1=\hat{y}_1-y_1$, $\Delta y_2=\hat{y}_2-y_2$, in each one of obtained equations to get.

$$\langle\Delta y_{1t},v_1\rangle+a_1(t,\Delta y_1,v_1)+(b_1(t)\Delta y_1,v_1)-(b(t)\Delta y_2,v_1)=(\Delta u_1,v_1)_\Gamma \quad (30)$$

&

$$\langle\Delta y_{2t},v_2\rangle+a_2(t,\Delta y_2,v_2)+(b_2(t)\Delta y_2,v_2)+(b(t)\Delta y_1,v_2)=(\Delta u_1,v_2)_\Gamma \quad (29)$$

Substituting $v_1=\Delta y_1$ and $v_2=\Delta y_2$ in (29) and (30) respectively, using Lemma 1.2 in [12]. For the 1^{st} terms in the L.H.S. of each one of the above obtained equation, then adding the resulting equations, to get

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left\|\overrightarrow{\Delta y}\right\|_0^2+\alpha\left\|\overrightarrow{\Delta y}\right\|_1^2=(\Delta u_1,\Delta y_1)+(\Delta u_2,\Delta y_2) \quad (31)$$

Since the 2^{nd} term of (31) is positive, then (31) become

$$\frac{1}{2}\frac{d}{dt}\left\|\overrightarrow{\Delta y}\right\|_0^2\le(\Delta u_1,\Delta y_1)+(\Delta u_2,\Delta y_2)$$

Integrating both sides for t from 0 to t, then for the R.H.S we use the Cuschy Schwarz inequality and the trace theorem ,and finally using Bellman- Gronwall inequality in [11]to get,

$$\left\|\overrightarrow{\Delta y}\right\|_{L^\infty(I,L^2(\Omega))}\le K\left\|\overrightarrow{\Delta u}\right\|_\Sigma$$

Then easily, one can get

$\left\|\overrightarrow{\Delta y}\right\|_{L^2(Q)}\le K\left\|\overrightarrow{\Delta u}\right\|_\Sigma$,And$\left\|\overrightarrow{\Delta y}\right\|_{L^2(I,V)}\le K\left\|\overrightarrow{\Delta u}\right\|_\Sigma$

b-Let $\overrightarrow{\Delta u}=\vec{u}_1-\vec{u}_2$ and $\overrightarrow{\Delta y}=\vec{y}_1-\vec{y}_2$ where $\vec{y}_1$and $\vec{y}_2$ are the correspond states to the controls $\vec{u}_1$and $\vec{u}_2$, then using part (a) of this theorem, to get that

$$\|\vec{y}_1-\vec{y}_2\|_{L^\infty(I,L^2(\Omega))}\le K\|\vec{u}_1-\vec{u}_2\|_\Sigma$$

Which means the operator $\vec{u}\longmapsto\vec{y}$ is Lipschitz continuous from $(L^2(\Sigma))^2$ in to$(L^\infty(I,L^2(\Omega)))^2$ .The other result are obtained easily

Lemma (2): The cost function (7) is weakly lower semi continuous [13]

Theorem (2)

The cost function (7) has a classical boundary optimal control if it is coercive.

Proof:

Since $G_0(\vec{u})$ is non-negative and coercive, then there exists a minimizing sequence $\{\vec{u}_k\}=\{(u_{1k},u_{2k})\}\in\overrightarrow{W}_A,\forall k$, such that

$\lim_{n\to\infty}G_0(\vec{u}_k)=\inf_{\vec{u}\in\overrightarrow{W}_A}G_0(\vec{u})$, $\|\vec{u}_k\|\le c,\forall k$.

By the Alaoglu theorem there exists a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ such that $\vec{u}_k\longrightarrow\vec{u}$ weakly in$(L^2(\Sigma))^2$

Then there exist a sequence of solution $\{\vec{y}_k\}$, corresponding to the sequence of the control $\{\vec{u}_k\}$ and the norms $\|\vec{y}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$and$\|\vec{y}_k\|_{L^2(I,V)}$are bounded, then by the Alaoglu theorem there exist a subsequence of of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$, such that $\vec{y}_k \longrightarrow \vec{y}$ Weakly in$\left(L^\infty\left(I, L^2(\Omega)\right)\right)^2$, $\left(L^2(Q)\right)^2$,and in $\left(L^2(I,V)\right)^2$

The weak forms of the state equations (12a) & (13a) can be rewritten as

$$\langle y_{1kt}, v_1\rangle = -a_1(t, y_{1k}, v_1) - (b_1(t)y_{1k}, v_1) + (b(t)y_{2k}, v_1) + (f_1, v_1) + (u_1, v_1)_\Gamma$$

&

$$\langle y_{2kt}, v_2\rangle = -a_2(t, y_{2n}, v_2) - (b_2(t)y_{2n}, v_2) -(b(t)y_{1n}, v_2) + (f_2, v_2) + (u_2, v_2)_\Gamma$$

By adding the above equality and integrating both sides of the obtained equation w.r.t. t from 0 to T, taking the absolute value, using Cauchy Schwartz inequality, and finally using assumptions (A-iii), one obtain

$$\left|\int_0^T \langle \vec{y}_{kt}, \vec{v}\rangle dt\right| = | -\int_0^T c\left(t, \vec{y}_{k,} \vec{v}\right) dt + \int_0^T (f_1, y_1) dt + \int_0^T (u_1,, y_1)_\Gamma dt + \int_0^T (f_2, y_2) dt + \int_0^T (u_2,, y_2)_\Gamma]dt|$$

$$\leq \left|\int_0^T c\left(t, \vec{y}_{k,} \vec{v}\right) dt\right| + \left|\int_0^T (f_1, v_1) dt\right| + \left|\int_0^T (u_1, v_1) dt\right| + \left|\int_0^T (f_2, v_2)\, dt\right| + \left|\int_0^T (u_2, v_2)\, dt\right| \Rightarrow \|\vec{y}_{kt}\|_{L^2(I,V^*)} \leq h_7$$

$,\forall \vec{y}_{kt} \in V^* \times V^*$, where $h_7 \geq 0$

Since for each k, y_{1k} and y_{2k} are solutions of the state equations, then

$$\langle y_{1kt}, v_1\rangle + a_1(t, y_{1k}, v_1) + (b_1(t)y_{1k}, v_1) - (b(t)y_{2k}, v_1) = (f_1, v_1) + (u_{1k}, v_1)_\Gamma \quad (32)$$

&

$$\langle y_{2kt}, v_2\rangle + a_2(t, y_{2k}, v_2) + (b_2(t)y_{2k}, v_2) + (b(t)y_{1k}, v_2) = (f_2, v_2) + (u_{2k}, v_2)_\Gamma \quad (33)$$

Let $\varphi_i \in C^1[0,T]$, such that$\varphi_i(T) = 0, \forall i = 1,2$. Rewriting the 1^{st} terms in the L.H.S. of (32) & (33), multiplying both sides of them by $\varphi_1(t)$ and $\varphi_2(t)$ respectively, and integrating for both sides for t From 0 to, finally using integration by parts for the 1^{st} terms in the L.H.S. of each one of the above obtained equations, using the converges $\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^2(Q)\right)^2$ and $\vec{y}_k \longrightarrow \vec{y}$ weakly in$\left(L^2(I,V)\right)^2$,to get

$$-\int_0^T (y_{1k}, v_1)\varphi_1'(t) dt + \int_0^T [a_1(t, y_{1k}, v_1) + (b_1(t)y_{1k}, v_1) - (b(t)y_{2k}, v_1)]\varphi_1(t) dt \to -\int_0^T (y_1, v_1)\varphi_1'(t) dt + \int_0^T [a_2(t, y_1, v_1) + (b_2(t)y_1, v_1) - (b(t)y_2, v_1)]\varphi_1(t) dt \quad (34a)$$

&

$$-\int_0^T (y_{2k}, v_2)\varphi_2'(t) dt + \int_0^T [a_2(t, y_{2k}, v_1) + (b_2(t)y_{2k}, v_2) - (b(t)y_{1k}, v_2)]\varphi_2(t) dt \to -\int_0^T (y_2, v_2)\varphi_2'(t) dt + \int_0^T [a_2(t, y_2, v_1) + (b_2(t)y_2, v_2) - (b(t)y_1, v_2)]\varphi_2(t) dt \quad (35a)$$

Since $y_{1k}(0)\,, y_{2k}(0)$ are bounded in $L^2(\Omega)$ and from the Projection theorem, one has

$$(y_{1k}^0, v_1)\varphi_1(0) \to (y_1^0, v_1)\varphi_1(0) \quad (34b)$$

&

$$(y_{2k}^0, v_2)\varphi_2(0) \to (y_2^0, v_2)\varphi_2(0) \quad (35b)$$

Since $\vec{u}_k \longrightarrow \vec{u}$ weakly in $\left(L^2(\Sigma)\right)^2$, then

$$\int_0^T (f_1, v_1) dt + \int_0^T (u_{1k}\,, v_1)_\Gamma dt \to \int_0^T (f_1,, v_1) dt + \int_0^T (u_1,, v_1)_\Gamma dt \quad (34c)$$

&

$$\int_0^T (f_2, v_2) dt + \int_0^T (u_{2k}\,, v_2)_\Gamma dt \to \int_0^T (f_2,, v_2) dt + \int_0^T (u_2,, v_2)_\Gamma dt \quad (35c)$$

Finally, from the converges (34a, b& c) and (35a, b& c), one get that

$$-\int_0^T (y_1, v_1)\varphi_1'(t) dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1, v_1) - (b(t)y_2, v_1)]\varphi_1(t) dt = \int_0^T (f_1, y_1)\varphi_1(t) dt + \int_0^T (u_1, y_1)_\Gamma \varphi_1(t) dt + (y_1^0, v_1)\varphi_1(0) \quad (36)$$

&

$$-\int_0^T (y_2, v_2)\varphi_2'(t) dt + \int_0^T [a_2(t, y_2, v_1) + (b_2(t)y_2, v_2) - (b(t)y_1, v_2)]\varphi_2(t) dt = \int_0^T (f_2, y_2)\varphi_2(t) dt + (y_2(0), v_2)\varphi_2(0) + \int_0^T (u_2, y_2)_\Gamma \varphi_2(t) dt \quad (37)$$

Now, one has the following two cases:

<u>Case 1</u>: Choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0, \forall i = 1,2$

Now, integration by parts for the 1^{st} terms in the L.H.S. of (36) and (37), once get

$$\int_0^T \langle y_{1t}, v_1\rangle \varphi_1(t) dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1, v_1) - (b(t)y_2, v_1)]\varphi_1(t) dt = \int_0^T (f_1, y_1)\varphi_1(t) dt + \int_0^T (u_1, y_1)_\Gamma \varphi_1(t) dt \;, \forall v_1 \in V, \quad (38)$$

&

$$\int_0^T \langle y_{2t}, v_2\rangle \varphi_2(t) dt + \int_0^T [a_2(t, y_2, v_2) + (b_2(t)y_2, v_2) + (b(t)y_1, v_2)]\varphi_2(t) dt = \int_0^T (f_2, y_2)\varphi_2(t) dt + \int_0^T (u_2, y_2)_\Gamma \varphi_2(t) dt \quad (39)$$

<u>Case 2</u>: Choose$\varphi_i \in C^1[I]$, such that $\varphi_i(T) = 0$ &$\varphi_i(0) \neq 0, \forall i = 1,2$

Integrating by parts for the 1^{st} terms in the L.H.S. of (38) and (39), one has $(y_1^0, v_1)\varphi_1(0) = (y_1(0), v_1)\varphi_1(0), \varphi_1(0) \neq 0\ \forall\ \varphi_1\ \in C^1[0,T]$

$\Rightarrow\ y_1^0 = y_1(0) = y_1^0(x)$

&

$(y_2^0, v_2)\varphi_2(0) = (y_2(0), v_2)\varphi_2(0), \varphi_2(0) \neq 0\ \forall\ \varphi_2\ \in C^1[0,T]$

$\Rightarrow\ y_2^0 = y_2(0) = y_2^0(x)$

Then $y_1 = y_{u1}$ & $y_2 = y_{u2}$ are the solutions of the state equations (from case 1& case2)

Since $G_0(\vec{u})$ is W.L.S.C., i.e.

$G_0(\vec{u}) \leq \lim_{k\to\infty} \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{u}_k)$

$\Rightarrow\ G_0(\vec{u}) = \inf_{\vec{\bar{u}} \in \vec{W}_A} G_0(\vec{\bar{u}})$

$\therefore \vec{u}$ is a continuous classical boundary optimal control .

***Theorem (3)*:**

Consider the cost function (7), and the following adjoins equations $(z_1, z_2) = (z_{u1}, z_{u2})$ (where $(y_1, y_2) = (y_{u1}, y_{u2})$)of the state equations (1-6) are given by

$-z_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial z_1}{\partial x_j}) + b_1(x,t)z_1 + b(x,t)z_2 = (y_1 - y_{1d})$, in Ω (40)

$-z_{2t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}(x,t)\frac{\partial z_2}{\partial x_j}) + b_2(x,t)z_2 - b(x,t)z_1 = (y_2 - y_{2d})$,onΩ (41)

$z_1(x,T) = 0$ onΩ (42)

$\frac{\partial z_1}{\partial n} = 0$,on Σ (43)

$z_2(x,T) = 0$ on Ω (44) $\frac{\partial z_2}{\partial n} = 0$,on$\Sigma$ (45)

Then for $(u_1, u_2) \in \overrightarrow{W_A}$, the Frechet derivative of G_0 is given by

$(\acute{G_0}(\overrightarrow{u}), \overrightarrow{\Delta u}) = (\vec{z} + \beta\vec{u}, \overrightarrow{\Delta u})$

***Proof* :**

So as in the state equation (1-6) the weak form of the adjoint equations is:

$-\langle z_{1t}, v_1\rangle + a_1(t, z_1, v_1) + (b_1(t)z_1, v_1) - (b(t)z_2, v_1) = (y_1 - y_{1d}, v_1), \forall v_1 \in V$ (46)

&

$-\langle z_{2t}, v_2\rangle + a_2(t, z_2, v_2) + (b_2(t)z_2, v_2) + (b(t)z_1, v_2) = (y_2 - y_{2d}, v_2), \forall v_2 \in V$ (47)

Now, substituting $v_1 = z_1$and $v_2 = z_2$ in (29) and (30) respectively, yield

$\langle \Delta y_{1t}, z_1\rangle + a_1(t, \Delta y_1, z_1) + (b_1(t)\Delta y_1, z_1) - (b(t)\Delta y_2, z_1) = (\Delta u_1, z_1)$ (48)

&

$\langle \Delta y_{2t}, z_2\rangle + a_2(t, \Delta y_2, z_2) + (b_2(t)\Delta y_2, z_2) + (b(t)\Delta y_1, z_2) = (\Delta u_2, z_2)$ (49)

Also, substituting $v_1 = \Delta y_1$ and $v_2 = \Delta y_2$ in (46) and (47) respectively, one get

$-\langle z_{1t}, \Delta y_1\rangle + a_1(t, z_1, \Delta y_1) + (b_1(t)z_1, \Delta y_1) + (b(t)z_2, \Delta y_1) = (y_1 - y_{1d}, \Delta y_1)$ (50)

&

$-\langle z_{2t}, \Delta y_2\rangle + a_2(t, z_2, \Delta y_2) + (b_2(t)z_2, \Delta y_2) - (b(t)z_1, \Delta y_2) = (y_2 - y_{2d}, \Delta y_2)$ (51)

Now, integrating both sides of (48-51), for t from 0 to T, using integration by parts for the 1^{st}terms of the L.H.S. of each obtained equations, to get

$\int_0^T \langle \Delta y_{1t}, z_1\rangle dt + \int_0^T [a_1(t, \Delta y_1, z_1) + (b_1(t)\Delta y_1, z_1) - (b(t)\Delta y_2, z_1)]dt = \int_0^T (\Delta u_1, z_1)dt$, (52)

$\int_0^T \langle \Delta y_{2t}, z_2\rangle dt + \int_0^T [a_2(t, \Delta y_2, z_2) + (b_2(t)\Delta y_2, z_2) + (b(t)\Delta y_1, z_2)]dt = \int_0^T (\Delta u_2, z_2)dt$, (53)

$\int_0^T \langle \Delta y_{1t}, z_1\rangle\, dt + \int_0^T [a_1(t, z_1, \Delta y_1) + (b_1(t)z_1, \Delta y_1) + (b(t)z_2, \Delta y_1)]dt = \int_0^T (y_1 - y_{1d}, \Delta y_1)dt$, (54)

$\int_0^T \langle \Delta y_{2t}, z_2\rangle\, dt + \int_0^T [a_2(t, z_2, \Delta y_2) + (b_2(t)z_2, \Delta y_2) - (b(t)z_1, \Delta y_2)]dt = \int_0^T (y_2 - y_{2d}, \Delta y_2)dt$ (55)

And

By subtracting (52) and (53) from (54) and (55) respectively and by adding the two resulting equations, yields

$-\int_0^T (\Delta u_1, z_1)dt - \int_0^T (\Delta u_2, z_2)dt = \int_0^T (y_1 - y_{1d}, \Delta y_1)dt + \int_0^T (y_2 - y_{2d}, \Delta y_2)dt$ (56)

$y_1 + \Delta y_1$, $y_2 + \Delta y_2$ are the corresponding solutions of the control $u_1 + \Delta u_1$ and$u_2 + \Delta u_2$, then

$G_0(\vec{u} + \overrightarrow{\Delta u}) - G_0(\vec{u}) = \frac{1}{2}\|y_1 - y_{1d}\|_Q^2 + \int_0^T \int_\Omega (y_1 - y_{1d})\Delta y_1 dxdt + \frac{1}{2}\|\Delta y_1\|_Q^2 + \frac{1}{2}\|y_2 - y_{2d}\|_Q^2 + \int_0^T \int_\Omega (y_2 - y_{2d})\Delta y_2 dxdt + \frac{1}{2}\|\Delta y_2\|_Q^2 + \frac{\beta}{2}\|u_1\|_\Sigma^2 + \int_0^T \int_\Gamma \beta u_1 \Delta u_1 dxdt + \frac{\beta}{2}\int_0^T \int_\Gamma (\Delta u_1)^2\, dxdt + \frac{\beta}{2}\|u_2\|_\Sigma^2 +$

$\int_0^T \int_\Gamma \beta u_2\, \Delta u_2\, dxdt + \frac{\beta}{2}\int_0^T \int_\Gamma (\Delta u_2)^2\, dxdt - \frac{1}{2}\|y_1 - y_{1d}\|_Q^2 - \frac{1}{2}\|y_2 - y_{2d}\|_Q^2 - \frac{\beta}{2}\|u_1\|_\Sigma^2 - \frac{\beta}{2}\|u_2\|_\Sigma^2$

$G_0(\vec{u} + \overrightarrow{\Delta u}) - G_0(\vec{u}) = (\Delta u_1, z_1) + \beta(u_1, \Delta u_1) + (\Delta u_2, z_2) + \beta(u_2, \Delta u_2) + \frac{1}{2}\|\overrightarrow{\Delta y}\|_Q^2 + \frac{\beta}{2}\|\overrightarrow{\Delta u}\|_\Sigma^2$

$\therefore G_0(\vec{u} + \overrightarrow{\Delta u}) - G_0(\vec{u}) = (\vec{z} + \beta\vec{u}, \overrightarrow{\Delta u}) + \frac{1}{2}\|\overrightarrow{\Delta y}\|_Q^2 + \frac{\beta}{2}\|\overrightarrow{\Delta u}\|_\Sigma^2$ (57)

$\because \|\overrightarrow{\Delta y}\|_Q \le k\|\overrightarrow{\Delta u}\|_\Sigma, k > 0$

$(G_0'(\vec{u}), \overrightarrow{\Delta u}) = (\vec{z} + \beta\vec{u}, \overrightarrow{\Delta u})$

***Theorem (4)*:**

The classical boundary optimal control of the above problem is $G_0'(\vec{u}) = \vec{z} + \beta\vec{u} = 0$ with $\vec{y} = \vec{y}_{\vec{u}}$ and $\vec{z} = \vec{z}_{\vec{u}}$.

Proof:

If $\vec{u}$ is an boudary optimal control of the problem

$G_0(\vec{u}) = \min_{\vec{u}\in\overrightarrow{W}_A} G_0(\vec{u}), \forall\vec{u} \in (L^2(\Sigma))^2$, i.e. $G_0'(\vec{u}) = 0$

$\Rightarrow \vec{z} + \beta\vec{u} = 0 \Rightarrow u_1 = \frac{-z_1}{\beta}$ And $u_2 = \frac{-z_2}{\beta}$

$\overrightarrow{\Delta u} = \vec{w} - \vec{\vec{u}} \Rightarrow$ The necessary optimality is $(G_0'(\vec{u}), \overrightarrow{\Delta u}) \ge 0$

$\Rightarrow (\vec{z} + \beta\vec{u}, \overrightarrow{\Delta u}) \ge 0$

$\Rightarrow (\vec{z} + \beta\vec{u}, \vec{u}) \le (\vec{z} + \beta\vec{u}, \vec{w}), \forall\vec{w} \in (L^2(\Sigma))^2$

Conclusion

The Galerkin method is suitable to prove the existence of a unique solution of a couple of linear parabolic partial differential when the continuous boundary control vector is fixed. The existence theorem of a continuous classical boundary optimal control vector governing by the considered couple of linear partial differential equations of parabolic type is developed and proved. The existence and uniqueness solution of the couple of adjoint equations associated with the considered couple equations of the state are studied through derivation the Fréchet derivative. The necessary conditions theorem for optimality of the problem is developed and proved.

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