Al-Mustansiriyah Journal of Science

Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print) 2521-3520 (electronic)

ORIGINAL ARTICLE *OPEN ACCESS*

Quaternary Boundary Optimal Control Problem for Quaternary Nonlinear Elliptic System with Constraints

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ARTICLE INFO
Received: March 27, 2024
Revised: June 14, 2024
Accepted: June 23, 2024
Published:September 30, 2024

ABSTRACT: ***Background:*** *Boundary optimal control problems governed by nonlinear elliptic systems are complex, involving equality and inequality constraints.* ***Objective:*** *This paper examines a quaternary boundary optimal control vector problem (QBOCVP) regulated by a quaternary nonlinear elliptic system (QNES) and subject to equality and inequality constraints (EINC).* ***Methods:*** *A weak formulation of the QBOCVP is developed, along with a mathematical representation of the quaternary adjoint equations (QAEs) associated with the QNES.* ***Results:*** *An existence theorem for a QBOCV addressing the constrained problem is established and rigorously proven under appropriate assumptions. The QAEs corresponding to the QNES are mathematically formulated. The Fréchet derivative (FD) of the cost function (CF) and the EINC is also derived. Furthermore, the necessary condition theorem (NCTH) and the sufficient condition theorem (SCTH) for optimality are presented and proved.* ***Conclusions:*** *This work provides a rigorous analysis of the QBOCVP with EINC controlled by QNES. It establishes the existence theorem and optimality conditions, providing a theoretical framework for addressing such constrained problems.*

KEYWORDS: *Fréchet derivative; Optimality conditions; Quaternary classical boundary control vector; Nonlinear elliptic system*

INTRODUCTION

Optimal control problems (OCPs) have been widely applied across various real-world domains, including medicine [1], economics [2], robotics [3], and aircraft [4], among others. Over time, researchers have shown significant interest in studying OCPs broadly, with a particular focus on optimal classical continuous control problems (OCCCPs). In the last decade, considerable attention has been directed toward OCCCPs governed by three main types of nonlinear partial differential equations (NPDES): elliptic [5], hyperbolic [6], and parabolic [7].

Subsequently, this research extended to systems governed by coupled NPDES of these three types [8]–[10], and later to systems controlled by triple NPDES of these types [11].

These advancements motivated us to investigate the quaternary boundary optimal control vector problem (QBOCVP) with equality and inequality constraints (EINC) governed by quaternary nonlinear elliptic systems (QNES), a topic previously unexplored by other researchers. In this study, we establish and prove an existence theorem for a QBOCV under appropriate assumptions. Additionally, the mathematical formulation of the quaternary adjoint equations (QAEs) associated with QNES is presented. The Fréchet derivative (FD) for the cost function (CF) and the EINC is derived. Finally, the necessary condition theorem (NCTH) and the sufficient condition theorem (SCTH) for optimality are formulated and proven.

DESCRIPTION OF THE PROBLEM

Let $\Omega \subset R^2$ be an open and bounded subset with boundary $\partial\Omega$, the QBOCVP includs the QNES:

$$C_1y_1 + c_1y_1 + \sigma_1y_2 + \sigma_2y_3 + \sigma_3y_4 + a_1(x, y_1) = \rho_1(x) \ \ in \ \Omega \tag{1}$$

$$C_2y_2 - \sigma_1y_1 + c_2y_2 + \sigma_4y_3 - \sigma_5y_4 + a_2(x, y_2) = \rho_2(x) \ \ in \ \Omega \tag{2}$$

$$C_3y_3 - \sigma_2y_1 - \sigma_4y_2 + c_3y_3 - \sigma_6y_4 + a_3(x, y_3) = \rho_3(x) \ \ in \ \Omega \tag{3}$$

$$C_4y_4 + \sigma_3y_1 + \sigma_5y_2 + \sigma_6y_3 + c_4y_4 + a_4(x, y_4) = \rho_4(x) \ \ in \ \Omega \tag{4}$$

with

$$\frac{\partial y_r}{\partial n_r} = \sum_{i,j=1}^{2} c_{rij}\frac{\partial y_r}{\partial x_j}\cos(n_r, x_i) = w_r, \forall r = 1,2,3,4 \text{ on } \partial\Omega, \tag{5}$$

where $C_ry_r = -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(c_{rij}(x)\frac{\partial y_r}{\partial x_j}\right)$, $\forall r = 1,2,3,4 \ on \ \partial\Omega$, $\forall \ i,j = 1,2, c_r, \rho_r \in L_2(\Omega)$, for $r = 1,2,3,4, \sigma_l = \sigma_l(x) \in L_\infty(\Omega)$,$\forall l = 1,2,3,4,5,6$, $\vec{y} = (y_1, y_2, y_3, y_4) \in (H_2(\Omega))^4$ is the quaternary state vector solution (QSVS), $\vec{w} = (w_1, w_2, w_3, w_4) \in L_2(\partial\Omega) = (L_2(\partial\Omega))^4$ denotes its corresponding QBCV, the functions $a_r(x, y_r)$ and $\rho_r(x)$ ($\forall \ r = 1,2,3,4$) will be defined later.

The QBCV Constraint is $\vec{U} = \{\vec{u} \in L_2(\partial\Omega) : \vec{u} \in \vec{W} \ a.e \ in \ \partial\Omega\}$, with $W_1 \times W_2 \times W_3 \times W_4 = \vec{W} \subset R^4$ is convex and bounded .

The CF ($r = 1,2,3,4$) is

$$H_0(\vec{w}) = \sum_{r}^{4}\int_\Omega h_{0r}(x, y_r)\,dx + \sum_{r=1}^{4}\int_{\partial\Omega}\bar{h}_{0r}(x, w_r)\,d\omega, \tag{6}$$

the constraints on QBCV are

$$H_1(\overrightarrow{w}) = \sum_{r}^{4}\int_\Omega h_{1r}(x, y_r)\,dx + \sum_{r}^{4}\int_{\partial\Omega}\bar{h}_{1r}(x, w_r)\,d\omega, \tag{7}$$

$$H_2(\vec{w}) = \sum_{r}^{4}\int_\Omega h_{2r}(x, y_r)\,dx + \sum_{r}^{4}\int_{\partial\Omega}\bar{h}_{2r}(x, w_r)\,d\omega, \tag{8}$$

the set of the admissible QBCV is

$$\vec{U}_A = \{\vec{w} \in \vec{U} \mid H_1(\vec{w}) = 0, H_2(\vec{w}) \le 0\}.$$

The QBOCVP is to minimize (6) subject to the EINC (7)-(8), i.e. to find $\vec{w} \in \vec{U}_A$, s.t.:

$$H_0(\vec{w}) = \min_{\vec{u}\in\vec{U}_A} H_0(\vec{u}).$$

The Weak Formulation (WF) of the QNES
Let $\vec{V} = V_1 \times V_2 \times V_3 \times V_4 = (H_1(\Omega))^4 = H_1(\Omega)$, where

$$H_1(\Omega) = \{\vec{v} : \vec{v} \in H_1(\Omega), \text{ with } \frac{\partial v_r}{\partial n} = w_r, r = 1,2,3,4, \text{ on } \partial\Omega\}.$$

The WF of (1)-(5) is

$$\begin{aligned}&C_1(y_1, v_1) + (c_1y_1, v_1)_\Omega + (\sigma_1y_2, v_1)_\Omega + (\sigma_2y_3, v_1)_\Omega - \\ &(\sigma_3y_4, v_1)_\Omega + (a_1(y_1), v_1)_\Omega = (\rho_1, v_1)_\Omega + (w_1, v_1)_{\partial\Omega}\end{aligned} \tag{9}$$

$$C_2(y_2, v_2) + (c_2y_2, v_2)_\Omega - (\sigma_1y_1, v_2)_\Omega + (\sigma_4y_3, v_2)_\Omega - (\sigma_5y_4, v_2)_\Omega + (a_2(y_2), v_2)_\Omega = (\rho_2, v_2)_\Omega + (w_2, v_2)_{\partial\Omega} \tag{10}$$

$$C_3(y_3, v_3) + (c_3y_3, v_3)_\Omega - (\sigma_2y_1, v_3)_\Omega + (\sigma_4y_2, v_3)_\Omega - (\sigma_6y_4, v_3)_\Omega + (a_3(y_3), v_3)_\Omega = (\rho_3, v_3)_\Omega + (w_3, v_3)_{\partial\Omega} \tag{11}$$

$$C_4(y_4, v_4) + (c_4y_4, v_4)_\Omega - (\sigma_3y_1, v_4)_\Omega + (\sigma_5y_2, v_4)_\Omega - (\sigma_6y_3, v_4)_\Omega + (a_4(y_4), v_4)_\Omega = (\rho_4, v_4)_\Omega + (w_4, v_4)_{\partial\Omega} \tag{12}$$

Adding the above four equalities (9)-(12), to get

$$C(\vec{y}, \vec{v}) + (a_1(y_1), v_1)_\Omega + (a_2(y_2), v_2)_\Omega + (a_3(y_3), v_3)_\Omega + (a_4(y_4), v_4)_\Omega = (\rho_1, v_1)_\Omega + (w_1, v_1)_{\partial\Omega} + (\rho_2, v_2)_\Omega + (w_2, v_2)_{\partial\Omega} + (\rho_3, v_3)_\Omega + (w_3, v_3)_{\partial\Omega} + (\rho_4, v_4)_\Omega + (w_4, v_4)_{\partial\Omega} \tag{13}$$

Presumption 1:

P1) $\frac{C(\vec{y},\vec{y})}{\|\vec{y}\|_V} \geq d_1\|\vec{y}\|_V, \forall\vec{y} \in \vec{V}$.

P2)$|\ C(\vec{y}, \vec{v})| \leq d_2\|\vec{y}\|_V\|\vec{v}\|_V, \forall \vec{y}, \vec{v} \in \vec{V}$.

P3) $a_r(x, y_r)$ and $\rho_r(x)\,(\forall r = 1, 2, 3, 4)$ are of Carathéodory type (CTHT) on $\Omega \times R$ and on Ω resp. and satisfy $(\forall r = 1, 2, 3, 4)$.

$$|a_r(x, y_r)| \leq \theta_r(x) + d_r|y_r|, |\rho_r(x)| \leq \bar{\theta}_r(x), \text{ with } \theta_r, \ \bar{\theta}_r \in L_2(\Omega), \forall (x, y_r) \in \Omega \times R \times W_r, \ d_r \geq 0.$$

P4) $a_r(x, y_r)$ is monotone w.r.t. y_r resp. $\forall\ x \in \Omega$.

P5) $a_r(x, 0) = 0, \ \forall\ x \in \Omega, \forall r = 1, 2, 3, 4$

Theorem 1 [12]: In additions to the Presumption 1, if a_1 is strictly monotone. Then the WF 13 has a unique QSVS $\vec{y} \in \vec{V}$, for a given QBCV $\vec{w} \in \vec{U}_A$.

Lemma 1 [12]: In addition to the Presumption 1, if the function a_r is Lipchitz (LIP) $(\forall r = 1, 2, 3, 4)$ w.r.t. y_r resp., the function $\rho_r(\forall r = 1, 2, 3, 4)$ is bounded . Then the operator $\vec{w} \to \vec{y}_{\vec{w}}$ from $\vec{U}_A$ to $L_\mathbf{2}(\partial\Omega)$ is LIP continuous (LIPC) , i.e., $\|\Delta\vec{y}\|_\Omega \leq L\|\Delta\vec{w}\|_{\partial\Omega}, L > 0$.

Presumptions 2: Assume that h_{lr} and $\bar{h}_{lr}$ are of CTHT on $\Omega \times R$ and $\Omega{\times}W_r$ resp., for $r = 1, 2, 3, 4$, $l = 0, 1, 2$, and satisfy :

$$|h_{lr}(x, y_r)| \leq \beta_{lr}(x) + d_{lr}y_r^2, \left|\bar{h}_{lr}(x, w_r)\right| \leq \bar{\beta}_{lr}(x) + \bar{d}_{lr}w_r^2$$

where $(y_r, w_r) \in R \times W_r$ with $\beta_{lr}, \bar{\beta}_{lr} \in L_1(\partial\Omega)$ and $d_{lr}, \bar{d}_{lr} > 0$.

Lemma 2 [12]: With Presumption 2, the functional $\vec{w} \mapsto H_l(\vec{w})$, for each l=0,1,2 , defines on $L_\mathbf{2}(\partial\Omega)$ is continuous.

Theorem 2 [8]: In addition to the assumptions. (A) and (B), if $\vec{U}$ in the $\vec{W}_A$ is compact, $\vec{W}_A \neq \emptyset$. If for each $i = 1, 2, 3, G_1(\vec{u})$ is independent of $u_i, G_0(\vec{u})$ and $G_2(\vec{u})$ are convex w.r.t u_i, for fixed (x, t, y_i). Then there exists a CCCBOTCV for the considered problem.

Proposition 1 [13]: Let f and $f_y : D \times \mathbb{R}^n \to \mathbb{R}^m$ are of CTHT, let $F : L^p(D) \to \mathbb{R}$ be a functional, s.t $F(y) = \int_D f(x, y(x))\,dx$, where $D \subset \mathbb{R}^{\,d}$, and $\forall (x, y) \in D \times \mathbb{R}^{\,n}, p, q \neq 0 : \|f_y(x, y)\| \leq \zeta(x) + \eta(x)\|y\|^{\frac{\beta}{q}}$, where $\zeta \in L^q(D \times R)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta \in L^{\frac{pq}{p-\beta}}(D \times\ R), \beta \in [0, p]$ if $p \neq \infty, and\ \eta \equiv 0$, if $p = \infty$. Then the FD of F exists $\forall y \in L^p(D \times \mathbb{R}^{\,n})$ and is given by $\acute{G}(y)\,h = \int_D f_y(x, y(x))\,h(x)dx$.

RESULTS AND DISCUSSION

Existence of an QBOCV

Theorem 3: In addition to the Presumptions 1 and 2, assume that $\vec{U}_A \neq \emptyset$, H_1 is independent of w_r, (for $r = 1, 2, 3, 4$), $\bar{h}_{lr}$ is convex w.r.t. w_r for fixed x resp. for $r = 1, 2, 3, 4$ and $l = 0, 2$. Then there exists a QBOCV.

Proof: The continuity of $H_l(\vec{w})$ (for each $l = 0, 1, 2$) on $L_\mathbf{2}(\Omega)$ is obtained by Lemma 2.

Now, since for each $r = 1, 2, 3, 4$, that $y_{rn} \xrightarrow{S} y_r$ in $L_2(\Omega)$, (by using the same technique which is used in the proof of theorem 1, for more details see ref. [12]), then

$$H_1(\vec{w}_n) = \sum_{r=1}^{4} \int_\Omega h_{1r}(x, y_{rn})\, dx \to \sum_{r=1}^{4} \int_\Omega h_{1r}(x, y_r)\, dx = H_1(\vec{w})$$

But $H_1(\vec{w}_n) = 0$, for each n, hence $H_1(\vec{w}) = 0$.

From the other side, since $\int_\Omega h_{lr}(x, y_{rn})(\forall l = 0, 2$ and $\forall r = 1, 2, 3, 4)$ is continuous w.r.t. y_r, and $\vec{W}$ is compact, hence $h_l(y_r)$ is satisfied the presumptions of Theorem 2, to get that

$$\sum_{r=1}^{4} \int_\Omega h_{lr}(x, y_{rn})\, dx \to \sum_{r=1}^{4} \int_\Omega h_{lr}(x, y_r)\, dx, \forall l = 0, 2.$$

Since $\int_\Omega h_{l1}(x, y_1)dx(\int_{\partial\Omega} \bar{h}_{l1}(x, w_1)d\omega)$ is continuous w.r.t. y_1 (w.r.t. w_1 and is weakly lower semi continuous (W.L.S.C) w.r.t. w_1), i.e.

$$\begin{aligned}\int_\Omega h_{lr}(x, y_r)\, dx + \int_{\partial\Omega} \bar{h}_{lr}(x, w_r)\, d\omega &\leq \int_\Omega h_{lr}(x, y_r)\, dx + \lim_{n\to\infty} \inf \int_{\partial\Omega} \bar{h}_{lr}(x, w_{rn})\, d\omega \\ &= \lim_{n\to\infty} \inf \int_\Omega \left[h_{lr}(x, y_r) - \bar{h}_{lr}(x, y_{rn})\right] dx + \\ &\liminf_{n\to\infty} \int_\Omega h_{lr}(x, y_{rn})\, dx + \lim_{n\to\infty} \int_{\partial\Omega} \bar{h}_{lr}(x, w_{rn})\, d\omega \\ &= \lim_{n\to\infty} \inf \left[\int_\Omega h_{lr}(x, y_{rn})\, dx + \int_{\partial\Omega} \bar{h}_{lr}(x, w_{rn})\, d\omega\right]\end{aligned}$$

Hence $H_l(\vec{w})$ is W.L.S.C w.r.t. $(\vec{y}, \vec{w})$, $\forall\, l = 0, 2$.

Then $H_2\left(\overrightarrow{w}_n\right) \leq \lim_{n\to\infty} H_2\left(\overrightarrow{w}_n\right) = 0$,

Beside these results, one has

$$H_0\left(\vec{\vec{w}}\right) \leq \lim_{n\to\infty} inf H_0(\vec{w}_n) = \lim_{n\to\infty} H_0(\vec{w}_n) = \min_{\vec{w}\ \epsilon \vec{U}_A} H_0(\vec{w})$$

$\vec{\vec{w}}$ is a QBOCV

The NCTH and THE SCTH for Optimality

The following Presumptions are useful to study the NCTH and SCTH.

Presumptions 3:

P1) a_{1yr}, (for $r = 1, 2, 3, 4$) is of CTHT on $\Omega \times R$ and satisfies $|a_{1yr}(x, y_r)| \leq \widetilde{d}_r, a_{1yr}(x, y_r) \geq 0, for\ x \in \Omega$ and $\widetilde{d}_r \geq 0$.

P2) ρ_r(for $r = 1, 2, 3, 4$) is of the CTHT type on Ω and satisfies: $|\ \rho_r(x)| \leq \check{d}_r$, for $x \in \Omega$ and $\check{d}_r \geq 0$

P3) h_{lyr} and $\bar{h}_{lwr}$, $(r = 1, 2, 3, 4\&l = 0, 1, 2)$ are of the CTHT type on $\Omega \times R$ and satisfy $|h_{lryr}| \leq \beta_{lr} + d_{lr}\,|y_r|$, and $\left|\bar{h}_{lrwr}\right| \leq \bar{\beta}_{lr} + \bar{d}_{lr}\,|w_r|$ with $d_{lr}, \bar{d}_{lr} \geq 0,\ \beta_{lr}, \bar{\beta}_{lr} \in L_2(\Omega),\ r = 1, 2, 3, 4\ and\ \ l = 0, 1, 2.$

Theorem 4: With Presumptions (1,2 and 3), the Hamiltonian is defined by:

$$\chi\left(x, \vec{y}, \vec{z}, \vec{w}\right) = z_1\left(\rho_1\left(x\right) - a_1\left(x, y_1\right)\right) h_{01}\left(x,\ y_1\right) + \bar{h}_{01}\left(x,\ w_1\right) + z_2\left(\rho_2\left(x\right) - a_2\left(x, y_2\right)\right) + h_{02}\left(x,\ y_2\right) + \bar{h}_{02}\left(x,\ w_2\right) + z_3((\rho_3((x) - a_3\left(x, y_3\right)) + h_{03}\left(x,\ y_3\right) + \bar{h}_{03}\left(x,\ w_3\right) + z_4\left(\rho_4\left(x\right) - a_4\left(x, y_4\right)\right) + h_{04}\left(x,\ y_4\right) + \bar{h}_{04}\left(x,\ w_4\right)$$

The QAES of (1)-(5) are given by

$$C_1 z_1 + c_1 z_1 - \sigma_1 z_2 - \sigma_2 z_3 - \sigma_3 z_4 + z_1 a_{1y_1}\left(x, y_1\right) = h_{01y_1}\left(x, y_1\right) \tag{14}$$

$$C_2 z_2 + \sigma_1 z_1 + c_2 z_2 - \sigma_4 z_3 + \sigma_5 z_4 + z_2 a_{2y_2}\left(x, y_2\right) = h_{02y_2}\left(x, y_2\right) \tag{15}$$

$$C_3 z_3 + \sigma_2 z_1 + \sigma_4 z_2 + c_3 z_3 + \sigma_6 z_4 + z_3 a_{3y_3}\left(x, y_3\right) = h_{03y_3}\left(x, y_3\right) \tag{16}$$

$$C_4 z_4 - \sigma_3 z_1 - \sigma_5 z_2 - \sigma_6 z_3 + c_4 z_4 + z_4 a_{4y_4}\left(x, y_4\right) = h_{04y_4}\left(x, y_4\right) \tag{17}$$

$$\frac{\partial z_r}{\partial n} = 0\ \ ,\ \forall\ r = 1, 2, 3, 4\ on\ \partial\Omega \tag{18}$$

Then the FD of H_0 is given by

$$\dot{\vec{H}}_0(\vec{w})\Delta\vec{w} = \int_{\partial\Omega} \chi_{\vec{w}}'^{T} \cdot \Delta\vec{w}\,d\omega, \text{ where } \chi'_{\vec{w}} = \begin{pmatrix} \chi'_{w_1}(x, \vec{y}, \vec{z}, \vec{w}) \\ \chi_{w_2}(x, \vec{y}, \vec{z}, \vec{w}) \\ \chi'_{w_3}(x, \vec{y}, \vec{z}, \vec{w}) \\ \chi'_{w_4}(x, \vec{y}, \vec{z}, \vec{w}) \end{pmatrix} = \begin{pmatrix} z_1 + \bar{h}_{01w_1} \\ z_2 + \bar{h}_{02w_2} \\ z_3 + \bar{h}_{03w_3} \\ z_4 + \bar{h}_{04w_4} \end{pmatrix}$$

Where $\vec{y} = (y_1, y_2, y_3, y_4), \vec{z} = (z_1, z_2, z_3, z_4), and \vec{w} = (w_1, w_2, w_3, w_4)$.

Proof: The WF of the QAEs (14)-(18) is:

$$C_1(z_1, v_1) + (c_1 z_1, v_1)_\Omega - (\sigma_1 z_2, v_1)_\Omega - (\sigma_2 z_3, v_1)_\Omega - (\sigma_3 z_4, v_1)_\Omega + (z_1 a_{1y_1}(x, y_1), v_1)_\Omega = (h_{01y_1}(x, y_1), v_1)_\Omega, \tag{19}$$

$$C_2(z_2, v_2) + (\sigma_1 z_1, v_2)_\Omega + (c_2 z_2, v_2)_\Omega - (\sigma_4 z_3, v_2)_\Omega + (\sigma_5 z_4, v_2)_\Omega + (z_2 a_{2y_2}(x, y_2), v_2)_\Omega = (h_{02y_2}(x, y_2), v_2)_\Omega, \tag{20}$$

$$C_3(z_3, v_3) + (\sigma_2 z_1, v_3)_\Omega + (\sigma_4 z_2, v_3)_\Omega + (c_3 z_3, v_3)_\Omega + (\sigma_6 z_4, v_3)_\Omega + (z_3 a_{3y_3}(x, y_3), v_3)_\Omega = (h_{03y_3}(x, y_3), v_3)_\Omega, \tag{21}$$

$$C_4(z_4, v_4) - (\sigma_3 z_1, v_4)_\Omega - (\sigma_5 z_2, v_4)_\Omega - (\sigma_6 z_3, v_4)_\Omega + (c_4 z_4, v_4)_\Omega + (z_4 a_{4y_4}(x, y_4), v_4)_\Omega = (h_{04y_4}(x, y_4), v_4)_\Omega, \tag{22}$$

Adding the above four equality, using $\vec{v} = \Delta\vec{y}$, to get

$$C_1(z_1, \Delta y_1) + (c_1 z_1, \Delta y_1)_\Omega - (\sigma_1 z_2, \Delta y_1)_\Omega - (\sigma_2 z_3, \Delta y_1)_\Omega - (\sigma_3 z_4, \Delta y_1)_\Omega + C_2(z_2, \Delta y_2) + (\sigma_1 z_1, \Delta y_2)_\Omega + (c_2 z_2, \Delta y_2)_\Omega - (\sigma_4 z_3, \Delta y_2)_\Omega + (\sigma_5 z_4, \Delta y_2)_\Omega + C_3(z_3, \Delta y_3) + (\sigma_2 z_1, \Delta y_3)_\Omega + (\sigma_4 z_2, \Delta y_3)_\Omega + (c_3 z_3, \Delta y_3)_\Omega + (\sigma_6 z_4, \Delta y_3)_\Omega + C_4(z_4, \Delta y_4) - (\sigma_3 z_1, \Delta y_4)_\Omega - (\sigma_5 z_2, \Delta y_4)_\Omega - (\sigma_6 z_3, \Delta y_4)_\Omega + (c_4 z_4, \Delta y_4)_\Omega + (z_1 a_{1y_1}(x, y_1), \Delta y_1)_\Omega + (z_2 a_{2y_2}(x, y_2), \Delta y_2)_\Omega + (z_3 a_{3y_3}(x, y_3), \Delta y_3)_\Omega + (z_4 a_{4y_4}(x, y_4), \Delta y_4)_\Omega = (h_{01y_1}(x, y_1), \Delta y_1)_\Omega + (h_{02y_2}(x, y_2), \Delta y_2) + (h_{03y_3}(x, y_3), \Delta y_3)_\Omega + (h_{04y_4}(x, y_4), \Delta y_4)_\Omega \tag{23}$$

Using the QSVS $\vec{y}$ in the WF of the QNES (9)-(12) resp. once and once again the QSVS $\vec{y}+\Delta\vec{y}$ resp. Then subtracting each obtained equation from the other, and then using $\vec{v}=\vec{z}$ in the resulting equation , to obtain

$$\begin{aligned}&C_1(\Delta y_1,z_1)+(c_1\Delta y_1,z_1)_\Omega+(\sigma_1\Delta y_1,z_1)_\Omega+(\sigma_1\Delta y_2,z_1)_\Omega+(\sigma_2\Delta y_3,z_1)_\Omega+(\sigma_3\Delta y_4,z_1)_\Omega+\\&C_2(\Delta y_2,z_2)+(c_2\Delta y_2,z_2)_\Omega-(\sigma_1\Delta y_1,z_2)_\Omega+(\sigma_4\Delta y_3,z_2)_\Omega-(\sigma_5\Delta y_4,z_2)_\Omega+\\&C_3(\Delta y_3,z_3)+(c_3\Delta y_3,z_3)_\Omega-(\sigma_2\Delta y_1,z_3)_\Omega-(\sigma_4\Delta y_3,z_3)_\Omega-(\sigma_6\Delta y_4,z_3)_\Omega+\\&C_4(\Delta y_4,z_4)+(c_4\Delta y_4,z_4)_\Omega+(\sigma_3\Delta y_1,z_4)_\Omega+(\sigma_5\Delta y_2,z_4)_\Omega+(\sigma_6\Delta y_3,z_4)_\Omega+(a_1(y_1+\Delta y_1)-\\&a_1(y_1),z_1)_\Omega+(a_2(y_2+\Delta y_2)-a_2(y_2),z_2)_\Omega+(a_3(y_3+\Delta y_3)-a_3(y_3),z_3)_\Omega+(a_4(y_4+\Delta y_4)-\\&a_4(y_4),z_4)_\Omega=(\Delta w_2,z_2)_\Omega+(\Delta w_2,z_2)_\Omega+(\Delta w_3,z_3)_\Omega+(\Delta w_4,z_4)_\Omega\end{aligned}\tag{24}$$

From Presumptions (P1&P3) on $a_r(\forall\ r=1,2,3,4)$, and Proposition 1, the FD of a_r exists, i.e.

$$\int_\Omega\left(a_r(x,y_r+\Delta y_r)-a_r(x,y_r)\right)z_r dx=(a_{ry_r},\Delta y_r,z_r)+\tilde{\delta}_r(\Delta\vec{w})\|\Delta\vec{w}\|_{\partial\Omega}$$

By replacing this result in (24), to obtain

$$\begin{aligned}&C_1(\Delta y_1,z_1)+(c_1\Delta y_1,z_1)_\Omega+(\sigma_1\Delta y_1,z_1)_\Omega+(\sigma_2\Delta y_3,z_1)_\Omega+(\sigma_3\Delta y_4,z_1)_\Omega+\\&C_2(\Delta y_2,z_2)+(c_2\Delta y_2,z_2)_\Omega-(\sigma_1\Delta y_1,z_2)_\Omega+(\sigma_4\Delta y_3,z_2)_\Omega-(\sigma_5\Delta y_4,z_2)_\Omega+\\&C_3(\Delta y_3,z_3)+(c_3\Delta y_3,z_3)_\Omega-(\sigma_2\Delta y_1,z_3)_\Omega-(\sigma_4\Delta y_2,z_3)_\Omega-(\sigma_6\Delta y_4,z_3)_\Omega+\\&C_4(\Delta y_4,z_4)+(c_4\Delta y_4,z_4)_\Omega+(\sigma_3\Delta y_1,z_4)_\Omega+(\sigma_5\Delta y_2,z_4)_\Omega+(\sigma_6\Delta y_3,z_4)_\Omega+\\&(a_1y_1\Delta y_1,z_1)_\Omega+\tilde{\delta}_1(\Delta\tilde{w})\|\Delta\tilde{w}\|_{\partial\Omega}+(a_2y_2\Delta y_2,z_2)_\Omega+\tilde{\delta}_2(\Delta\tilde{w})\|\Delta\tilde{w}\|_{\partial\Omega}+\\&(a_3y_3\Delta y_3,z_3)_\Omega+\tilde{\delta}_3(\Delta\tilde{w})\|\Delta\tilde{w}\|_{\partial\Omega}+(a_4y_4\Delta y_4,z_4)_\Omega+\tilde{\delta}_4(\Delta\tilde{w})\|\Delta\tilde{w}\|_{\partial\Omega}=\\&(\Delta w_2,z_2)_\Omega+(\Delta w_2,z_2)_\Omega+(\Delta w_3,z_3)_\Omega+(\Delta w_4,z_4)_\Omega\end{aligned}\tag{25}$$

Subtracting (23) from (25), it gives

$$\sum_{r=1}^{4}(h_{0ry_r}(x,y_r),\Delta y_r)_\Omega+\check{\delta}_5(\Delta\vec{w})\|\Delta\vec{w}\|_{\partial\Omega}=\sum_{r=1}^{4}(\Delta w_r,z_r)_{\partial\Omega}\tag{26}$$

with

$$\check{\delta}_5(\Delta\vec{w})\|\Delta\vec{w}\|_{\partial\Omega}=\sum_{r=1}^{4}\check{\delta}_r(\Delta\vec{w})\|\Delta\vec{w}\|_{\partial\Omega}$$

From Presumption 3 and Lemma 1,

$$\begin{aligned}&H_0(\vec{w}+\overrightarrow{\Delta w})-H_0(\vec{w})=\\&\sum_{r=1}^{4}\int_\Omega h_{0ry_r}(x,y_1)\Delta y_r\,dx+\sum_{r=1}^{4}\int_{\partial\Omega}\bar{h}_{0rw_r}(x,w_r)\Delta w_r\,d\omega+\check{\delta}_6(\overrightarrow{\Delta w})\|\overrightarrow{\Delta w}\|_{\partial\Omega}\end{aligned}\tag{27}$$

where $\check{\delta}_6(\overrightarrow{\Delta w})\to 0$ and $\|\overrightarrow{\Delta w}\|_{\partial\Omega}\to 0$ as $\overrightarrow{\Delta w}\to 0$. From (26) and (27), it yields

$$\begin{aligned}&H_0(\vec{w}+\overrightarrow{\Delta w})-H_0(\vec{w})=\\&\sum_{r=1}^{4}\int_{\partial\Omega}\left(z_r+\bar{h}_{0rw_r}\right)\Delta w_r\,d\omega+\check{\delta}_7(\overrightarrow{\Delta w})\|\overrightarrow{\Delta w}\|_{\partial\Omega}\\&\text{where }\check{\delta}_7(\overrightarrow{\Delta w})\|\overrightarrow{\Delta w}\|_{\partial\Omega}=\check{\delta}_6(\overrightarrow{\Delta w})\|\overrightarrow{\Delta w}\|_{\partial\Omega}-\sum_{r=1}^{4}\check{\delta}_r(\overrightarrow{\Delta w})\|\overrightarrow{\Delta w}\|_{\partial\Omega}\end{aligned}\tag{28}$$

But from definition of the FD of H_0, we obtain

$$H_0(\vec{w}+\overrightarrow{\Delta w})-H_0(\vec{w})=H_0(\vec{w})\overrightarrow{\Delta w}+\check{\delta}_7(\overrightarrow{\Delta w})\|\overrightarrow{\Delta w}\|_{\partial\Omega} \tag{29}$$

Finally, (28) & (29), gives

$$\vec{H}_0\,(\vec{w})\cdot\overrightarrow{\Delta w}=\int_{\partial\Omega}\chi_{\vec{W}}'^{T}\cdot\overrightarrow{\Delta w}d\omega \text{ where}$$

$$\chi_{\vec{w}}'=\begin{pmatrix}\chi_{w_1}'(x,\vec{y},\vec{z},\vec{w})\\ \chi_{w_2}'(x,\vec{y},\vec{z},\vec{w})\\ \chi_{w_3}'(x,\vec{y},\vec{z},\vec{w})\\ \chi_{w_4}'(x,\vec{y},\vec{z},\vec{w})\end{pmatrix}=\begin{pmatrix}z_1+\bar{h}_{01w_1}\\ z_2+\bar{h}_{02w_2}\\ z_3+\bar{h}_{03w_3}\\ z_4+\bar{h}_{04w_4}\end{pmatrix}$$

Theorem 5: The NCTH for optimality

i) Under the Presumptions 1,2 and 3 , if $\vec{w}\in\vec{U}_A$ is a QBOCV , then there exist "multiplies" $\gamma_l\in R, l=0,1,2$ with $\gamma_0,\gamma_2\geq 0, \sum_{l=0}^{2}|\gamma_l|=1$, for which the following Kuhn-Tucker-Lagrange conditions(K-T-L-C) are held (for $\overrightarrow{\Delta w}=\vec{u}-\vec{w}$)

$$\int_{\partial\Omega}\chi_{\vec{w}}'^{T}\cdot\Delta\vec{w}d\omega\geq 0,\forall\vec{u}\in\vec{U}, \tag{30a}$$

where $\bar{h}_{rw_r}=\sum_{l=0}^{2}\gamma_l\bar{h}_{lrw_r}, z_r=\sum_{l=0}^{2}\gamma_l z_{rl}, (r=1,2,3,4)$ in the definition of the χ (Theorem 4), and also

$$\gamma_2 H_2(\vec{w})=0, \tag{30b}$$

ii) Inequality (30a) is equivalent to

$$\chi_{\vec{w}}'^{T}\cdot\vec{w}=\min_{\vec{u}\in\vec{U}_A}\chi_{\vec{w}}'^{T}\cdot\vec{u}\qquad \text{a. e. on }\partial\Omega. \tag{31}$$

Proof: i) From Lemma 2, $H_l\,(\vec{w})$ is continuous "in an open neighbored "and it is $\rho-$ local continuous at each $\vec{w}\in\vec{U}$ for each $l=0,1,2$ for each ρ. Also from Theorem 2, $H_l\,(\vec{w})$ has a continuous FD (for each $l=0,1,2$) at each $\vec{w}\in\vec{U}$, hence $H_l\,(\vec{w})$ is ρ- differentiable there for each ρ. Since $\vec{w}\in\vec{U}_A$ is QBOCV, then by the K-T-L-C with $\gamma_l\in R$, $l=0,1,2$, with $\gamma_0\geq 0$, $\gamma_2\geq 0$, $\sum_{l=0}^{2}|\gamma_l|=1$, one has

$$\left(\left(\gamma_0\dot{\vec{H}}_0'(\vec{w})+\gamma_1'\,\vec{H}_1'\,(\vec{w})+\gamma_2\,\dot{\vec{H}}_2'(\vec{w})\right),(\vec{u}-\vec{w})\right)_{\partial\Omega}\geq 0,\forall\vec{u}\in\vec{U} \tag{32a}$$

and

$$\gamma_2\mathrm{H}_2(\vec{w})=0 \tag{32b}$$

Utilizing Theorem 4 , to find the FD of H_l, for $l=0,1,2$, in (32a), with setting $\Delta w_r=u_r-w_r$, for $r=1,2,3,4$, to get

$$\sum_{r=1}^{4}\int_{\partial\Omega}\left[(\gamma_0 z_{0r}+\gamma_1 z_{1r}+\gamma_2 z_{2r})+\left(\gamma_0\bar{h}_{0rw_r}+\gamma_1\bar{h}_{1rw_r}+\gamma_2\bar{h}_{2rw_r}\right)\Delta w_r\right]d\omega\geq 0$$

$$\Rightarrow\sum_{r=1}^{4}\int_{\partial\Omega}\left[z_r+\bar{h}_{rw_r}\Delta w_r\right]d\omega\geq 0,\ \text{with } z_r=\sum_{l=0}^{2}\gamma_l z_{rl},\bar{h}_{rw_r}=\sum_{l=0}^{2}\gamma_l\bar{h}_{lrw_r}\Rightarrow$$

$$\int_{\partial\Omega}\chi_{\vec{w}}'^{T}\cdot\overrightarrow{\Delta w}d\omega\geq 0,\forall\vec{u}\in\vec{U},\overrightarrow{\Delta w}=\vec{u}-\vec{w}.$$

 DOI: https://doi.org/10.23851/mjs.v35i4.1546

ii) First, let $\vec{U} = \{\vec{u} \in L_2(\partial\Omega, R) \mid u_r(x) \in W_r$, a.e. on $\partial\Omega\}$, with $W_r \subset R, \mu$ is a Lebesgue measure on $\partial\Omega$, $\{\vec{w}_n\}$ be a dense sequence in $\vec{U}_A$ and let $S \subset \partial\Omega$ be a measurable set, s.t.

$$\vec{u}(x) = \begin{cases} \vec{w}_n(x), & \text{if } x \in S \\ \vec{w}(x), & \text{if } x \notin S \end{cases}$$

Hence (30a) becomes

$$\int_S \chi_{\vec{w}}^{\prime T} \cdot (\vec{w}_n - \vec{w})\, ds \geq 0, \text{ for each } S \subset \partial\Omega$$

Then from Egorov's Theorem [13] once get that

$$\chi_{\vec{w}}^{\prime T} \cdot (\vec{w}_n - \vec{w}) \geq 0, \text{ a.e. on } \partial\Omega. \text{ hence}$$

$$\chi'_{\vec{w}}(x, \vec{y}, \vec{z}, \vec{w}) \cdot (\vec{w}_n - \vec{w}) \geq 0, \text{ in } Q = \cap_n Q_n, \text{ where } Q_n = \partial\Omega - \partial\Omega_n \text{ with } \mu(\partial\Omega_n) = 0.$$

And this hold for each n, since Q is independent of n and

$$\mu(\partial\Omega/Q) = \mu\left(\cup_{n=1}^{\infty} \partial\Omega_n\right) = 0$$

But $\{\vec{w}_n\}$ is dense in $\vec{U}$, then

$$\chi_{\vec{w}}^{\prime T} \cdot (u - \vec{w}) \geq 0 \text{ in } Q, \text{ i.e. a. e. on } \partial\Omega, \text{ or}$$
$$\chi_{\vec{w}}^{\prime T} \cdot \vec{w} = \min_{\vec{u} \in \vec{U}_A} \chi_{\vec{w}}^{\prime T} \cdot \vec{u}, \text{ a. e. on } \partial\Omega.$$

The converse is obtained directly.

Theorem 6: In addition to the Presumptions 1,2, and 3), if a_r, h_{1r} are affine w.r.t. $y_r, \bar{h}_{1r}$ is affine w.r.t. w_r, ρ_r is bounded for each x, and $h_{lr}, \bar{h}_{lr}(r = 1, 2, 3, 4, l = 0, 2)$ are convex w.r.t. y_r and w_r resp. for each x. Then the NCTH in Theorem 5 , with $\gamma_0 > 0$ is also sufficient.

Proof: From proof of Theorem 5, one has that

$$\int_{\partial\Omega} \chi'_{\vec{w}}(x, z_r, w_r) \cdot \overrightarrow{\Delta w} d\omega \geq 0, \forall r = 1, 2, 3, 4, \forall \vec{u} \in \vec{U}$$

Now, assume $\vec{w} \in \vec{U}_A$, and let $H(\vec{w}) = \sum_{l=0}^{2} \gamma_l H_l(\vec{w}), , \forall r = 1, 2, 3, 4.$ then

$$\dot{\vec{H}}(\vec{w})\overrightarrow{\Delta w} = \sum_{l=0}^{2} \gamma_l \dot{\vec{H}}_l(\vec{w})\overrightarrow{\Delta w} = \sum_{l=0}^{2}\sum_{r=1}^{4} \int_{\partial\Omega} \gamma_l \left(z_{rl} + h_{lrw_r}\right) \Delta w_r d\omega$$
$$= \int_{\partial\Omega} \chi'_{\vec{w}}(x, \vec{z}, \vec{w}) \cdot \overrightarrow{\Delta w} d\omega \geq 0$$

From the Presumptions on a_r, $\forall\ r = 1, 2, 3, 4$.

$$a_r(x, y_r) = a_{r1}(x)y_1 + a_{r2}(x),$$

Let w_r and $\bar{w}_r (\forall r = 1, 2, 3, 4)$ are given QBCV, hence $y_r = y_{rw_r}$ and $\bar{y}_r = \bar{y}_{r\bar{w}_r}, (\forall r = 1, 2, 3, 4)$ are their conforming QSVS (Theorem 1), i.e.

$$\mathrm{C}_1 y_1 + c_1 y_1 + \sigma_1 y_2 + \sigma_2 y_3 + \sigma_3 y_4 + a_{11}(x)y_1 + a_{12}(x) = \rho_1(x), \tag{33a}$$
$$C_2 y_2 - \sigma_1 y_1 + c_2 y_2 + \sigma_4 y_3 - \sigma_5 y_4 + a_{21}(x)y_2 + a_{22}(x) = \rho_2(x), \tag{33b}$$
$$C_3 y_3 - \sigma_2 y_1 - \sigma_4 y_2 + c_3 y_3 - \sigma_6 y_4 + a_{31}(x)y_3 + a_{32}(x) = \rho_3(x), \tag{33c}$$
$$C_4 y_4 + \sigma_3 y_1 + \sigma_5 y_2 + \sigma_6 y_3 + c_4 y_4 + a_{41}(x)y_4 + a_{42}(x) = \rho_4(x), \tag{33d}$$
$$\frac{\partial y_r}{\partial n_r} = \sum_{i,j=1}^{2} c_{rij} \frac{\partial \partial_r}{\partial x_j} \cos(n_r, x_i) = w_r, \forall r = 1, 2, 3, 4 \tag{33e}$$

and

$$C_1\bar{y}_1 + c_1\bar{y}_1 + \sigma_1\bar{y}_2 + \sigma_2\bar{y}_3 + \sigma_3\bar{y}_4 + a_{11}(x)\bar{y}_1 + a_{12}(x) = \rho_1(x), \tag{34a}$$
$$C_2\bar{y}_2 - \sigma_1\bar{y}_1 + c_2\bar{y}_2 + \sigma_4\bar{y}_3 - \sigma_5\bar{y}_4 + a_{21}(x)\bar{y}_2 + a_{22}(x) = \rho_2(x), \tag{34b}$$
$$C_3\bar{y}_3 - \sigma_2\bar{y}_1 - \sigma_4\bar{y}_2 + c_3\bar{y}_3 - \sigma_6\bar{y}_4 + a_{31}(x)\bar{y}_3 + a_{32}(x) = \rho_3(x) \tag{34c}$$
$$C_4\bar{y}_4 + \sigma_3\bar{y}_1 + \sigma_5\bar{y}_2 + \sigma_6\bar{y}_3 + c_4\bar{y}_4 + a_{41}(x)\bar{y}_4 + a_{42}(x) = \rho_4(x), \tag{34d}$$
$$\frac{\partial \bar{y}_r}{\partial n_r} = \sum_{i,j=1}^{2} c_{rij}\frac{\partial \bar{y}_r}{\partial x_j}\cos(n_r, x_i) = \bar{w}_r, \forall r = 1,2,3,4 \tag{34e}$$

Multiplying (33) by $\alpha \in [0,1]$ and (34) by $(1-\alpha)$, then combining the obtained equalities from each pair of $((33),(34))$, we get

$$\mathrm{C}_1(\alpha y_1 + (1-\alpha)\bar{y}_1) + c_1(\alpha y_1 + (1-\alpha)\bar{y}_1) + \sigma_1(\alpha y_2 + (1-\alpha)\bar{y}_2) + \sigma_2(\alpha y_3 + (1-\alpha)\bar{y}_3) + \sigma_3(\alpha y_4 + (1-\alpha)\bar{y}_4) + a_{11}(x)(\alpha y_1 + (1-\alpha)\bar{y}_1) + a_{12}(x) = \rho_1(x) \tag{35a}$$

$$\mathrm{C}_2(\alpha y_2 + (1-\alpha)\bar{y}_2) - \sigma_1(\alpha y_1 + (1-\alpha)\bar{y}_1) + c_2(\alpha y_2 + (1-\alpha)\bar{y}_2) + \sigma_4(\alpha y_3 + (1-\alpha)\bar{y}_3) - \sigma_5(\alpha y_4 + (1-\alpha)\bar{y}_4) + a_{21}(x)(\alpha y_2 + (1-\alpha)\bar{y}_2) + a_{22}(x) = \rho_2(x) \tag{35b}$$

$$C_3(\alpha y_3 + (1-\alpha)\bar{y}_3) - \sigma_2(\alpha y_1 + (1-\alpha)\bar{y}_1) - \sigma_4(\alpha y_2 + (1-\alpha)\bar{y}_2) + c_3(\alpha y_3 + (1-\alpha)\bar{y}_3) - \sigma_6(\alpha y_4 + (1-\alpha)\bar{y}_4) + a_{31}(x)(\alpha y_3 + (1-\alpha)\bar{y}_3) + a_{32}(x) = \rho_3(x) \tag{35c}$$

$$C_4(\alpha y_4 + (1-\alpha)\bar{y}_4) + \sigma_3(\alpha y_1 + (1-\alpha)\bar{y}_1) + \sigma_5(\alpha y_2 + (1-\alpha)\bar{y}_2) + \sigma_6(\alpha y_3 + (1-\alpha)\bar{y}_3) + c_4(\alpha y_4 + (1-\alpha)\bar{y}_4) + a_{41}(x)(\alpha y_4 + (1-\alpha)\bar{y}_4) + a_{42}(x) = \rho_4(x) \tag{35d}$$

$$\frac{\partial}{\partial n_r}(\alpha y_r + (1-\alpha)\bar{y}_r) = \sum_{i,j=1}^{2} c_{rij}\frac{\partial}{\partial x_j}(\alpha y_r + (1-\alpha)\bar{y}_r)\cos(n_r, x_i) = (\alpha w_r + (1-\alpha)\bar{w}_r), \quad \forall r = 1,2,3,4 \tag{35e}$$

It means the QBCV $\vec{\bar{\bar{w}}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \bar{\bar{w}}_3, \bar{\bar{w}}_4)$ with $\bar{\bar{w}}_r = \alpha w_r + (1-\alpha)\bar{w}_r$, has QSVS

$$\bar{\bar{y}}_r = y_{r\bar{\bar{w}}_r} = y_{r(\alpha w_r + (1-\alpha)\bar{w}_r)} = \alpha y_{rw_r} + (1-\alpha)y_r\bar{w}_r = \alpha y_r + (1-\alpha)\bar{y}_r, \text{ for each } r = 1,2,3,4, \text{ i.e.}$$

$$C_1\bar{\bar{y}}_1 + c_1\bar{\bar{y}}_1 + \sigma_1\bar{\bar{y}}_2 + \sigma_2\bar{\bar{y}}_3 + \sigma_3\bar{\bar{y}}_4 + a_{11}(x)\bar{\bar{y}}_1 + a_{12}(x) = \rho_1(x) \tag{36a}$$
$$C_2\bar{\bar{y}}_2 - \sigma_1\bar{\bar{y}}_1 + c_2\bar{\bar{y}}_2 + \sigma_4\bar{\bar{y}}_3 - \sigma_5\bar{\bar{y}}_4 + a_{21}(x)\bar{\bar{y}}_2 + a_{22}(x) = \rho_2(x) \tag{36b}$$
$$C_3\bar{\bar{y}}_3 - \sigma_2\bar{\bar{y}}_1 - \sigma_4\bar{\bar{y}}_2 + c_3\bar{\bar{y}}_3 - \sigma_6\bar{\bar{y}}_4 + a_{31}(x)\bar{\bar{y}}_3 + a_{32}(x) = \rho_3(x) \tag{36c}$$
$$C_4\bar{\bar{y}}_4 + \sigma_3\bar{\bar{y}}_1 + \sigma_5\bar{\bar{y}}_2 + \sigma_6\bar{\bar{y}}_3 + c_4\bar{\bar{y}}_4 + a_{41}(x)\bar{\bar{y}}_4 + a_{42}(x) = \rho_4(x) \tag{36d}$$
$$\frac{\partial \bar{y}_r}{\partial n_r} = \sum_{i,j=1}^{2} c_{rij}\frac{\partial \bar{y}_r}{\partial x_j}\cos(n_r, x_i) = \bar{\bar{w}}_r, \forall r = 1,2,3,4 \tag{36e}$$

i.e. the operator $w_r \to y_{rw_r}$ is convex-linear w.r.t. (y_r, w_r) resp. , for each $x \in \Omega$, and for each $r = 1,2,3,4$.

Also, from the Presumptions on $h_{1r}(x, y_r)$ and $\bar{h}_{1r}(x, y_r)$ for each $r = 1,2,3,4$ one gets that $H_l(\vec{w})$ is convex w.r.t. $(\vec{y}, \vec{w})$ for each $x \in \Omega$, and so $H(\vec{w})$ is convex w.r.t $(\vec{y}, \vec{w})$. On the other hand $H_l(\vec{w})$, for each , $l = 0,1,2$ has the FD and continuous for each $\vec{w} \in \vec{U}$, and $\vec{U}$ is convex. hence

$$\dot{\vec{H}}(\vec{w})\overrightarrow{\Delta w} \geq 0.$$

Thus, $H(\vec{w})$ has a minimum at $\vec{w}$, i.e.

$$H(\vec{w}) \leq H(\vec{u}), \forall \vec{u} \in \vec{U}$$
$$\Rightarrow \sum_{l=0}^{2} \gamma_l H_l(\vec{w}) \leq \sum_{l=0}^{2} \gamma_l H_l(\vec{u}) \tag{37}$$

Now, let $\vec{u} \in \vec{U}_{\vec{A}}$, then (37) becomes

$$\gamma_0 H_0(\vec{w}) + \gamma_2 H_2(\vec{w}) \leq \gamma_0 H_0(\vec{u}), \forall \vec{u} \in \vec{U},$$

and from (30b),

$$\gamma_0 H_0(\vec{w}) \leq \gamma_0 H_0(\vec{u}), \forall \vec{u} \in \vec{U} \Longrightarrow H_0(\vec{w}) \leq H_0(\vec{u}), \forall \vec{u} \in \vec{U}$$

i.e. $\vec{w}$ is a QBOC for the problem.

CONCLUSION

The existence theorem for a QBOCV that satisfies the EINC of the problem is established and proven under appropriate assumptions. The mathematical formulation of the AQEs associated with the QNES is derived, along with the Fréchet derivative for the CF and the EINC. Finally, the NCTH and SCTH for optimality are formulated and proven through the application of the Kuhn-Tucker-Lagrange Theorem.

SUPPLEMENTARY MATERIAL

None.

AUTHOR CONTRIBUTIONS

Alaa Khneab: Conceptualization, methodology, writing—original draft preparation, writing—review and editing, validation. Jamil Al-Hawasy: Conceptualization, methodology, writing—original draft preparation, writing—review and editing, validation, formal analysis, resources, visualization, supervision, and project administration. Ion Chryssoverghi: Validation.

FUNDING

None.

DATA AVAILABILITY STATEMENT

None.

ACKNOWLEDGMENTS

None.

CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

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