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On Best Multiplier Approximation of k −Monotone by Trigonometric Polynomial

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Abstract

The main goal of this paper is to study the degree of the best multiplier approximation of monotone unbounded functions in L_{p,λ_n} −space on the closed interval $[-\pi,\pi]$ by means of K-functional, which we represented with, $K(f,L_{p,\lambda_n},W^1_{p,\lambda_n},\widetilde{W}^1_{p,\lambda_n})$, defined by the W^1_{p,λ_n} and $\widetilde{W}^1_{p,\lambda_n}$ that are referred to during this research. In addition, we have established a set of definitions, concepts and some useful lemmas that are needed in our work.

Keywords: Multiplier integral, Multiplier averaged modulus of smoothness, Multiplier norm.

الخلاصة

ان الغرض الاساسي من هذا البحث هو الحصول على درجة افضل تقريب مضاعف للدوال الرتيبة الدورية غير المقيدة في الفضاء $L_{P,\lambda_n}(X)$ حيث ان $X=[-\pi,\pi]$ بواسطة استخدام $K(f,L_{p,\lambda_n},W^1_{p,\lambda_n},\widetilde{W}^1_{p,\lambda_n})$ معرفه بواسطة الدوال W^1_{p,λ_n} و $\widetilde{W}^1_{p,\lambda_n}$ والتي سيشار اليها خلال هذا البحث . علاوة على ذلك اسسنا مجموعه من التعاريف والمفاهيم والقضايا المساعدة المفيده والتي سنحتاج لها خلال هذا العمل.

INTRODUCTION

Many researchers and specialists have been interested in the problems of approximation and finding approximate solutions to them through analysis and study using multiple different methods according to type and shape of the functions and their properties, once for bounded functions and again for unbounded functions and in different spaces. For example, in 2010 [1], Guven, A. and Israfilav, D. M., studied trigonometric approximation of bounded functions in generalized Lebesque $L_{P,\alpha}(X)$ −space. Also in 2010 [2], Kadhim studied and obtained results with respect to approximation of bounded μ −measurable functions using trigonometric polynomials. On the other hand, there are those who studied approximation of unbounded functions such as in 2020 [3] where S. K. Jassim and A. Zoboon studied and discussed finding the best multiplier approximation of periodic unbounded functions using Fejer operators. In 2022 [4], S. K. Jassim and R. F. Hassan studied and obtained some results about finding the degree of the best one-sided multiplier approximation of periodic unbounded function in $L_{P,\Psi_n}(X)$ −space, where $X=[0,2\pi]$. They estimated the degree of the best one-sided multiplier approximation in terms of averaged modulus.

In this work, we study and discuss finding the degree of the best multiplier approximation of monotone periodic unbounded functions in $L_{P,\lambda_n}(X)$ −space, where $X=[-\pi,\pi]$ in terms of K −functional. Thus, some definitions, concepts, and important lemmas are needed in this work.

DEFINITIONS AND LEMMAS

Each of the following definitions that is recalled, we establish a similar definition listed below by dealing with certain conditions in our work and in the space we have defined.

Definition (1) [5]

A series $\sum_{a=0}^{\infty} a_n$ is called a multiplier convergent series if there is a convergent sequence of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ such that $\sum_{a=0}^{\infty} a_n \lambda_n \leq \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ is called multiplier for the convergence.

Let $L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$, be the space of all real valued unbounded function.

Definition (2)

For any real valued function $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$, if there is a sequence $\{\lambda_n\}_{n=0}^{\infty}$, such that:

$$\int_{-\pi}^{\pi} f(x)\lambda_n dx < \infty \qquad (1)$$

Then f is called a Multiplier integrable function, λ_n is called a Multiplier integrable sequence.

Definition (3)

A [6]. Let $f \in L_p[a,b]$, where $1 \leq p \leq \infty$, be all bounded functions' space by the norm

$$\| f \|_p = [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} < \infty \qquad (2)$$

B. Let $f \in L_{p,\lambda_n}(X)$,where $X = [-\pi, \pi]$, then: $\| f \|_{p,\lambda_n}$, is given by the below definite, Multiplier integral norm:

$$\| f \|_{p,\lambda_n} = [\int_{-\pi}^{\pi} |(\lambda_n f)(x)|^p dx]^{\frac{1}{p}} \qquad (3)$$

Definition (4) [7]

Let $f \in L_p[a,b]$, where $1 \leq p \leq \infty$, then the integral modulus (L_p-modulus or p-modulus) of order k of the function f is the following function of $\delta \in [0, (b-a)/k]$:

$$\omega_k(f,\delta)_{L_p} = sup_{0 \leq h \leq \delta} \left\{ \int_a^{b-kh} \left|\Delta_h^k f(x)\right|^p dx \right\}^{1/p} \qquad (4)$$

Definition (5)

Let $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$, $1 \leq p < \infty$. Thus, the following formula defines the multiplier integral modulus of order κ of the function f where $0 \leq \delta \leq b - a\kappa$, according to:

$$\omega_k(f,\delta)_{p,\lambda_n} = \sup_{h \in [0,\delta]} (\int_a^{b-kh} | \triangle_h^k (\lambda_n f)(x)|^p dx)^{\frac{1}{p}} \qquad (5)$$

Where:

$$\triangle_h^k (\lambda_n f)(x) = \sum_{m=i}^{k} (-1)^{m+k} \binom{k}{m} (\lambda_n f) \cdot (x+mh); \binom{k}{m} = \frac{k!}{m!\,(k-m)!} \qquad (6)$$

Definition (6) [7]

Let $f \in L_p(X)$, where $X = [a,b]$ and $1 \leq p \leq \infty$. The local modulus of smoothness of the function f of order k at a point $x \in [a,b]$ is the following function of $\delta \in [0, (b-a)/k]$:

$$\omega_\kappa(f,x;\delta) = sup\{|\Delta_h^\kappa f(t)|: t, t+\kappa h \in \left[x - \frac{\kappa\delta}{2}, x + \frac{\kappa\delta}{2}\right] \cap [a,b]\right\} \qquad (7)$$

Definition (7)

If $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$, $1 \leq p < \infty$, then the multiplier local smoothness modulus of an order k function f at a point $x \in [a,b]$, $0 \leq \delta \leq \frac{b-a}{k}$, according to:

$$\omega_k(f,x,\delta)_{p,\lambda_n} = \sup_{h \in [0,\delta]} \{\triangle_h^k (\lambda_n f)(t): t, t+kh \in [x - \frac{k\delta}{2}, x + \frac{k\delta}{2}] \cap [a,b]\} \qquad (8)$$

Definition (8) [7]

The smoothness of order k is averaged modulus (or τ-modulus) of the function $f \in M[a,b]$ is the following function of $\delta \in [0, (b-a)/\kappa]$:

$$\tau_k(f;\delta)_p = \|\omega_k(f,.,;\delta)\|_{L_p} = |(\omega_k(f,x;\delta))^p dx|^{1/p} \qquad (9)$$

Definition (9)

If $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$, $1 \leq p < \infty$, then the multiplier's averaged order k of $f \in L_{p,\lambda_n}(X)$, smoothness modulus is defined by:

$$\tau_k(f,\delta)_{p,\lambda_n} = \| \omega_k(f,.,\delta) \|_{p,\lambda_n} = \left(\int_a^b [\omega_k(\lambda_n f, x, \delta)]^p dx\right)^{\frac{1}{p}} \quad (10)$$

Definition (10) [7]
If $f \in L_p(X), X = [a,b]$, then:

$$E_n(f)_p = inf\{\|f - P_n\|_p : P_n \in P\} \quad (11)$$

such that $E_n(f)_P$ is referred to as the degree of the best polynomial P_n approximation of f a monotone multiplier.

Definition (11)
If $f \in L_{\rho,\lambda_n}(X), X = [-\pi,\pi]$, then:

$$E_n(f)_{P,\lambda_n} = inf\{\|f - S_n\|_{P,\lambda_n} : S_n \in P\} \quad (12)$$

like that $E_n(f)_{P,\lambda_n}$ is called the degree of the best monotone multiplier approximationof f by polynomial S_n.

Definition (12) [7]
If $f \in L_\rho(X)$, then using trigonometric polynomials of order n in $L_p(X)$, the best one-sided approximation of f is as follows:

$$\tilde{E}_n(f)_\rho = inf\{\|P - Q\|_P : P, Q \in T, Q(x) \le f(x) \le P(x); \forall x\} \quad (13)$$

Definition (13)
If $f \in L_p(X), X = [-\pi,\pi]$, then:

$$\tilde{E}_n(f)_{p,\lambda_n} = inf\{\|S_n - G_n\|_{P,\lambda_n} : S_n, G_n \in T, G_n(x) \le f(x) \le S_n(x);\ \forall x\} \quad (14)$$

such that $\tilde{E}_n(f)_{p,\lambda_n}$ is referred to as the degree of the most accurate polynomial approximation of f the one-sided monotone multiplier, S_n and G_n.

Definition (14)
Let $f \in \mathrm{L}_{p,\lambda_n}(\mathrm{X})$, where $X = [-\pi,\pi]$, $n \in \mathbb{N}$, and

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} cos\ ku \quad (15)$$

be the Dirichlet kernel for u. Note that

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} cos\ ku = \frac{sin\ (n+\frac{1}{2})u}{2sin\ (\frac{u}{2})} = \frac{sin\ (2n+1)\ {}^{u}/_{2}}{2sin\ (\frac{u}{2})} \quad (16)$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(u)du = 1 \quad (17)$$

where:

$$K_n(u) = \left[\frac{sin\ (n+1)\ {}^{u}/_{2}}{sin^2\ (\frac{u}{2})}\right]\frac{2}{2(n+1)}, \quad n = 0,1,2,\dots. \quad (18)$$

This means

$$K_n(u) = \frac{1}{n+1}[D_0(u) + D_1(u) + \cdots + D_n(u)] = \frac{1}{n+1}\sum_{k=0}^{n} D_k(u) \quad (19)$$

Now to prove the following: $(1) D_n(u) = \frac{sin\ (2n+1)^{u}/_{2}}{2sin\ (\frac{u}{2})}$ $(2) \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(u)du = 1$ for every $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi,\pi]$.

Proof
To show that (1) holds, we start with:

$$sin\ (2n+1)u = sin\ \frac{u}{2} + sin\ \frac{3u}{2} - sin\ \frac{u}{2} + sin\ \frac{5u}{2} - sin\ \frac{3u}{2} + \cdots + sin\ \frac{2n+1}{2}u - sin\ (2n-1)\frac{u}{2}$$

$$sin\ (2n+1)u = sin\frac{u}{2} + sin\ \frac{u}{2}\cdot 2cos\ u + 2sin\ \frac{u}{2}\cdot cos\ u + 2sin\frac{u}{2}\cdot cos\ 3u + \cdots + sin\ \frac{u}{2}\cdot cos\ nu$$

$$sin\ (2n+1)u = sin\frac{u}{2}(1 + 2\sum_{k=1}^{n} cos\ ku) = sin\ (1 + 2\sum_{k=1}^{n} cos\ ku) \quad (20)$$

so that

$$\frac{\sin\,(2n+1)\,{}^{u}\!/_{2}}{\sin\,{}^{u}\!/_{2}} = (1 + 2\sum_{k=1}^{n} \cos ku)$$

then

$$\frac{\sin\,(2n+1)\,{}^{u}\!/_{2}}{2\sin\,{}^{u}\!/_{2}} = \frac{1}{2} + \sum_{k=1}^{n} \cos ku$$

$$= D_n(u) \qquad (21)$$

Now, to show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(u)du = 1 \qquad (22)$$

Proof

We start the proof with the below definite integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(u)du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + 2\sum_{k=1}^{n}\cos ku\right]du$$

Then

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[1 + 2(\cos u + \cos 2u + \cdots + \cos nu)]du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\left[\frac{1}{2} + (\cos u + \cos 2u + \cdots + \cos nu)\right]du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + (\cos u + \cos 2u + \cdots + \cos nu)\right]du$$

$$= \frac{1}{\pi}\left[\frac{1}{2}u + \sin u + \frac{1}{2}\sin 2u + \cdots + \frac{1}{n}\sin nu\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{2}(\pi - (-\pi)) + 0 + + \cdots + 0\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}(2\pi)\right] = 1 \qquad (23)$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)du = 1$$

Now, to show that (2) holds start with:

$$K_n(u) = \frac{1}{n+1}[D_0(u) + D_1(u) + \cdots + D_n(u)]$$

$$= \frac{1}{n+1}\sum_{k=0}^{n} D_k(u)$$

and since $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)du = 1$, for $n \in N$, then:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)du = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{n+1}(D_o(u) + D_1(u) + \cdots + D_n(u))\right]du$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K(u)du = \frac{1}{n+1}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} D_0(u)du + \frac{1}{\pi}\int_{-\pi}^{\pi} D_1(u)du + \cdots + \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)du\right]$$

$$= \frac{1}{n+1}[1 + 1 + \cdots + 1]_{n+1 times}$$

$$= \frac{1}{n+1}(n+1) = 1$$

the proof is completed.

Note:

Let

$$J_n(\lambda_n f, x) = \frac{2}{n+1}\sum_{k=0}^{n}(\lambda_n f)(x_k)K_n(x - x_n) \qquad (24)$$

Now for $x_k = x_{k,n}$, let

$$\sigma_n = (f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi}(\lambda_n f)(x+n) \cdot K_n(u)du \qquad (25)$$

Thus for $\sigma_n \in T_n$ and for $f \in L_{p,\lambda_n}(X), X = [-\pi, \pi], 1 \le P < \infty$.

Lemma (1)

For $f \in L_{p,\lambda_n}(X), X = [-\pi, \pi]$, $1 \le P < \infty$, then

$$\|\sigma_n(f)\|_{P,\lambda_n} \le C\|f\|_{P,\lambda_n} \qquad (26)$$

Where $\sigma_n \in T_n$, C is a constant.

Proof

Subsequently

$$\|\sigma_n(f)\|_{p,\lambda_n} = \left[\int_{-\pi}^{\pi}\left[\frac{1}{\pi}\int_{-\pi}^{\pi}((\lambda_n f)(x + u)K_n(u)du)^p\,dx\right]\right] \qquad (27)$$

By using Minkowski's inequality [8] which is given by the following:
aimed at $f, g \in L_p(X)$ where $X = [a, b]$, we have:

$$\left[\int_a^b |f(x) + g(x)|^p dx\right]^{1/p} \leq \left[\int_a^b |f(x)|^p dx\right]^{1/p} + \left[\int_a^b |g(x)|^p dx\right]^{1/p}, p > 1 \quad (28)$$

and for $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)$ are proportional, we have:

$$\left[\sum_{k=1}^{n} |a_k + b_k|^p\right]^{1/p} \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{1/p}, p > 1, a_k, b_k > 0 \quad (29)$$

Hence, we achieve:

$$\|\sigma_n(f)\|_{p,\lambda_n} \leq \left[\int_{-\pi}^{\pi} \frac{1}{\pi} \int_{-\pi}^{\pi} |f\lambda_n(x+u)|^p \, dx \cdot K_n(t)dt\right]^{1/p}$$

$$\|\sigma_n(f)\|_{p,\lambda_n} \leq \frac{1}{\pi}\left[\int_{-\pi}^{\pi} |(f\lambda_n)(x+u)|^p dx \cdot C\right]^{1/p}$$

$$\|\sigma_n(f)\|_{p,\lambda_n} \leq \left[\int_{-\pi}^{\pi} |(f\lambda_n)(x+u)|^p dx\right]^{1/p} \cdot C(P) = C(P)\|f\|_{p,\lambda_n}$$

Then

$$\|\sigma_n(f)\|_{p,\lambda_n} \leq C\|f\|_{p,\lambda_n} \quad (30)$$

the proof is completed.

Definition (15)

Let $f \in L_{p,\lambda_n}(X)$, where $X = [a, b]$, let us define the following:

1. $W^1_{p,\lambda_n} = \{f \in L_{p,\lambda_n}(X), X = [a, b]\}, \acute{f}$ is $a$absolutely continuous.
2. $\widetilde{W}^1_{p,\lambda_n} = \{\tilde{f} \in L_{p,\lambda_n}(X), X = [a, b]\}, \tilde{f}'$ is $a$absolutely continuous, where $\tilde{f}$ is conjugate of function f.

While

$$W^1_p = \{f \in L_p(X), X = [a, b], \acute{f} \text{ is absolutely continuous}\} \quad (31)$$

and

$$\widetilde{W}^1_p = \{\tilde{f}^1 \in L_p(X), X = [a, b], \tilde{f}' \text{ is absolutely continous}\} \quad (32)$$

where $\tilde{f}$ is conjucate of the function f.
Note: Jenson Inequality [9]:
For $P \geq 1$ and $a_i > 0$, and for every $i = 1, 2, \ldots, n$, there are:

$$\left|\sum_{i=1}^{n} a_i b_i\right|^p \leq \sum_{i=1}^{n} a_i |B_i|^p \quad (33)$$

Where $\sum_{i=1}^{n} a_i = 1$.

Lemma (2)

For

$$J_n(f, x) = \frac{2}{n+1} \sum_{k=0}^{n} f(x_\kappa) K_n \cdot (x - x_\kappa) \quad (34)$$

Where:

$$K_n(x - x_k) = \frac{\sin^2 (n+1)(x - x_k)}{2n \sin^2 \left[{(x - x_k)}/{2}\right]} \quad (35)$$

Then

$$\|J_n(f, x)\|_{\rho,\lambda_n} \leq C\|f\|_{\rho,\lambda_n} \quad (36)$$

Proof

With the aid of Jensen's inequality [9], we get:

$$\|J_n(f, x)\|_{p,\lambda_n} = \left[\int_{-\pi}^{\pi} \left|\frac{2}{n+1} \sum_{k=0}^{n} f(x_k) K_n(x - x_k)\right|^p dx\right]^{1/p}$$

$$\leq \left[\left|\frac{2}{n+1} \sum_{k=0}^{n} f(x_k)\right|^p \cdot \int_{-\pi}^{\pi} |K_n(x - x_k)| dx\right]^{1/p} \quad (37)$$

Since

$$\int_{-\pi}^{\pi} |K_n(x - x_n)| dx \leq 1$$

then

$$\|J_n(f,x)\|_{p,\lambda_n} \leq \left[\left|\frac{2}{n+1}\sum_{k=0}^{n} f(x)\right|^p\right]^{1/p} O(1)$$
$$\leq \|f\|_{p,\lambda_n} O(1) \qquad (38)$$

Hence:

$$\|J_n(f,x)\|_{p,\lambda_n} \leq C\|f\|_{p,\lambda_n}$$

Definition (16)

For $f \in L_{p,\lambda_n}(X)$ where $X = [a,b]$ for $g \in W^1_{p,\lambda_n} \cap \widetilde{W}^1_{p,\lambda_n}$, let us define the K-functional as follows:

$$K(f, L_{p,\lambda_n}, W^1_{p,\lambda_n}, \widetilde{W}^1_{p,\lambda_n}) = inf\left\{\|fh\|_{p,\lambda_n} + (\left\|\acute{h}\right\|_{p,\lambda_n} + \left\|\tilde{h}'\right\|_{p,\lambda_n}) : h \in T_n\right\},$$
$$1 \leq p < \infty \qquad (39)$$

RESULTS AND DISCUSSION

In this Section, all above lemmas are used to state and prove the theorems of the best multiplier approximation of monotone of $f \in L_{P,\lambda_n}(X) -$ space, where $X = [-\pi, \pi]$ in terms of K −functional.

Theorem (1)

For $f \in W^1_{p,\lambda_n}, n \in \mathbb{N}, 1 \leq p < \infty$, then:

$$\|f - \sigma_n(f)\|_{P,\lambda_n} \leq \frac{C}{n}\left\{\|f'\|_{P,\lambda_n} + \left\|\widetilde{f}'\right\|_{P,\lambda_n}\right\} (40)$$

Proof

Let $P_n \in T_n$ be the best trigonometric polynomial for f such that:

$$\|f - P_n(f)\|_{p,\lambda_n} = E_n^T(f)_{p,\lambda_n} \qquad (41)$$

Now

$$\|f - \sigma_n(f)\|_{P,\lambda_n} \leq \|f - P_n(f)\|_{P,\lambda_n} + \|P_n(f) - (\sigma_n f)\|_{P,\lambda_n}$$
$$\leq E_n^T(f)_{P,\lambda_n} + C\|P_n(f) - \sigma_n(f)\|_{P,\lambda_n}$$
$$\leq E_n^T(f)_{P,\lambda_n} + C\|P_n(f) - f\|_{P,\lambda_n} + C\|f - \sigma_n(f)\|_{P,\lambda_n} \qquad (42)$$

Now for $P \in T_n, f \in \widetilde{W}^1$, from [9] it is known that:

$$n\|f - \sigma_n(f)\|_P \leq C\left\|\tilde{f}'\right\|_P$$

And

$$\|J_n(P) - \sigma_n(P)\|_P = \frac{1}{n+1}[\tilde{P}'cos\,(n+1)x - P'sin\,(n+1)x]$$

Thus:

$$n\|f - \sigma_n(f)\|_{P,\lambda_n} \leq C\left\|\tilde{f}'\right\|_{P,\lambda_n} \qquad (43)$$

And

$$\|J_n(P) - \sigma_N(P)\|_{p,\lambda_n} = \frac{1}{n+1}[\tilde{P}'cos\,(n+1)x - P'sin\,(n+1)x] \qquad (44)$$

Thus

$$\|f - \sigma_n(f)\|_{p,\lambda_n} \leq E_n^T + C\|P_n(f) - f\|_{p,\lambda_n} + \frac{C}{n}\left\|\tilde{f}'\right\|_{p,\lambda_n} \qquad (45)$$

using the inequality [9]

$$E_n^T(f) \leq \frac{C}{n}\left\|\acute{f}\right\|_{p,\lambda_n} \qquad (46)$$

$$\|f - \sigma_n(f)\|_{p,\lambda_n} \leq \frac{C}{n}\left\|\acute{f}\right\|_{p,\lambda_n} + C\|P_n(f) - f\|_{p,\lambda_n} + \frac{C}{n}\left\|\tilde{f}'\right\|_{p,\lambda_n}$$
$$\leq \frac{C}{n}\left\{\|f\|_{p,\lambda_n} + \left\|\tilde{f}'\right\|_{p,\lambda_n}\right\} \quad (47)$$

Using $\acute{\sigma}_n(f) = \sigma_n(\acute{f})$ and $\tilde{\sigma}_n(f) = \sigma_n(\tilde{f})$, thus:

$$\|f - \sigma_n(f)\|_{p,\lambda_n} \leq \frac{C}{n}\left\{\left\|\acute{f}\right\|_{p,\lambda_n} + \left\|\tilde{f}'\right\|\right\}$$
$$\leq \|f - \sigma_n(f)\|_{p,\lambda_n} + \frac{C}{n}\cdot \left\{\|\acute{\sigma}_n(f)\|_{p,\lambda_n} + \|\tilde{\sigma}_n'(f)\|_{p,\lambda_n}\right\}$$

Let $g = \sigma_n(f), g \in T_n$, be the trigonometric polynomial:

$$\|f - \sigma_n(f)\|_{p,\lambda_n} \leq \|f - g\|_{p,\lambda_n} + \frac{C}{n}\cdot \left\{\|\acute{g}\|_{p,\lambda_n} + \|\tilde{g}\|_{p,\lambda_n}\right\}$$
$$= CK(f, \frac{1}{n}, L_{p,\lambda_n}, W^1_{p,\lambda_n}, \widetilde{W}_{p,\lambda_n}) \qquad (48)$$

Theorem (2)

For $f \in L_{p,\lambda_n}(X), X = [-\pi, \pi], 1 \leq P < \infty$ and we have $K(f, \frac{1}{n}, L_{p,\lambda_n}, W^1_{p,\lambda_n}, \widetilde{W}^1_{p,\lambda_n})$, of K-functional on T_n, then:

$$E_n^T(f)_{p,\lambda_n} \leq CK(f, \frac{1}{n}, L_{p,\lambda_n}, W^1_{p,\lambda_n}, \widetilde{W}^1_{p,\lambda_n}) (49)$$

Proof

$$E_n^T(f)_{p,\lambda_n} = inf\{\|f - g\|_{p,\lambda_n} : g \in W^1_{p,\lambda_n} \cap \widetilde{W}^1_{p,\lambda_n}\} \qquad (50)$$

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Let $\acute{h} \in W^1_{p,\lambda_n}, \tilde{h}' \in \widetilde{W}^1_{p,\lambda_n}$ and since $\frac{1}{n}(\|\acute{g}\|_{p,\lambda_n} + \|\tilde{g}'\|_{p,\lambda_n}) \geq 0$, we have:

$$E^T_n(f)_{p,\lambda_n} = inf\{\|f-g\|_{p,\lambda_n}: g \in W^1_{p,\lambda_n} \cap \widetilde{W}^1_{p,\lambda_n}\}$$
$$\leq inf\left\{\|f-g\|_{p,\lambda_n} + \frac{1}{n}(\|\acute{g}\|_{p,\lambda_n} + \|\tilde{g}'\|_{p,\lambda_n})\right\}$$
$$= inf\left\{\|f-g\|_{p,\lambda_n} + \frac{1}{n}\|\acute{g}\|_{p,\lambda_n} + \frac{1}{n}\|\tilde{g}'\|_{p,\lambda_n}\right\}$$
$$= K(f, \frac{1}{n}, L_{p,\lambda_n}, W^1_{p,\lambda_n}, \widetilde{W}^1_{p,\lambda_n}) \quad (51)$$

Then

$$E^T_n(f)_{p,\lambda_n} \leq CK(f, \tfrac{1}{n}, L_{p,\lambda_n}, W^1_{p,\lambda_n}, \widetilde{W}^1_{p,\lambda_n})$$

the proof is completed.

Theorem (3)

For $f \in L_{P,\lambda_n}(X), X = [-\pi, \pi], 1 \leq P < \infty$, then:

$$\|f - \sigma_n(f)\|_{p,\lambda_n} \leq CE^T_n(f)_{p,\lambda_n} \quad (52)$$

Proof

Let $P \in T_n$ be the best trigonometric polynomial for f.

$$\|f - \sigma_n(f)\|_{p,\lambda_n} = \|f - P + P - \sigma_n(f)\|_{p,\lambda_n}$$
$$\leq \|f-P\|_{p,\lambda_n} + \|P - \sigma_n(f)\|_{p,\lambda_n}$$
$$= \|f-P\|_{p,\lambda_n} + \|\sigma_n(f) - \sigma_n(P)\|_{p,\lambda_n}$$
$$\leq E^T_n(f)_{p,\lambda_n} + \|\sigma_n(f-P)\|_{,\lambda_n}$$
$$\leq E^T_n(f)_{p,\lambda_n} + C\|f-P\|_{p,\lambda_n} \leq E^T_n(f)_{p,\lambda_n} + CE^T_n(f)_{p,\lambda_n} \leq CE^T_n(f)_{p,\lambda_n} \quad (53)$$

Hence, we get:

$$\|f - \sigma_n(f)\|_{p,\lambda_n} \leq CE^T_n(f)_{p,\lambda_n}$$

The proof is completed.

CONCLUSIONS

In this paper, we have established several definitions and proved some useful lemmas that we recall in our proofs. We obtain the degree of the best multiplier approximation of monotone unbounded periodic functions $f \in L_{p,\lambda_n}|-\pi,\pi|$-space in terms of k-functional.

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