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The Compactness of the Family of $A(z)$ −Analytic Functions

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ABSTRACT

In recent years, the Beltrami equation has garnered the attention of numerous researchers for the study of its analytical properties. Therefore, in this paper, we investigate certain properties of an analogue of the Cauchy integral for *A(z)*-analytic functions, utilizing the analytical properties of the Beltrami equation. Additionally, we obtain the compactness conditions for a family of functions within an *A(z)*-lemniscate.

KEYWORDS: $A(z)$ −analytic function, Cauchy Integral Formula, Taylor Series.

الخلاصة

في السنوات الأخيرة، لقت معادلة بلترامي انتباه العديد من الباحثين لدراسة خصائصها التحليلية. في هذه المقالة، نحقق في بعض خصائص نظير تكامل كوشي للوظائف التحليلية A(z) بتوظيف معادلة بلترامي. كذلك حصلنا على شروط الانضغاط لعائلة من الدوال في $A\ (z) - lemniscate$.

INTRODUCTION

Eugenio Beltrami [1], an Italian mathematician, built a local model of Lobachevski's geometry on the pseudo sphere in 1868, proving that Lobachevski's geometry is a consistent theory as Euclidean geometry. He was the first to consider the form of an elliptic system of partial differential equations:

$$\begin{Bmatrix} \alpha u_x + \beta u_y = v_y \\ \beta u_x + \gamma u_y = -v_x \end{Bmatrix} \quad (1)$$

− Beltrami equation

where $\alpha(x,y)\gamma(x,y) - \beta^2(x,y) = 1$.

These equations' homeomorphic solutions determine quasiconformal mappings. *Otto Teichmüller* solved the famous problem of the moduli of Riemann surfaces by employing quasiconformal mappings in the early forties of the last century [2], and thus interest in the Beltrami equations. The application of their generalizations to tomography problems has sparked interest in the Beltrami equation. The current work is devoted to the investigation of some properties of Beltrami equation solutions using recently discovered approaches. Recent studies have focused on the Beltrami equation in the theory of generalized analytic functions, for example, one may see [3], [4] and [5].

The derivatives formula for $A(z)$ −analytic functions.

We can consider the basis $(z, \bar{z})$ in the space $\mathbb{C} \approx R^2$ that associates the basis (x, y). Then we have $z = x + iy$, $\bar{z} = x - iy$, and we can find the formal derivatives by comparing the equalities.

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial}{\partial x}\frac{1}{2} + \frac{\partial}{\partial y}\frac{1}{2i},$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{1}{2} + \frac{\partial}{\partial y}\frac{-1}{2i}.$$

Let $w = u + iv$, then the Beltrami equation can be written in complex form:

$$w_{\bar{z}} = A(z)w_z \qquad (2)$$

where $A(z) = -\frac{p(z)-1}{p(z)+1}e^{2i\theta(z)}$, $p = \frac{\alpha+\gamma}{2} + \sqrt{(\frac{\alpha-\gamma}{2})^2 + \beta^2}$, $p \geq 1$, $0 \leq \theta \leq \pi$.

Note that the complex representation of the Cauchy-Riemann equations has the form $f_{\bar{z}} = 0$. The Beltrami equation then becomes the Cauchy-Riemann equations when $A(z) = 0$. Green's formula can be written using formal derivatives as:

$$\frac{1}{2i}\int_{\partial G} f(\zeta)d\,\zeta = \iint_G \frac{\partial f}{\partial \bar{z}}dxdy \qquad (3')$$

$$\frac{1}{2i}\int_{\partial G} f(\zeta)d\,\bar{\zeta} = -\iint_G \frac{\partial f}{\partial z}dxdy \qquad (3'')$$

Green's formula can be written using formal derivatives as, it is required that the function $f(z)$ have continuous first-order partial derivatives. The Beltrami equation was generalized in [1] to the case where $w(z)$ is a vector-valued function and $A(z)$ is a linear operator in the corresponding vector space. The methodological advances discovered in [6] for obtaining analogues of classical results were used in [7] for the usual Beltrami equation (2), and new non-trivial results for $A(z)$ −analytic functions were obtained.

Consider G be a simply connected domain with boundary G a piecewise smooth closed curve. $A(z)$ is given as anti-analytic function; that is the function $A(z)$ is holomorphic in G, and in everywhere in this region. As a result, it satisfies the condition $|A(z)| \leq q < 1, (0 < q < 1)$. Hence the integral

$$\int_{\breve{az}} \overline{A(z)}\mathrm{dz},$$

where $\breve{az}$ is an arbitrary piecewise-smooth path in G connecting the points $a, z \in G$, does not depend on the shape of the path and for a fixed point a is a holomorphic function of z, and

$$\frac{\partial}{\partial z}\left(\int_{\breve{az}} \overline{A(z)}\mathrm{dz}\right) = \overline{A(z)},$$

$$\frac{\partial}{\partial \bar{z}}\left(\int_{\breve{az}} \overline{A(z)}\mathrm{dz}\right) = 0 \qquad (4)$$

Consider the function:

$$\psi(z,\zeta) = z - \zeta + \overline{\int_{\gamma(\zeta,z)} \overline{A(\tau)}\, d\tau} \qquad (5)$$

where $\gamma(\zeta, z)$ is an arbitrary piecewise smooth path in G connecting the points ζ and z. Let's do some calculations

$$\frac{\partial\psi}{\partial z} = 1 + \frac{\partial}{\partial z}\overline{\left(\int_{\gamma(\zeta,z)} \overline{A(\tau)}\, d\tau\right)} = 1 + \overline{\frac{\partial}{\partial \bar{z}}\left(\int_{\gamma(\zeta,z)} \overline{A(z)}dz\right)} = 1 + 0 = 1,$$

$$\frac{\partial\psi}{\partial z} = 0 + \frac{\partial}{\partial \bar{z}}\overline{\left(\int_{\gamma(\zeta,z)} \overline{A(\tau)}\, d\tau\right)} = \overline{\frac{\partial}{\partial z}\left(\int_{\gamma(\zeta,z)} \overline{A(z)}dz\right)} = A(z).$$

Therefore,

$$\frac{\partial\psi}{\partial \bar{z}} = A(z).1 = A(z)\frac{\partial\psi}{\partial z}, \qquad (6)$$

In other words, the function $\psi(z,\zeta)$ fulfills the Beltrami equation $\frac{\partial\psi}{\partial \bar{z}} = A(z)\frac{\partial\psi}{\partial z}$, and by the definition of $A(z)$-analytic function with respect to the first variable, we get:

$$\psi(\zeta, z) = -\psi(z,\zeta) \qquad (7)$$

then this function is $A(z)$ −analytic with respect to the second variable. Note that if the functions $f(z)$ and $g\,(z)$ are $A(z)$ −analytic for the same function $A(z)$, then the functions $f(z) \pm g(z), f(z).g(z), \frac{f(z)}{g(z)}$ $with\ g(z) \neq 0$, are also $A(z)$ −analytic functions. We will verify this for the product of functions, thus we have

$$f_{\bar{z}} = Af_z\ , g_{\bar{z}} = Ag_z\ .$$

Then,

$(fg)_{\bar{z}} = f_{\bar{z}}g + fg_{\bar{z}} = A(f_zg + fg_z) = A(fg)_z$.

Therefore, $f(z) \cdot g\,(z)$ is an A-analytic function. As a consequence, it is extremely useful to mention the $Goursa$ lemma in this sense. The following well-known $Goursa$ lemma is useful.

Lemma (1.1) $Goursat's$ Lemma [3].
If the function $f(z)$ is continuous in the domain G and γ is a closed piecewise-smooth Jordan curve lying in G, then for any given $\varepsilon > 0$, we can indicate a polygon P lying in G with vertices on γ such that $\left|\int_{\gamma} f(z)dz - \int_{\gamma p} f(z)dz\right| < \varepsilon$, where γp is the contour of the polygon P.
Green's formula is obviously applicable for a polygon, and thus it is applicable for a region bounded by a piecewise-smooth contour. In the book of I.N. Vekua ([3], Chapter 2), the following statement regarding the solutions of the Beltrami equation is made regarding the degree of smoothness of the functions included in the green formula.
If in the domain G the function $A(z)$ belongs to the space $C^m(G)$, then all continuous solutions of the Beltrami equation belong to the space $C^{m+1}(G)$. As an antiholomorphic function $A(z)$ belongs to the space C^∞.
Following are some theorems that will be useful during our subsequent results.

Theorem 1. ([2], Analogue of the Cauchy integral theorem)
Let $f(z)$ be $A(z)$-analytic in the domain G and $D \subset G$ is a simply connected domain bounded by a piecewise-smooth contour $\Gamma = \partial D \subset G$. Then

$$\int_{\Gamma} f(z)(dz + A(z)d\bar{z}) = 0.$$

Proof: from the previous arguments it follows that the green formula can be applied to the integral under consideration. Then, taking into account the equalities

$$dzd\bar{z} = -2idxdy \, , dz \wedge d\bar{z} = -d\bar{z} \wedge dz \, ,$$
$$\frac{\partial A(z)}{\partial z} = \overline{\left(\frac{\partial A(z)}{\partial \bar{z}}\right)} = 0 \, ,$$

and Green's formulas (3'), (3 ''), we obtain

$$\int_{\Gamma} f(z)(dz + A(z)d\bar{z}) = \iint_{D} \left(-\frac{\partial f}{\partial \bar{z}} + \frac{\partial}{\partial z}(f(z)A(z)\right) dz \wedge d\bar{z} = \iint_{D} \left(-\frac{\partial f}{\partial \bar{z}} + A(z)\frac{\partial f}{\partial z}\right) dz \wedge d\bar{z} = 0 \, .$$

An alternative of the Cauchy integral formula for $A(z)$ −analytic functions was proved in the work [2] under the conditions of Theorem 1, we find the following formula holds:

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)(d\zeta + A(\zeta)d\bar{\zeta})}{\psi(\zeta, z)} \qquad (8)$$

Analogue form for the Cauchy integral formula of $A(z)$ −analytic functions.
In regard to formula (8), the following question arises: what can be said about the integral if the function $f(\zeta)$is only given on the curve Γ? Consider a piecewise smooth curve $\gamma \subset \mathbb{C}$, that is not necessarily closed and a continuous function $f(\zeta)$ on it. Then, it comes the integral.

$$F(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)(d\zeta + A(\zeta)d\bar{\zeta})}{\psi(\zeta, z)} \qquad (9)$$

An $A(z)$ −Cauchy type integral is one that exists for all $z \notin \gamma$. In the integral on the right, the variable z is a parameter. The integral can be differentiated with respect to the parameter in the neighborhood of any point z that does not intersect γ. As a result of differentiation, we get

$$A(z)F_z = \frac{1}{2\pi i}\int_{\gamma} \left(\left(\frac{1}{\psi(\zeta,z)}\right) - A(z)\left(\frac{1}{\psi(\zeta,z)}\right)_z\right) f(\zeta)(d\zeta + A(\zeta)d\bar{\zeta}) =$$
$$= \frac{1}{2\pi i}\int_{\gamma} \frac{-A(z) + A(z)}{\psi(\zeta,z)^2} f(\zeta)(d\zeta + A(\zeta)d\bar{\zeta}) = 0 \, .$$

Thus, an $A(z)-$ integral of Cauchy type is an $A(z)$ −analytic function in the neighborhood of any point $z \notin \gamma$, that does not intersect γ. Since the point $z \notin \gamma$ is arbitrary, the Cauchy type's $A(z)$ −integral is an $A(z)$ −analytic function everywhere outside the curve γ. If $z_0 \in \gamma$, then the integral in (9) may not exist for $z = z_0$,because the denominator vanishes for $\zeta = z_0$. As in the classical case, the integral can be given meaning by imposing additional constraints on the function $f(\zeta)$. If the function $f(\zeta)$ satisfies the $H\ddot{o}lder$ condition on the curve γ, then the integral in (9) exists in the sense of the Cauchy principal value. This means that for sufficiently small δ , we denote by γ_δ the part of the curve γ lying outside the circle $|\zeta - z_0| \leq \delta$, for sufficiently small δ. The integral,

$$\frac{1}{2\pi i}\int_{\gamma_\delta}\frac{f(\zeta)\left(d\zeta+A(\zeta)d\bar{\zeta}\right)}{\psi(\zeta,z_0)}$$

makes sense in the usual sense. If the following limit exists

$$\lim_{\delta\to 0}\frac{1}{2\pi i}\int_{\gamma_\delta}\frac{f(\zeta)\left(d\zeta+A(\zeta)d\bar{\zeta}\right)}{\psi(\zeta,z_0)},$$

Then it is referred to as the integral in the sense of the Cauchy principal value and is denoted by

$$\frac{1}{2\pi i}\int_{\gamma_\delta}\frac{f(\zeta)(d\zeta+A(\zeta)d\bar{\zeta})}{\psi(\zeta,z_0)}.$$

Theorem 2. If the function $f(\zeta)$ satisfies the $H\ddot{o}lder$ condition $|f(\zeta_1) - f(\zeta_2)| \leq L|\zeta_1 - \zeta_2|^\alpha$ with exponent $\alpha \in (0,1]$ on the curve γ, then the integral in (9) exists in the sense of the Cauchy principal value.

Proof: It must be proved that the above limit exists. To begin, instead of using the circle $|\zeta - z_0| \leq \delta$ in the definition of the curve γ, we can use the $A(z)-$lemniscate $L(z_0,\delta) = \{z: |\psi(z, z_0)| \leq \delta\}$ centered at z_0 and radius δ. Taking into account that the equality $d\psi(z,\zeta) = dz + A(z)d\bar{z}$ holds, we obtain

$$\int_{\partial L(z_0,\delta)}\frac{d\zeta+A(\zeta)d\bar{\zeta}}{\psi(\zeta,z_0)} = \int_{|\psi(\zeta,z_0)|=\delta}\frac{d\psi(\zeta,z_0)}{\psi(\zeta,z_0)} = 2\pi i.$$

We can substitute any curve homotopic to the boundary of the $A(z)$ −lemniscate $\partial(z_{0,}\delta)$ by using the analogue of the Cauchy theorem for $A(z)$ −analytic functions in the last integral. As a result, the equality

$$\frac{1}{2\pi i}\int_{\gamma_\delta}\frac{f(\zeta)(d\zeta+A(\zeta)d\bar{\zeta})}{\psi(\zeta,z_0)} = \frac{1}{2\pi i}\int_{\gamma_\delta}\frac{[f(\zeta)-f(z_0)](d\zeta+A(\zeta)d\bar{\zeta})}{\psi(\zeta,z_0)} + f(z_0) - f(z_0)\frac{1}{2\pi i}\int_{\gamma_1}\frac{d\zeta+A(\zeta)d\bar{\zeta}}{\psi(\zeta,z_0)},$$

where γ_1 is the part of the boundary of the $A(z)-$lemniscate $\partial L(z_{0,}\delta)$, lying outside the finite region bounded by γ. By assumption, the function $A(z)$ satisfies the inequality

$$|A(z)| \leq q < 1;\ |\psi(\zeta,z_0)| = \left|\zeta - z_0 + \overline{\int_{\overline{z_0\zeta}}\overline{A(\tau)}\,d\tau}\right| \geq |\zeta - z_0| - \left|\int_{\overline{z_0\zeta}}\overline{A(\tau)}\,d\tau\right| \geq (1-q)|\zeta - z_0|,$$

from which we derive using the $Holder$ condition on the function $f(\zeta)$, we have the estimate

$$\left|\frac{f(\zeta)-f(z_0)}{\psi(\zeta,z_0)}\right| \leq \frac{L}{1-q}|\zeta - z_0|^{\alpha-1}.$$

Therefore, the improper integral

$$\frac{1}{2\pi i}\int_{\gamma}\frac{[f(\zeta)-f(z_0)](d\zeta+A(\zeta)d\bar{\zeta})}{\psi(\zeta,z_0)}$$

converges uniformly and

$$\lim_{\delta\to 0}\left\{\frac{1}{2\pi i}\int_{\gamma_\delta}\frac{[f(\zeta)-f(z_0)](d\zeta+A(\zeta)d\bar{\zeta})}{\psi(\zeta,z_0)}\right\} = \frac{1}{2\pi i}\int_{\gamma}\frac{[f(\zeta)-f(z_0)](d\zeta+A(\zeta)d\bar{\zeta})}{\psi(\zeta,z_0)}.$$

The smoothness of the curve γ ensures the equality

$$lim_{\delta\to 0}\left\{\frac{1}{2\pi i}\int_{\gamma_1}\frac{d\zeta+A(\zeta)d\bar{\zeta}}{\psi(\zeta,z_0)}\right\} = \frac{1}{2}.$$

Therefore, we get:

$$\lim_{\delta\to 0}\frac{1}{2\pi i}\int_{\gamma_\delta}\frac{f(\zeta)(d\zeta+A(\zeta)d\bar{\zeta})}{\psi(\zeta,z_0)} = \frac{1}{2}f(z_0) + \frac{1}{2\pi i}\int_{\gamma}\frac{[f(\zeta)-f(z_0)](d\zeta+A(\zeta)d\bar{\zeta})}{\psi(\zeta,z_0)}.$$

Compactness of the family of $A(z)$ −analytic functions

First, we provide supporting information about $A(z)$ −analytic functions that will be needed later. An analog of the Taylor formula for $A(z)$ −analytic functions is obtained in [2]as follows: f the function $f(z)$ is $A(z)$ −analytic in the $A(z) - lemniscate$ $L(a,R) = \{z: |\psi(z,a)| < R\}$ and continuous on its closure, then it can be expanded into a Taylor series analog of the form:

$$\sum_{k=0}^{\infty} C_k\, \psi^k(z,a) \qquad (10)$$

where $C_k = \frac{1}{2\pi i}\int_{\partial L(a,\rho)} \frac{f(\zeta)}{[\psi(\zeta,a)]^k}(d\zeta + A(\zeta)d\bar{\zeta})$, $0<\rho<R$, $k=0,1,\ldots$

The generalized power series (10) converges in $the\ A-lemniscate\ L(a,r)$ and the radius of convergence is found by the $Cauchy-Hadamard$ formula

$$\frac{1}{r} = \overline{\lim_{k\to\infty}}\sqrt[k]{|C_k|}.$$

The Cauchy inequality for the coefficients of the series (10) was also proved there.

$$|C_k| \le \frac{max\{|f(z)|: z\in \partial L(a,\rho)\}}{\rho^k} \qquad (11)$$

Where, $0<\rho<r$, $k=0,1,2,\ldots$

A collection of functions $\{f\}$, defined in some domain D is said to be compact in D if from each sequence $\{f_n\}$ of functions of this collection one can extract a subsequence $\{f_{n_k}\}$ converging uniformly on any compact set $K\subset D$ [8].

Theorem 3. If the set of $A(z)$-analytic functions $\{f(z)\}$ is uniformly bounded in the $A(z)-lemniscate\ L(a,r)$, that is, there exists a positive number M such that the inequalities $|\,f^*(z)\,| < M$ holds at all points of the $A(z)-lemniscate\ L(a,r)$ and for all functions $f^*(z)$ from a given collection $\{f(z)\}$, then this collection is compact in the $A(z)-lemniscate\ L(a,r)$.

Proof: Consider an arbitrary sequence of functions $\{f_n(z)\}$ from a given collection $\{f(z)\}$ and the corresponding Taylor series (10) $f_n(z) = \sum_{k=0}^{\infty} C_k^{(n)}\psi^k(z,a)$. By the Cauchy inequality (11) and the uniform boundedness of the set of functions, we have estimates for each $\rho, 0<\rho<r$: $\left|C_k^{(n)}\right| \le \frac{M_n(\rho)}{\rho^k}$, where $M_n(\rho) = max\{|f_n(z)|: \ z\in\partial L(a,\rho\} < M$. Tending ρ to the limit r, we obtain for any non-negative integer k the inequality,

$$\left|C_k^{(n)}\right| \le \frac{M}{r^k}, \qquad n=1,2,\ldots \qquad (12)$$

Thus, for a fixed k, the numerical sequence $\{C_k^{(n)}\}$ is bounded and a convergent subsequence can be extracted from it. For $k=0$, from the sequence $\{f_n(z)\}$, we choose a subsequence $\left\{f_{n_j^{(0)}}(z)\right\}$such that the numerical subsequence $\{C_0^{n_j^{(0)}}(z)\}$ converges. For $k=1$, from the chosen subsequence $\left\{f_{n_j^{(0)}}(z)\right\}$ we extract a new subsequence $\left\{f_{n_j^{(1)}}(z)\right\}$such that the numerical subsequence $\left\{C_1^{n_j^{(1)}}(z)\right\}$ converges. And so on, if for $k=m$ a subsequence $\left\{f_{n_j^{(m)}}(z)\right\}$ is chosen such that the numerical subsequences

$$\left\{C_0^{n_j^{(m)}}\right\}.\left\{C_1^{n_j^{(m)}}\right\},\ldots,\left\{C_m^{n_j^{(m)}}(z)\right\}$$

converge, then we extract from it a subsequence $\left\{f_{n_j^{(m+1)}}(z)\right\}$ such that the subsequence $\left\{C_{m+1}^{n_j^{(m+1)}}\right\}$ converges. Let the subsequences $\left\{f_{n_j^{(m)}}(z)\right\}$ be constructed for all m. We now make a diagonal sampling, i.e. in the new sequence, the $m-th$ element of the $m-th$ subsequence will stand at $m-th$ place. Denote the resulting sequence by $\{f_{\nu k}(z)\}$, where $\nu k = n_k^{(k)}$. By construction, we have $f_{\nu k}(z) = \sum_{j=0}^{\infty} C_j^{(\nu k)}\psi^j(z,a)$ and for any fixed j the sequence $\left\{C_j^{(\nu k)}\right\}$ is converge as $k\to\infty$. Denote $\lim_{k\to\infty} C_j^{(\nu k)} = C_j$. From the Cauchy inequality (12) we obtain the estimate:

$$\left|C_j\right| \le \frac{M}{r^j} \qquad (13)$$

When follows $\overline{\lim_{j\to\infty}}\sqrt[j]{\left|C_j\right|} \le \frac{1}{r}$. Therefore, the radius of convergence of the series $\sum_{j=0}^{\infty} C_j\psi^j(z,a)$ is not less than r and the sum of $F(z)$ of this series represents an A-analytic function in the $A(z) - lemniscate\ L(a,r)$. Let us show that the sequence $\{f_{vk}(z)\}$ converges uniformly to $F(z)$ on any compact in $L(a,r)$. To do this, it is enough to show the uniform convergence on any $A(z) - lemniscate\ L(a,\rho)$, $\rho < r$. For any $\varepsilon > 0$, we find an integer n_0 such that the inequality $\sum_{n=n_{0+1}}^{\infty} M\left(\frac{\rho}{r}\right)^n < \frac{\varepsilon}{3}$. Then we have from (12) and (13)

$$\sum_{n=n_0+1}^{\infty} |C_n^{(vk)}|\,\rho^n \le \sum_{n=n_0+1}^{\infty} M\left(\frac{\rho}{r}\right)^n < \frac{\varepsilon}{3},$$

and

$$\sum_{n=n_0+1}^{\infty} |C_n|\,\rho^n \le \sum_{n=n_0+1}^{\infty} M\left(\frac{\rho}{r}\right)^n < \frac{\varepsilon}{3}.$$

Thus, for $z \in L(a,\rho)$, i.e. for $|\psi(z,a)| < \rho$, We obtain

$$|f_{vk}(z) - F(z)| = \left|\sum_{n=0}^{\infty}(|C_n^{vk} - C_n)\psi^n(z,a)\right|$$
$$\le \sum_{n=0}^{n_0}|C_n^{vk} - C_n|\rho^n$$
$$+ \sum_{n=n_0+1}^{\infty}|C_n^{vk}|\rho^n$$
$$+ \sum_{n=n_0+1}^{\infty}|C_n|\rho^n$$
$$< \sum_{n=0}^{\infty}|C_n^{vk} - C_n|\rho^n + \frac{2}{3}\varepsilon.$$

Since n_0 is fixed and $C_n^{vk} \to C_n$ as $k \to \infty$, for sufficiently large $k > k_0$, we have $\sum_{n=0}^{n_0}|C_n^{vk} - C_n|\rho^n < \frac{\varepsilon}{3}$. Therefore, $|f_{vk}(z) - F(z)| < \varepsilon$ for all $z \in L(a,\rho), \forall\, m > m_0, i.e. \{f_{vk}(z)\}$ converges uniformly to $F(z)$ on $L(a,\rho), \forall\, \rho < r$, which means the uniform convergence of $\{f_{vk}(z)\}$ to $F(z)$ on any compact set $K \subset L(a,r)$.

CONCLUSION

This paper aims to study some properties of A(z)-analytic functions by employing the well-known Beltrami equation and its analytical properties. It is widely known that there is a connection between the Beltrami equation and quasiconformal mappings. We have analytically derived a solution to the Beltrami equation and identified an analogue of the *A(z)*-analytic function. Additionally, we have investigated the compactness of the family of *A(z)*-analytic functions and have uncovered some valuable properties.

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