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Quaternary Boundary Optimal Control Problem Dominating by Quaternary Nonlinear Parabolic System

Fetan J. Naji[1], Jamil A. Ali Al-Hawasy[1*], Ion Chryssoveghi[2]

[1]Department of Mathematics, College of Science, Mustansiriyah University, 10052 Baghdad, IRAQ.
[2]Depart. of Mathematics, School of Applied mathematical and Physical Sciences, National Technical University of Athens, Athens, Greece.

*Correspondent contact: jhawassy17@uomustansiriyah.edu.iq

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ABSTRACT

In this paper, our purpose is to study the quaternary continuous classical boundary optimal control vector problem (QCCBOCVP) dominated by the quaternary nonlinear parabolic boundary value problem (QNLPBVP). Under suitable assumptions and with given quaternary continuous classical boundary control vector (QCCBCV), the existence theorem for a unique quaternary state vector solution (QSVS) of the weak form (WF) for the QNLPBVP is stated and demonstrated via the Method of Galerkin and the first compactness theorem. Furthermore, the continuity of the Lipchitz operator between the QSVS of the WF for the QLPBVP and the corresponding QCCBCV is proved. The existence of a quaternary continuous classical boundary optimal control vector (QCCBOVC) is stated and demonstrated under suitable assumptions.

KEYWORDS: Quaternary continuous classical boundary optimal control vector problem, Quaternary nonlinear parabolic boundary value problem, Method of Galerkin, Lipschitz continuity.

الخلاصة

الهدف من هذا البحث هو دراسة مسالة السيطرة الامثلية الحدودية التقليدية المستمرة المسيطرة بمسالة قيم حدودية مكافئة رباعية غير خطية . تم صياغة نص وبرهان مبرهنة وجود حل المتجه الحالة الرباعي للصيغة الضعيفة لمسالة القيم الحدودية المكافئة الرباعية الغير خطية بوجود شروط مناسبة وعندما يكون متجه السطيرة الحدودية التقليدية المستمرة الرباعية ثابتا" باستخدام طريقة كاليركن. تم برهان استمرارية مؤثر ليبشتز بين الحل المتجه الرباعي للصيغة الضعيفة وبين متجه السيطرة الحدودية الرباعية التقليدية المتسمرة المقابلة له. كذلك تم ذكر نص وبرهان مبرهنة وجود متجه سيطرة امثلية حدودية رباعية تقليدية مستمرة بوجود شروط مناسبة .

1.INTRODUCTION

Optimal control problems (OCPs) play an important role in many practical applications, such as in medicine [1], aircraft [2], economics [3], robotics [4], weather conditions [5] and many other scientific fields. There are two types of OCPs; the classical and the relax type, each one of these two types is dominated either by ODEqs [6] or by PDEqs [7]. The Continuous Classical boundary optimal control problem (CCBOCP) is dominated by a couple of nonlinear parabolic, elliptic or hyperbolic PDEqs that were studied in [8-10]. Later, these studies for these three types were generalized to deal with CCBOCP dominated by triple nonlinear PDEqs (TNLPDEqs) so as [11-13].

All of the above- mentioned studies and many others encouraged us to think about generalizing the study of the CCBOCP dominated by TNLPDEqs of parabolic type to a CCBOQCP dominated by QNLPBVP. According to this generalization, the mathematical model for the dominating equation is needed to be reformulated, as well as the objective function (OBF). The study of the QCCBOCVP dominated by QNLPBVP which is proposed in this paper and it starts with the statement and proof of the existence theorem of the QSVS for the WF of the QNLPBVP using the Method of Galerkin (GAM) and the first compactness theorem (FCT), under suitable assumptions when the QCCBCV is known. The continuity of the Lipchitz operator (LIO) between the QSVS of the WF for the QNLPBVP and the corresponding QCCBCV is proved. The existence theorem of a QCCBOCV is stated and demonstrated under suitable assumption.

MATERIALS AND METHODS

Problem Description

Let $\Omega \subset \mathbb{R}^2$ be a bounded open region with boundary Γ, $Q = I \times \Omega$, $\Sigma = \Gamma \times I$, and $x = (x_1, x_2)$. The QCCBOCVP consists of the state quaternary equations (SQEqs), which are considered as (in Q):

$$y_{1t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{1ij}\frac{\partial y_1}{\partial x_j}\right) + b_1 y_1 - b_5 y_2 + b_6 y_3 + b_7 y_4 = f_1(x,t,y_1) \tag{1}$$

$$y_{2t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{2ij}\frac{\partial y_2}{\partial x_j}\right) + b_2 y_2 + b_5 y_1 - b_9 y_3 - b_{11} y_4 = f_2(x,t,y_2) \tag{2}$$

$$y_{3t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{3ij}\frac{\partial y_3}{\partial x_j}\right) + b_3 y_3 + b_9 y_2 - b_6 y_1 + b_{15} y_4 = f_3(x,t,y_3) \tag{3}$$

$$y_{4t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{4ij}\frac{\partial y_1}{\partial x_j}\right) + b_4 y_4 - b_7 y_1 + b_{11} y_2 - b_{15} y_3 = f_4(x,t,y_4) \tag{4}$$

With the following boundary conditions (BCs) on Σ and initial conditions (ICs):

$$\frac{\partial y_r}{\partial n_r} = \sum_{l,j=1}^{2} a_{rij}\frac{\partial y_r}{\partial x_j}\cos(n_r, x_i) = u_r(x,t) \tag{5}$$

$$y_r(x,0) = y_r^0(x), \text{ on } \Omega \tag{6}$$

where $\vec{f} = (f_1, f_2, f_3, f_4) \in \left(L^2(Q)\right)^4 = \boldsymbol{L^2(Q)}$, is a vector of function for each $x = (x_1, x_2) \in \Omega$, $\vec{u} = (u_1, u_2, u_3, u_4) \in \left(L^2(\Sigma)\right)^4 = \boldsymbol{L^2(\Sigma)}$ is a QCCBCV and $\vec{y} \in \left(H^2(\Omega)\right)^4 = \boldsymbol{H^2(\Omega)}$is the QSVS corresponding to the QCCBCV$\vec{u}$,$a_{ij}(x,t)$,$b_{ij}(x,t)$, $c_{ij}(x,t)$,$d_{ij}(x,t)$, $b_{r+3}(x,t)$, $b_{2r+3}(x,t)$, $b_{3r+3}(x,t)$, $b_{4r+3}(x,t) \in C^\infty(Q)$, n_r (for $r = 1,2,3,4$) is an outer normal vector on the boundary Σ, and (n_r, x_j) is the angle between n_r and $x_j - axis$.

The Set of Admissible QCCBCV is defined by:

$$\overrightarrow{W_A} = \{\vec{u} \in \boldsymbol{L^2(\Sigma)} \,|\, \vec{u} \in \vec{U} \subset \mathbb{R}^4 \}$$

The OBF is defined by:

$$G_0(\vec{u}) = \int_Q [g_{01}(x,t,y_1) + g_{02}(x,t,y_2) + g_{03}(x,t,y_3) + g_{04}(x,t,y_4)]dxdt + \int_\Sigma [h_{01}(x,t,u_1) + h_{02}(x,t,u_2) + h_{03}(x,t,u_3) + h_{04}(x,t,u_4)]d\sigma \tag{7}$$

Let$\vec{V} = V_1 \times V_2 \times V_3 \times V_4 = \{\vec{v} : \vec{v} = (v_1, v_2, v_3, v_4) \in \left(H^1(\Omega)\right)^4 = \boldsymbol{H^1(\Omega)} \}$.

The WF:

The WF of the QNLPBVP, when $\vec{y} \in H^2(\Omega)$ is given by:

$$(y_{1t}, v_1) + a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega = (f_1, v_1)_\Omega + (u_1, v_1)_\Gamma \tag{8a}$$

$$(y_1(0), v_1)_\Omega = (y_1^0, v_1)_\Omega \tag{9b}$$

$$(y_{2t}, v_2) + a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega = (f_2, v_2)_\Omega + (u_2, v_2)_\Gamma \tag{9a}$$

$$(y_2(0), v_2)_\Omega = (y_2^0, v_2)_\Omega \tag{9b}$$

$$(y_{3t}, v_3) + a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega = (f_3, v_3)_\Omega + (u_3, v_3)_\Gamma \tag{10a}$$

$$(y_3(0), v_3)_\Omega = (y_3^0, v_3)_\Omega \tag{10b}$$

$$(y_{4t}, v_4) + a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega + (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega = (f_4, v_4)_\Omega + (u_4, v_4)_\Gamma \quad (11a)$$

$$(y_4(0), v_4)_\Omega = (y_4^0, v_4)_\Omega \quad (11b)$$

Where, $\forall r = 1,2,3,4$:

$$a_r(t, y_r, v_r) = \int_\Omega \sum_{i,j=1}^n a_{rij}(x,t) \frac{\partial y_r}{\partial x_j} \frac{\partial v_r}{\partial x_i} dx,$$

Assumptions (A):

(i) f_r ($\forall r = 1,2,3,4$) satisfies:

$| f_r(x,t) | \leq \eta_r(x,t) + c_r|y_r|$

where $, y_r \in \mathbb{R}, c_r > 0$ and $\eta_r \in L^2(Q, \mathbb{R})$

(ii) f_r is LI w.r.t. y_r, i.e.

$| f_r(x,t, y_r) - f_r(x,t,\hat{y}_r)| \leq L_r| y_r - \hat{y}_r|$

where $\hat{y}_r, y_r \in \mathbb{R}$ and $L_r > 0, \forall r = 1,2,3,4$,

(iii) $| a_r(t, y_r, v_r)| \leq \alpha_r\| y_r\|_1\| v_r\|_1$,

$|(b_r(\mathrm{t})\, y_r, v_r)_\Omega| \leq \beta_r\| y_r\|_0\| v_r\|_0$,

$a_r(t, y_r, y_r) \geq \bar{\alpha}_r\| y_r\|_1^2, (b_r(\mathrm{t})\, y_r, y_r)_\Omega \geq \bar{\beta}_r\| y_r\|_0^2, \forall r = 1,2,3,4$,

$|(b_{r+3}(\mathrm{t})\, y_r, v_1)_\Omega| \leq \epsilon_r\| y_r\|_0\| v_1\|_0, \forall r = 2,3,4$,

$| (b_{2r+3}(\mathrm{t})\, y_r, v_2)_\Omega | \leq \bar{\epsilon}_r\| y_r\|_0\| v_2\|_0, \forall r = 1,3,4$,

$| (b_{3r+3}(\mathrm{t})\, y_r, v_3)_\Omega | \leq \hat{\epsilon}_r\| y_r\|_0\| v_3\|_0, \forall r = 1,2,4$,

$| (b_{4r+3}(\mathrm{t})\, y_r, v_4)_\Omega | \leq \tilde{\epsilon}_r\| y_r\|_0\| v_4\|_0, \forall r = 1,2,3$,

$c(t,\vec{y},\vec{y}) = a_1(t, y_1, y_1) + (b_1(\mathrm{t})\, y_1, y_1)_\Omega + a_2(t, y_2, y_2) + (b_2(t)\, y_2, y_2)_\Omega + a_3(t, y_3, y_3) + (b_3(\mathrm{t})\, y_3, y_3)_\Omega + a_4(t, y_4, y_4) + (b_4(\mathrm{t})\, y_4, y_4)_\Omega$,

and $(t,\vec{y},\vec{y}) \geq \bar{\alpha}\|\vec{y}\|_1^2$, $\|\vec{y}\|_1^2 = \sum_{r=1}^4 \| y_r\|_1^2$

with $\alpha_r, \beta_r, \epsilon_r$ $(r = 1,2,3,4)$ and $\bar{\alpha}$ are real positive constants.

Lemma (2.1) [14]: Let V, H, V' be three Hilbert spaces, each space is included in the following one as in $\|u\|_{L^\infty(\Omega)} = ess.sup|u(x)|$ or $\|u\|_{L^P(\Omega)} = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}}$

V' being the dual of V. If a function u belongs to $L^2(0,T;V)$, and its derivative u' belongs to $L^2(0,T;V')$, then u is almost every equal to a function continuous from [0, T] into H and one have the following equality, which holds in the scalar distribution sense on (0, T) $\frac{d}{dt}|u|^2 = 2\langle u', u\rangle$.

Proposition (2.1)[9]: Let $f: D \times \mathbb{R}^n \to \mathbb{R}^m$ is Carathèodory type(Ca-T), let F be a functional, satisfy that $F(y) = \int_Q f(x, y(x))\, dx$, where D is measurable subset of $\mathbb{R}^n$, and suppose that $\|f(x,y)\| \leq \zeta(x) + \eta(x)\|y\|^\alpha$, $\forall (x,y) \in D \times \mathbb{R}^n$, $y \in L^p(D \times \mathbb{R}^n)$ where $\zeta \in L^1(D \times \mathbb{R})$, $\eta \in L^{\frac{p}{p-\alpha}}(D \times \mathbb{R})$, and $\alpha \in N$. If $p \in [1, \infty)$, and $\eta = 0$, if $p = \infty$.

Then $\boldsymbol{F}$ is continuous on $\boldsymbol{L^p}(\boldsymbol{D} \times \mathbb{R}^{\boldsymbol{n}})$

RESULTS AND DISCUSSION

In the following section, the existence of a unique QSVS for the WF for the QNLPBVP under suitable assumptions and with given QCCBCV is stated and proved using the method of Galerkin with utilizing the first compactness theorem.

The Existence of a unique QSVS for the WF

Theorem (3.1): with assumptions (A), for each fixed QCCBCV $\vec{u} \in \boldsymbol{L^2(\Omega)}$, the WF ((8) - (11)) has a unique QSVS $\vec{y} \in \left(L^2(\mathrm{I}, \mathrm{V})\right)^4 = \boldsymbol{L^2(\mathrm{I}, \mathrm{V})}$, $\vec{y_t} \in \left(L^2(\mathrm{I}, V^*)\right)^4 = \boldsymbol{L^2(I, V^*)}$.

Proof: Let $\vec{V_n} \subset \vec{V}$ be the set of piecewise affine in Ω, let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be the basis of $\vec{V_n}$ where $n = 4N$, then the approximate QSVS $\vec{y}$ of ((3.16) - (3.19)) is expressed by:

$$\vec{y}_n(x,t) = (y_{1n}, y_{2n}, y_{3n}, y_{4n}) = \sum_{j=1}^n c_j(t)\, \vec{v}_j(x) \quad (12)$$

Where,

$\vec{V_j} = ((2^{r-1} mod 2) v_{1k}, (2^{(r-2)^2} mod 2) v_{2k}, (2^{(r-3)^2} mod 2) v_{3k}, (2^{4-r}\ mod 2) v_{4k})$,

$c_j = c_{rj}$, for $\mathrm{k} = (1, \dots, \mathrm{N})$, $\forall \mathrm{r} = 1,2,3,4$, $\mathrm{j} = \mathrm{k} + \mathrm{N}(\mathrm{r} - 1)$ and $c_{rj}(t)$ is unknown functions of t.

The WF of the SQEqs ((8) - (11)) can be approximated with respect to the space variable, using the GAM, to get:

$$(y_{1nt}, v_1)_\Omega + a_1(t, y_{1n}, v_1) + (b_1(t)y_{1n}, v_1)_\Omega - (b_5(t)y_{2n}, v_1)_\Omega + (b_6(t)y_{3n}, v_1)_\Omega + (b_7(t)y_{4n}, v_1)_\Omega = (f_1(y_{1n}), v_1)_\Omega + (u_1, v_1)_\Gamma \quad (13a)$$

$$(y_{1n}^0, v_{1n})_\Omega = (y_1^0, v_1)_\Omega, \forall v_1 \in V_n \quad (13b)$$

$$(y_{2nt}, v_2)_\Omega + a_2(t, y_{2n}, v_2) + (b_2(t)y_{2n}, v_2)_\Omega + (b_5(t)y_{1n}, v_2)_\Omega - (b_9(t)y_{3n}, v_2)_\Omega - (b_{11}(t)y_{4n}, v_2)_\Omega = (f_2(y_{2n}), v_2)_\Omega + (u_2, v_2)_\Gamma \quad (14a)$$

$$(y_{2n}^0, v_{2n})_\Omega = (y_2^0, v_2)_\Omega, \forall v_2 \in V_n \quad (14b)$$

$$(y_{3nt}, v_3)_\Omega + a_3(t, y_{3n}, v_3) + (b_3(t)y_{3n}, v_3)_\Omega + (b_9(t)y_{2n}, v_3)_\Omega - (b_6(t)y_{1n}, v_3)_\Omega + (b_{15}(t)y_{4n}, v_3)_\Omega = (f_3(y_{3n}), v_3)_\Omega + (u_3, v_3)_\Gamma \quad (15a)$$

$$(y_{3n}^0, v_{3n})_\Omega = (y_3^0, v_3)_\Omega, \forall v_3 \in V_n \quad (15b)$$

$$(y_{4nt}, v_4)_\Omega + a_4(t, y_{4n}, v_4) + (b_4(t)y_{4n}, v_4)_\Omega - (b_7(t)y_{1n}, v_4)_\Omega + (b_{11}(t)y_{2n}, v_4)_\Omega - (b_{15}(t)y_{3n}, v_4)_\Omega = (f_4(y_{4n}), v_4)_\Omega + (u_4, v_4)_\Gamma \quad (16a)$$

$$(y_{4n}^0, v_{4n})_\Omega = (y_4^0, v_4)_\Omega, \forall v_4 \in V_n \quad (16b)$$

Where $y_{rn}^0 = y_{rn}^0(x) = y_{rn}(x,0) \in V_n \subset V \subset L^2(\Omega)$ is the Pro of y_r^0 with respect to the norm $\|.\|_0$, i.e., $\forall r = 1,2,3,4$,
$(y_{rn}^0, v_r)_\Omega = (y_r^0, v_r)_\Omega \Leftrightarrow \|y_{rn}^0 - y_r^0\|_0 \leq \|y_r^0 - v_r\|_0, \forall v_r \in V_n$.
Substituted (12) in ((13) - (16)) with setting $v_r = v_{ri}, \forall\, r = 1,2,3,4$. So, the WF can be written as the following nonlinear system (NLAGS) of ODEqs with its ICs which has a unique QSVS $\vec{y}_n$:

$$A_1 C'_{1j}(t) + D_1 C_1(t) - E_1\, C_2(t) + F_1 C_3(t) - H_1 C_4(t) = b_1(\bar{V}_1^T(x) C_1(t)) \quad (17a)$$

$$A_1 C_1(0) = b_1^0 \quad (17b)$$

$$A_2 C'_{2j}(t) + D_2 C_2(t) + E_2\, C_1(t) - F_2 C_3(t) - H_2 C_4(t) = b_2\left(\bar{V}_2^T(x) C_2(t)\right) \quad (18a)$$

$$A_2 C_2(0) = b_2^0 \quad (18b)$$

$$A_3 C'_{3j}(t) + D_3 C_3(t) + E_3\, C_2(t) - F_3 C_1(t) + H_3 C_4(t) = b_3(\bar{V}_3^T(x) C_3(t)) \quad (19a)$$

$$A_3 C_3(0) = b_3^0 \quad (19b)$$

$$A_4 C'_{4j}(t) + D_4 C_4(t) - E_4\, C_1(t) + F_4 C_2(t) - H_4 C_3(t) = b_4(\bar{V}_4^T(x) C_4(t)) \quad (20a)$$

$$A_4 C_4(0) = b_4^0 \quad (20b)$$

where $\forall r = 1,2,3,4$
$A_r = (a_{irj})_{n\times n}$, $a_{irj} = (v_{rj}, v_{ri})_\Omega$, $D_r = (d_{irj})_{n\times n}$, $d_{irj} = [a_r(t, v_{rj}, v_{ri}) + (b_r(t)v_{rj}, v_{ri})_\Omega]$, $E_1 = (e_{ij})_{n\times n}$, $e_{ij} = (b_5(t)v_{2j}, v_{1i})_\Omega$, $E_2 = (s_{ij})_{n\times n}$, $s_{ij} = (b_5(t)v_{1j}, v_{2i})_\Omega$, $E_3 = (p_{ij})_{n\times n}$, $p_{ij} = (b_9(t)v_{2j}, v_{3i})_\Omega$, $E_4 = (w_{ij})_{n\times n}$, $w_{ij} = (b_7(t)v_{1j}, v_{4i})_\Omega$, $F_1 = (f_{ij})_{n\times n}$, $f_{ij} = (b_4(t)v_{3j}, v_{1i})_\Omega$, $F_2 = (m_{ij})_{n\times n}$, $m_{ij} = (b_9(t)v_{3j}, v_{2i})_\Omega$, $F_3 = (g_{ij})_{n\times n}$, $g_{ij} = (b_6(t)v_{1j}, v_{3i})_\Omega$, $F_4 = (k_{ij})_{n\times n}$, $k_{ij} = (b_{11}(t)v_{2j}, v_{4i})_\Omega$, $H_1 = (h_{ij})_{n\times n}$, $h_{ij} = (b_7(t)v_{4j}, v_{1i})_\Omega$, $H_2 = (l_{ij})_{n\times n}$, $l_{ij} = (b_{11}(t)v_{4j}, v_{2i})_\Omega$, $H_3 = (q_{ij})_{n\times n}$, $q_{ij} = (b_{15}(t)v_{4j}, v_{3i})_\Omega$, $H_4 = (x_{ij})_{n\times n}$, $x_{ij} = (b_{15}(t)v_{3j}, v_{4i})_\Omega$, $C_r(t) = (c_{rj}(t))_{n\times 1}$
$c'_r(t) = (c'_{rj}(t))_{n\times 1}$, $C_r(0) = (c_{rj}(0))_{n\times 1}$,
$b_r = (b_{ri})_{n\times 1}$, $b_{ri} = (f_r(\bar{V}_r^T C_r(t)), v_{ri})_\Omega + (u_r, v_{ri})_\Gamma$, $b_{ri}^0 = (y_r^0, v_{ri})_\Omega$

The norm $\left\|\vec{y}_n^{\mathbf{0}}\right\|_{\mathbf{0}}$ is bounded: Since $y_r^0 = y_r^0(x) \in L^2(\Omega)$, then there exists $\{v_{rn}^0\}$, with $v_{rn}^0 \in V_n \subset V \subset L^2(\Omega)$, $\forall r = 1,2,3,4$, s.t. $v_{rn}^0 \xrightarrow{s} y_r^0$ in $L^2(\Omega)$, and from the projection theorem, $\forall v_{rn}^0 \in V_n \subset \mathrm{V} \subset L^2(\Omega)$:
$\|y_{rn}^0 - y_r^0\|_0 \leq \|y_r^0 - v_{rn}^0\|_0$ Thus $\|y_{rn}^0 - y_r^0\|_0 \to 0$, so $y_{rn}^0 \xrightarrow{s} y_r^0$ in $L^2(\Omega)$ with $\|y_{rn}^0\|_0 \leq b_r$.

The norm $\|\overrightarrow{y_n}(t)\|_{L^\infty(I,L^2(\Omega))}$ and $\|\overrightarrow{y_n}(t)\|_Q$ are bounded:

Setting $v_r = y_{rn}$ where $\forall r$=1,2,3,4. In Eqs. ((13a) - (16a)) and integrating both sides with respect to t from 0 to T, and adding the above four equations, finally using assumption (A-ii):

$$\int_0^T (\overrightarrow{y_{nt}}, \overrightarrow{y_n})dt + \bar{\alpha}\int_0^T \|\overrightarrow{y_n}\|_1^2\, dt = \int_0^T (f_1(y_{1n}), y_{1n})_\Omega dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (f_2(y_{2n}), y_{2n})_\Omega\, dt + \int_0^T (u_2, y_{2n})_\Gamma\, dt + \int_0^T (f_3(y_{3n}), y_{3n})_\Omega dt + \int_0^T (u_3, y_{3n})_\Gamma dt + \int_0^T (f_4(y_{4n}), y_{4n})_\Omega dt + \int_0^T (u_4, y_{4n})_\Gamma\, dt \qquad (21)$$

Since $\vec{y}_{nt} \in \boldsymbol{L^2(I,V^*)} = \boldsymbol{L^2(I,V)}$ and $\vec{y}_n \in \boldsymbol{L^2(I,V)}$ in the first term of L.H.S. of (21), hence can be using Lemma (2.1), on the other hand since the second term is positive, taking $T = t \in [0,T]$, using assumption (A-i) for the terms f_r in the R.H.S., to get:

$$\int_0^t \frac{d}{dt}\|\overrightarrow{y_n}(t)\|_0^2\, dt \le \int_0^t \int_\Omega (\eta_1^2 + |y_{1n}|^2)\, dx\, dt + 2\int_0^t \int_\Omega c_1|y_{1n}|^2\, dx\, dt + \int_0^t \int_\Gamma (|u_1|^2 + |y_{1n}|^2)\, d\gamma\, dt + \int_0^t \int_\Omega (\eta_2^2 + |y_{2n}|^2)\, dx\, dt + 2\int_0^t \int_\Omega c_2|y_{2n}|^2\, dx\, dt + \int_0^t \int_\Gamma (|u_2|^2 + |y_{2n}|^2)\, d\gamma dt + \int_0^t \int_\Omega (\eta_3^2 + |y_{3n}|^2)\, dx\, dt + 2\int_0^t \int_\Omega c_3|y_{3n}|^2\, dx\, dt + \int_0^t \int_\Gamma (|u_3|^2 + |y_{3n}|^2)\, d\gamma dt + \int_0^t \int_\Omega (\eta_4^2 + |y_{4n}|^2)\, dx\, dt + 2\int_0^t \int_\Omega c_4|y_{4n}|^2\, dx\, dt + \int_0^t \int_\Gamma (|u_4|^2 + |y_{4n}|^2)\, d\gamma dt.$$

$$\Rightarrow \int_0^t \frac{d}{dt}\|\overrightarrow{y_n}(t)\|_0^2\, dt \le \|\eta_1\|_Q^2 + \|u_1\|_\Sigma^2 + \|\eta_2\|_Q^2 + \|u_2\|_\Sigma^2 + \|\eta_3\|_Q^2 + \|u_3\|_\Sigma^2 + \|\eta_4\|_Q^2 + \|u_4\|_\Sigma^2 + c_5\int_0^t \|\vec{y}_n\|_0^2\, dt + \int_0^t \|\vec{y}_n\|_\Gamma^2\, dt$$

Where $c_5 = \max((2c_1+1), (2c_2+1), (2c_3+1), (2c_4+1))$.

Since $\|\eta_r\|_Q^2 \le k_r (k_r > 0)$, $\|u_r\|_\Sigma^2 \le p_r$, $\forall$r = 1,2,3,4., $\|\overrightarrow{y_n}(0)\|_0^2 \le c$, using trace theorem for the last terms of the above inequality, to get:

$$\|\overrightarrow{y_n}(t)\|_0^2 \le s + c_5\int_0^t \|\vec{y}_n\|_0^2\, dt + c_6\int_0^t \|\vec{y}_n\|_0^2\, dt \le s + c_7\int_0^t \|\vec{y}_n\|_0^2\, dt$$

where, $s = c + \sum_{r=1}^4 (k_r + p_r)$, $c_7 = c_5 + c_6$

By using the Belman Grownwall Inequality, to obtain:

$\|\overrightarrow{y_n}(t)\|_0^2 \le z_1, \forall t \in [0,T] \Rightarrow \|\overrightarrow{y_n}(t)\|_{L^\infty(I,L^2(\Omega))} \le z_2$, where $z_2 = \sqrt{z_1}$, and

$\|\overrightarrow{y_n}(t)\|_Q^2 = \int_0^T \|\overrightarrow{y_n}\|_0^2\, dt \le Tz_1 = z_3^2 \Rightarrow \|\overrightarrow{y_n}(t)\|_Q \le z_3$

The norm $\|\overrightarrow{y_n}(t)\|_{L^2(I,V)}$ is bounded:

Using Lemma (2.1) for the 1st of L.H.S. in Eq. (21), then using the same results which were obtained from the R.H.S., setting $t = T$ and using $\|\overrightarrow{y_n}(t)\|_0^2$ is positive so Eq. (21) becomes:

$$\|\overrightarrow{y_n}(t)\|_0^2 + 2\bar{\alpha}\int_0^T \|\overrightarrow{y_n}\|_1^2\, dt \le s + c_7\|\overrightarrow{y_n}(t)\|_Q^2$$

$$\Rightarrow \int_0^T \|\overrightarrow{y_n}\|_1^2\, dt \le \frac{s + c_7 z_3^2}{2\bar{\alpha}} = z_4^2 \quad \Rightarrow \|\overrightarrow{y_n}(t)\|_{L^2(I,V)} \le z_4.$$

The Convergence of the Solution:

Let $\{\overrightarrow{V_n}\}_{n=1}^\infty$ be a sequence of subspace of $\vec{V}$ s.t. $\forall \vec{v} \in \vec{V}$, there exists a sequence $\{\overrightarrow{v_n}\}$ with $\overrightarrow{v_n} \in V_n \subset V \subset \boldsymbol{L^2(\Omega)}$, and $\overrightarrow{v_n} \xrightarrow{s} \vec{v}$ in $\vec{V}$ s.t. $\overrightarrow{v_n} \xrightarrow{s} \vec{v}$ in $\boldsymbol{L^2(\Omega)}$, Utilizing $\vec{v} = \overrightarrow{v_n}$ in ((13) - (16)), to obtain:

$$(y_{1nt}, v_{1n})_\Omega + a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega - (b_5(t)y_{2n}, v_{1n})_\Omega + (b_6(t)y_{3n}, v_{1n})_\Omega + (b_7(t)y_{4n}, v_{1n})_\Omega = (f_1(y_{1n}), v_{1n})_\Omega + (u_1, v_{1n})_\Gamma \qquad (22a)$$

$$(y_{1n}^0, v_{1n})_\Omega = (y_1^0, v_{1n})_\Omega \qquad (22b)$$

$$(y_{2nt}, v_{2n})_\Omega + a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega + (b_5(t)y_{1n}, v_{2n})_\Omega - (b_9(t)y_{3n}, v_{2n})_\Omega - (b_{11}(t)y_{4n}, v_{2n})_\Omega = (f_2(y_{2n}), v_{2n})_\Omega + (u_2, v_{2n})_\Gamma \qquad (23a)$$

$$(y_{2n}^0, v_{2n})_\Omega = (y_2^0, v_{2n})_\Omega \qquad (23b)$$

$$(y_{3nt}, v_{3n})_\Omega + a_3(t, y_{3n}, v_{3n}) + (b_3(t)y_{3n}, v_{3n})_\Omega + (b_9(t)y_{2n}, v_{3n})_\Omega - \qquad (24a)$$

$$(b_6(t)y_{1n},v_{3n})_\Omega+(b_{15}(t)y_{4n},v_{3n})_\Omega = (f_3(y_{3n}),v_{3n})_\Omega + (u_3,v_{3n})_\Gamma$$

$$(y^0_{3n},v_{3n})_\Omega=(y^0_3,v_{3n})_\Omega \quad (24b)$$

$$(y_{4nt},v_{4n})_\Omega+a_4(t,y_{4n},v_{4n})+(b_4(t)y_{4n},v_{4n})_\Omega-(b_7(t)y_{1n},v_{4n})_\Omega+(b_{11}(t)y_{2n},v_{4n})_\Omega-(b_{15}(t)y_{3n},v_{4n})_\Omega = (f_4(y_{4n}),v_{4n})_\Omega + (u_4,v_{4n})_\Gamma \quad (25a)$$

$$(y^0_{4n},v_{4n})_\Omega=(y^0_4,v_{4n})_\Omega \quad (25b)$$

The WF ((8)-(11))has a sequence of QSVS $\{\overrightarrow{y_n}\}_{n=1}^{\infty}$, with $\|\overrightarrow{y_n}(t)\|_{L^2(Q)}$, $\|\overrightarrow{y_n}(t)\|_{L^2(I,V)}$ are bounded, then by Alauglu's theorem, there exists a sub sequence of $\{\overrightarrow{y_n}\}_{n\in N}$, say again $\{\overrightarrow{y_n}\}_{n\in N}$ s.t. $\overrightarrow{y_n}\xrightarrow{w}\vec{y}$ in $\boldsymbol{L^2(Q)}$ and $\overrightarrow{y_n}\xrightarrow{w}\vec{y}$ in $\boldsymbol{L^2(I,V)}$**,** then by the first compactness theorem, assumption (A-i), and the above indicate norms were bounded, we get $\overrightarrow{y_n}\xrightarrow{S}\vec{y}$ in $\boldsymbol{L^2(Q)}$.
Now, multiplying both sides of Eqs. ((22)-(25)) by $\varphi_r(t)\in C^1[0,T]$, $\forall r=1,2,3,4$. S.t. $\varphi_r(T)=0$,$\varphi_r(0)\neq 0$ and Integrating by parts with respect to on [0, T], to obtain:

$$\int_0^T(y_{1nt},v_{1n})\varphi_1(t)\,dt+\int_0^T[a_1(t,y_{1n},v_{1n})+(b_1(t)y_{1n},v_{1n})_\Omega-(b_5(t)y_{2n},v_{1n})_\Omega+(b_6(t)y_{3n},v_{1n})_\Omega+(b_7(t)y_{4n},v_{1n})_\Omega]\varphi_1(t)dt = \int_0^T(f_1(y_{1n}),v_{1n})_\Omega\varphi_1(t)dt+\int_0^T(u_1,v_{1n})_\Gamma\varphi_1(t)dt \quad (26)$$

$$\int_0^T(y_{2nt},v_{2n})\,\varphi_2(t)dt+\int_0^T[a_2(t,y_{2n},v_{2n})+(b_2(t)y_{2n},v_{2n})_\Omega+(b_5(t)y_{1n},v_{2n})_\Omega-(b_9(t)y_{3n},v_{2n})_\Omega-(b_{11}(t)y_{4n},v_{2n})_\Omega]\varphi_2(t)dt = \int_0^T(f_2(y_{2n}),v_{2n})_\Omega\varphi_2(t)dt+\int_0^T(u_2,v_{2n})_\Gamma\varphi_2(t)dt \quad (27)$$

$$\int_0^T(y_{3nt},v_{3n})\varphi_3(t)\,dt+\int_0^T[a_3(t,y_{3n},v_{3n})+(b_3(t)y_{3n},v_{3n})_\Omega+(b_9(t)y_{2n},v_{3n})_\Omega-(b_6(t)y_{1n},v_{3n})_\Omega+(b_{15}(t)y_{4n},v_{3n})_\Omega]\varphi_3(t)dt = \int_0^T(f_3(y_{3n}),v_{3n})_\Omega\varphi_3(t)dt+\int_0^T(u_3,v_{3n})_\Gamma\varphi_3(t)dt \quad (28)$$

$$\int_0^T(y_{4nt},v_{4n})\varphi_4(t)dt+\int_0^T[a_4(t,y_{4n},v_{4n})+(b_4(t)y_{4n},v_{4n})_\Omega-(b_7(t)y_{1n},v_{4n})_\Omega+(b_{11}(t)y_{2n},v_{4n})_\Omega-(b_{15}(t)y_{3n},v_{4n})_\Omega]\varphi_4(t)dt = \int_0^T(f_4(y_{4n}),v_{4n})_\Omega\varphi_4(t)dt+\int_0^T(u_4,v_{4n})_\Gamma\varphi_4(t)dt \quad (29)$$

Integrating by parts the first terms in the L.H.S. i.e.,

$$-\int_0^T(y_{1n},v_{1n})\varphi_1'(t)\,dt+\int_0^T[a_1(t,y_{1n},v_{1n})+(b_1(t)y_{1n},v_{1n})_\Omega-(b_5(t)y_{2n},v_{1n})_\Omega+(b_6(t)y_{3n},v_{1n})_\Omega+(b_7(t)y_{4n},v_{1n})_\Omega]\varphi_1(t)dt = \int_0^T(f_1(y_{1n}),v_{1n})_\Omega\varphi_1(t)dt+\int_0^T(u_1,v_{1n})_\Gamma\varphi_1(t)dt+(y^0_{1n},v_{1n})_\Omega\,\varphi_1(0) \quad (30)$$

$$-\int_0^T(y_{2n},v_{2n})\,\varphi_2'(t)dt+\int_0^T[a_2(t,y_{2n},v_{2n})+(b_2(t)y_{2n},v_{2n})_\Omega+ \quad (31)$$

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$$(b_5(t)y_{1n},v_{2n})_\Omega-(b_9(t)y_{3n},v_{2n})_\Omega-(b_{11}(t)y_{4n},v_{2n})_\Omega]\varphi_2(t)dt=\int_0^T(f_2(y_{2n}),v_{2n})_\Omega\varphi_2(t)dt+\int_0^T(u_2,v_{2n})_\Gamma\varphi_2(t)dt+(y_{2n}^0,v_{2n})_\Omega\,\varphi_2(0)$$

$$-\int_0^T(y_{3n},v_{3n})\varphi_3'(t)\,dt+\int_0^T[\,a_3(t,y_{3n},v_{3n})+(b_3(t)y_{3n},v_{3n})_\Omega+(b_9(t)y_{2n},v_{3n})_\Omega-(b_6(t)y_{1n},v_{3n})_\Omega+(b_{15}(t)y_{4n},v_{3n})_\Omega]\varphi_3(t)dt=\int_0^T(f_3(y_{3n}),v_{3n})_\Omega\varphi_3(t)dt+\int_0^T(u_3,v_{3n})_\Gamma\varphi_3(t)dt+(y_{3n}^0,v_{3n})_\Omega\varphi_3(0) \quad (32)$$

$$-\int_0^T(y_{4n},v_{4n})\varphi_4'(t)dt+\int_0^T[\,a_4(t,y_{4n},v_{4n})+(b_4(t)y_{4n},v_{4n})_\Omega-(b_7(t)y_{1n},v_{4n})_\Omega+(b_{11}(t)y_{2n},v_{4n})_\Omega-(b_{15}(t)y_{3n},v_{4n})_\Omega]\varphi_4(t)dt=\int_0^T(f_4(y_{4n}),v_{4n})_\Omega\varphi_4(t)dt+\int_0^T(u_4,v_{4n})_\Gamma\varphi_4(t)dt+(y_{4n}^0,v_{4n})_\Omega\varphi_4(0) \quad (33)$$

Since $v_{rn}\xrightarrow{w}v_r$ in V, and in $L^2(\Omega)$, then $v_{rn}\varphi_r'(t)\xrightarrow{w}v_r\,\varphi_r'(t)$ in $L^2(I,V)$, and $v_{rn}\varphi_r(t)\xrightarrow{w}v_r\,\varphi_r(t)$ in $L^2(Q)$, and since $y_{rn}\xrightarrow{W}y_r$ in $L^2(Q)$, $y_{rn}^0\xrightarrow{s}y_r^0$ in $L^2(\Omega)$, $\forall r=1,2,3,4$. Then the following converges are held:

$$-\int_0^T(y_{1n},v_{1n})\varphi_1'(t)\,dt+\int_0^T[\,a_1(t,y_{1n},v_{1n})+(b_1(t)y_{1n},v_{1n})_\Omega-(b_5(t)y_{2n},v_{1n})_\Omega+(b_6(t)y_{3n},v_{1n})_\Omega+(b_7(t)y_{4n},v_{1n})_\Omega]\varphi_1(t)dt\to-\int_0^T(y_1,v_1)\varphi_1'(t)\,dt+\int_0^T[\,a_1(t,y_1,v_1)+(b_1(t)y_1,v_1)_\Omega-(b_5(t)y_2,v_1)_\Omega+(b_6(t)y_3,v_1)_\Omega+(b_7(t)y_4,v_1)_\Omega]\varphi_1(t)dt \quad (34a)$$

$$(y_{1n}^0,v_{1n})_\Omega\varphi_1(0)\to(y_1^0,v_1)_\Omega\varphi_1(0) \quad (34b)$$

$$-\int_0^T(y_{2n},v_{2n})\,\varphi_2'(t)dt+\int_0^T[\,a_2(t,y_{2n},v_{2n})+(b_2(t)y_{2n},v_{2n})_\Omega+(b_5(t)y_{1n},v_{2n})_\Omega-(b_9(t)y_{3n},v_{2n})_\Omega-(b_{11}(t)y_{4n},v_{2n})_\Omega]\varphi_2(t)dt\to-\int_0^T(y_2,v_2)\,\varphi_2'(t)dt+\int_0^T[\,a_2(t,y_2,v_2)+(b_2(t)y_2,v_2)_\Omega+(b_5(t)y_1,v_2)_\Omega-(b_9(t)y_3,v_2)_\Omega-(b_{11}(t)y_4,v_2)_\Omega]\varphi_2(t)dt \quad (35a)$$

$$(y_{2n}^0,v_{2n})_\Omega\varphi_2(0)\to(y_2^0,v_2)_\Omega\varphi_2(0) \quad (35b)$$

$$-\int_0^T(y_{3n},v_{3n})\varphi_3'(t)\,dt+\int_0^T[\,a_3(t,y_{3n},v_{3n})+(b_3(t)y_{3n},v_{3n})_\Omega+(b_9(t)y_{2n},v_{3n})_\Omega-(b_6(t)y_{1n},v_{3n})_\Omega+(b_{15}(t)y_{4n},v_{3n})_\Omega]\varphi_3(t)dt\to-\int_0^T(y_3,v_3)\varphi_3'(t)\,dt+\int_0^T[\,a_3(t,y_3,v_3)+(b_3(t)y_3,v_3)_\Omega+(b_9(t)y_2,v_3)_\Omega-(b_6(t)y_1,v_3)_\Omega+(b_{15}(t)y_4,v_3)_\Omega]\varphi_3(t)dt \quad (36a)$$

$$(y_{3n}^0,v_{3n})_\Omega\varphi_3(0)\to(y_3^0,v_3)_\Omega\varphi_3(0) \quad (36b)$$

$$-\int_0^T(y_{4n},v_{4n})\varphi_4'(t)dt+\int_0^T[\,a_4(t,y_{4n},v_{4n})+(b_4(t)y_{4n},v_{4n})_\Omega-(b_7(t)y_{1n},v_{4n})_\Omega+(b_{11}(t)y_{2n},v_{4n})_\Omega-(b_{15}(t)y_{3n},v_{4n})_\Omega]\varphi_4(t)dt\to-\int_0^T(y_4,v_4)\varphi_4'(t)dt+\int_0^T[\,a_4(t,y_4,v_4)+(b_4(t)y_4,v_4)_\Omega-(b_7(t)y_1,v_4)_\Omega+(b_{11}(t)y_2,v_4)_\Omega-(b_{15}(t)y_3,v_4)_\Omega]\varphi_4(t)dt \quad (37a)$$

$$(y_{4n}^0,v_{4n})_\Omega\varphi_4(0)\to(y_4^0,v_4)_\Omega\varphi_4(0) \quad (37b)$$

On the other hand, since f_r for $r=1,2,3,4$ is of Ca-T and from assumption (A-i) and

Proposition 2.1, $\int_Q f_r(y_{rn})v_{rn}\varphi_r\,dx\,dt$ is cont., and with $v_{rn}\varphi_r \overset{S}{\to} v_r$ in $L^2(Q)$ and since $\vec{y}_n \overset{S}{\to} \vec{y}$ in $\boldsymbol{L^2(Q)}$, we get that

$\int_0^T (f_r(y_{rn}), v_{rn})_\Omega \varphi_r(t)dt \to \int_0^T (f_r(y_r), v_r)_\Omega \varphi_r(t)dt.$

From Eqs. ((34) - (37) a and b) and from the above converge, so Eqs. ((30)-(33)) will be in the form:

$$-\int_0^T (y_1, v_1)\varphi_1'(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(y_1), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1^0, v_1)_\Omega\,\varphi_1(0) \quad (38)$$

$$-\int_0^T (y_2, v_2)\,\varphi_2'(t)dt + \int_0^T [\,a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(y_2), v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt + (y_2^0, v_2)_\Omega\,\varphi_2(0) \quad (39)$$

$$-\int_0^T (y_3, v_3)\varphi_3'(t)\,dt + \int_0^T [\,a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3(y_3), v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt + (y_3^0, v_3)_\Omega \varphi_3(0) \quad (40)$$

$$-\int_0^T (y_4, v_4)\varphi_4'(t)dt + \int_0^T [\,a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega + (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4(y_4), v_4)_\Omega \varphi_4(t)dt + \int_0^T (u_4, v_4)_\Gamma \varphi_4(t)dt + (y_4^0, v_4)_\Omega \varphi_4(0) \quad (41)$$

Case 1: Choose $\varphi_r \in D[0, T]$, i.e., $\varphi_r(T) = \varphi_r(0) = 0$,$\forall r = 1,2,3,4$. So, then (38)-(41)) became:

$$-\int_0^T (y_1, v_1)\varphi_1'(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(y_1), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt \quad (42)$$

$$-\int_0^T (y_2, v_2)\,\varphi_2'(t)dt + \int_0^T [\,a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(y_2), v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt \quad (43)$$

$$-\int_0^T (y_3, v_3)\varphi_3'(t)\,dt + \int_0^T [\,a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3(y_3), v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt \quad (44)$$

$$-\int_0^T (y_4, v_4)\varphi_4'(t)dt + \int_0^T [\,a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega\ (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4(y_4), v_4)_\Omega \varphi_4(t)dt + \int_0^T (u_4, v_4)_\Gamma \varphi_4(t)dt \quad (45)$$

Using IBP the first terms in the L.H.S. of the above Eqs.:

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$$\int_0^T (y_{1t}, v_1)\varphi_1(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(y_1), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt \quad (46)$$

$$\int_0^T (y_{2t}, v_2)\,\varphi_2(t)dt + \int_0^T [\,a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(y_2), v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt \quad (47)$$

$$\int_0^T (y_{3t}, v_3)\varphi_3(t)\,dt + \int_0^T [\,a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3(y_3), v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt \quad (48)$$

$$\int_0^T (y_{4t}, v_4)\varphi_4(t)dt + \int_0^T [\,a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega + (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4(y_4), v_4)_\Omega \varphi_4(t)dt + \int_0^T (u_4, v_4)_\Gamma \varphi_4(t)dt \quad (49)$$

i.e., $\vec{y}$ is App of WF Eqs. ((8)-(11)).

Case 2: Choose $\varphi_r \in C^1[0,\mathrm{T}]$, i.e., $\varphi_r(T) = 0$ and $\varphi_r(0) \neq 0, \forall r = 1,2,3,4,$
Using integrating by parts IBP for the first term in the L.H.S. of (46), to get:

$$-\int_0^T (y_1, v_1)\varphi_1'(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(y_1), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1(0), v_1)_\Omega \varphi_1(0) \quad (50)$$

By subtracting Eq. (38) from Eq. (50), we get:
$(y_1^0, v_1)_\Omega \varphi_1(0) = (y_1(0), v_1)_\Omega \varphi_1(0), \quad \varphi_1(0) \neq 0, \forall \varphi_1(0) \in [0,\mathrm{T}]$
$(y_1^0, v_1)_\Omega = (y_1(0), v_1)_\Omega$
i.e., the first ICs holds. Same manner can be utilized to get that the ICs are held.

The Strong Convergence for the Approximation Solution

Utilizing $v_r = y_{rn}$ and $v_r = y_r, \forall r = 1,2,3,4,$In ((8a) - (11a)) and ((13a) - (16a)) resp. and integrating by parts these equations. On [0, T], then adding all the equations together, finally, using assumption (A-iii), to get:

$$\int_0^T (\overrightarrow{y_{nt}}, \overrightarrow{y_n})dt + \int_0^T c(t, \overrightarrow{y_n}, \overrightarrow{y_n})\,dt = \int_0^T (f_1(y_{1n}), y_{1n})_\Omega dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (f_2(y_{2n}), y_{2n})_\Omega\,dt + \int_0^T (u_2, y_{2n})_\Gamma\,dt + \int_0^T (f_3(y_{3n}), y_{3n})_\Omega dt + \int_0^T (u_3, y_{3n})_\Gamma dt + \int_0^T (f_4(y_{4n}), y_{4n})_\Omega dt + \int_0^T (u_4, y_{4n})_\Gamma\,dt \quad (51a)$$

$$\int_0^T (\vec{y_t}, \vec{y})dt + \int_0^T c(t, \vec{y}, \vec{y})\,dt = \int_0^T (f_1(y_1), y_1)_\Omega dt + \int_0^T (u_1, y_1)_\Gamma dt + \int_0^T (f_2(y_2), y_2)_\Omega\,dt + \int_0^T (u_2, y_2)_\Gamma\,dt + \int_0^T (f_3(y_3), y_3)_\Omega dt + \int_0^T (u_3, y_3)_\Gamma dt + \int_0^T (f_4(y_4), y_4)_\Omega dt + \int_0^T (u_4, y_4)_\Gamma\,dt \quad (51b)$$

Using Lemma (2.1) for the first terms in the L.H.S. of Eq. ((51) a and b) it becomes:

$$\frac{1}{2}\|\overrightarrow{y_n}(T)\|_0^2 - \frac{1}{2}\|\overrightarrow{y_n}(0)\|_0^2 + \int_0^T c(t, \overrightarrow{y_n}, \overrightarrow{y_n})\,dt = \int_0^T (f_1(y_{1n}), y_{1n})_\Omega dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (f_2(y_{2n}), y_{2n})_\Omega\,dt + \int_0^T (u_2, y_{2n})_\Gamma\,dt + \quad (52a)$$

$\int_0^T (f_3(y_{3n}), y_{3n})_\Omega dt + \int_0^T (u_3, y_{3n})_\Gamma dt + \int_0^T (f_4(y_{4n}), y_{4n})_\Omega dt + \int_0^T (u_4, y_{4n})_\Gamma dt$

$$\frac{1}{2}\|\vec{y}(T)\|_0^2 - \frac{1}{2}\|\vec{y}(0)\|_0^2 + \int_0^T c(t,\vec{y},\vec{y})\, dt = \int_0^T (f_1(y_1), y_1)_\Omega dt + \int_0^T (u_1, y_1)_\Gamma dt + \int_0^T (f_2(y_2), y_2)_\Omega dt + \int_0^T (u_2, y_2)_\Gamma dt + \int_0^T (f_3(y_3), y_3)_\Omega dt + \int_0^T (u_3, y_3)_\Gamma dt + \int_0^T (f_4(y_4), y_4)_\Omega dt + \int_0^T (u_4, y_4)_\Gamma dt \quad (52b)$$

Now, consider the following equality:

$$\frac{1}{2}\|\overrightarrow{y_n}(T) - \vec{y}(T)\|_0^2 - \frac{1}{2}\|\overrightarrow{y_n}(0) - \vec{y}(0)\|_0^2 + \int_0^T c(t,\overrightarrow{y_n} - \vec{y},\overrightarrow{y_n} - \vec{y})\, dt = P_1 - P_2 - P_3 \quad (53)$$

Where

$P_1 = \frac{1}{2}\|\overrightarrow{y_n}(T)\|_0^2 - \frac{1}{2}\|\overrightarrow{y_n}(0)\|_0^2 + \int_0^T c(t,\overrightarrow{y_n},\overrightarrow{y_n})\, dt$

$P_2 = \frac{1}{2}(\overrightarrow{y_n}(T),\vec{y}(T))_\Omega - \frac{1}{2}(\overrightarrow{y_n}(0),\vec{y}(0))_\Omega + \int_0^T c(t,\overrightarrow{y_n}(t),\vec{y}(t))\, dt$

$P_3 = \frac{1}{2}(\vec{y}(T),\overrightarrow{y_n}(T) - \vec{y}(T))_\Omega - \frac{1}{2}(\vec{y}(0),\overrightarrow{y_n}(0) - \vec{y}(0))_\Omega + \int_0^T c(t,\vec{y}(t),\overrightarrow{y_n}(t) - \vec{y}(t))dt$

Since,

$$\overrightarrow{y_n^0} = \overrightarrow{y_n}(0) \xrightarrow{s} \overrightarrow{y^0} = \vec{y}(0), \text{ in } \boldsymbol{L^2(\Omega)}, \quad \overrightarrow{y_n}(T) \xrightarrow{s} \vec{y}(T) \text{ in } \boldsymbol{L^2(\Omega)} \quad (54)$$

Then,

$$(\vec{y}(0),\overrightarrow{y_n}(0) - \vec{y}(0))_\Omega \to 0 \text{ and } (\vec{y}(T),\overrightarrow{y_n}(T) - \vec{y}(T))_\Omega \to 0 \quad (55)$$

$$\|\overrightarrow{y_n}(0) - \vec{y}(0)\|_0^2 \to 0 \text{ and } \|\overrightarrow{y_n}(T) - \vec{y}(T)\|_0^2 \to 0 \quad (56)$$

and since $\overrightarrow{y_n} \xrightarrow{w} \vec{y}$ in $\boldsymbol{L^2(I,V)}$ $\Rightarrow \overrightarrow{y_n} \xrightarrow{w} \vec{y}$ in $\boldsymbol{L^2(\Sigma)}$, then

$$\int_0^T c(t,\vec{y}(t),\overrightarrow{y_n}(t) - \vec{y}(t))dt \to 0 \quad (57)$$

From proposition (2.1), $\int_0^T (f_r(y_{rn}), y_{rn})_\Omega dt$ is continuous with respect to y_r, hence:

$$\int_0^T (f_1(y_{1n}), y_{1n})_\Omega dt + \int_0^T (f_2(y_{2n}), y_{2n})_\Omega dt + \int_0^T (f_3(y_{3n}), y_{3n})_\Omega dt + \int_0^T (f_4(y_{4n}), y_{4n})_\Omega dt \to \int_0^T (f_1(y_1), y_1)_\Omega dt + \int_0^T (f_2(y_2), y_2)_\Omega dt + \int_0^T (f_3(y_3), y_3)_\Omega dt + \int_0^T (f_4(y_4), y_4)_\Omega dt \quad (58)$$

Now, when $n \to \infty$ in both sides of Eq. (53), the following results are considered:

1. The first two terms in L.H.S. of Eq. (53) are tending to zero from Eq. (56).
2. The L.H.S. of Eq. (P_1) →L.H.S. of Eq. (52b).
3. Eq. (P_2) → L.H.S. (52b)
4. The three terms in P_3 are tending to zero from Eqs. (55) and (57), from the above convergences. The above sides of Eq. (53) give (as $n \to \infty$):

$\int_0^T c(t,\overrightarrow{y_n} - \vec{y},\overrightarrow{y_n} - \vec{y})\, dt \to 0$, Eq. (57)

By assumption (A-iii):

$\bar{\alpha}\int_0^T \|\overrightarrow{y_n} - \vec{y}\|_1^2\, dt \le \int_0^T c(t,\overrightarrow{y_n} - \vec{y},\overrightarrow{y_n} - \vec{y})\, dt \to 0$, So, $\overrightarrow{y_n} \xrightarrow{s} \vec{y}$ in $\boldsymbol{L^2(I,V)}$.

Uniqueness of the Solution

Let $\vec{y}$ and $\vec{\hat{y}}$ be two QSVs for the WF, substituting these QSVs in the WF, subtracting each equation (for $\vec{y}$) from its Corresponding equation (for $\vec{\hat{y}}$), then setting $v_r = y_r - \hat{y}_r$ to obtain:

$$((y_1 - \hat{y}_1)_t, y_1 - \hat{y}_1) + a_1(t, y_1 - \hat{y}_1, y_1 - \hat{y}_1) + (b_1(t)y_1 - \hat{y}_1, y_1 - \hat{y}_1)_\Omega - (b_5(t)y_2 - \hat{y}_2, y_1 - \hat{y}_1)_\Omega + (b_6(t)y_3 - \hat{y}_3, y_1 - \hat{y}_1)_\Omega + (b_7(t)y_4 - \hat{y}_4, y_1 - \hat{y}_1)_\Omega = (f_1(y_1), y_1 - \hat{y}_1)_\Omega - (f_1(\hat{y}_1), y_1 - \hat{y}_1)_\Omega \quad (59)$$

$$((y_2 - \hat{y}_2)_t, y_2 - \hat{y}_2) + a_2(t, y_2 - \hat{y}_2, y_2 - \hat{y}_2) + (b_2(t)y_2 - \hat{y}_2, y_2 - \hat{y}_2)_\Omega \quad (60)$$

$$+(b_5(t)y_1-\hat{y}_1,y_2-\hat{y}_2)_\Omega-(b_9(t)y_3-\hat{y}_3,y_2-\hat{y}_2)_\Omega-(b_{11}(t)y_4-\hat{y}_4,y_2-\hat{y}_2)_\Omega=(f_2(y_2),y_2-\hat{y}_2)_\Omega-(f_2(\hat{y}_2),y_2-\hat{y}_2)_\Omega$$

$$((y_3-\hat{y}_3)_t,y_3-\hat{y}_3)+a_3(t,y_3-\hat{y}_3,y_3-\hat{y}_3)+(b_3(t)y_3-\hat{y}_3,y_3-\hat{y}_3)_\Omega+(b_9(t)y_2-\hat{y}_2,y_3-\hat{y}_3)_\Omega-(b_6(t)y_1-\hat{y}_1,y_3-\hat{y}_3)_\Omega+(b_{15}(t)y_4-\hat{y}_4,y_3-\hat{y}_3)_\Omega=(f_3(y_3),y_3-\hat{y}_3)_\Omega-(f_3(\hat{y}_3),y_3-\hat{y}_3)_\Omega \quad (61)$$

$$((y_4-\hat{y}_4)_t,y_4-\hat{y}_4)+a_4(t,y_4-\hat{y}_4,y_4-\hat{y}_4)+(b_3(t)y_4-\hat{y}_4,y_4-\hat{y}_4)_\Omega-(b_7(t)y_1-\hat{y}_1,y_4-\hat{y}_4)_\Omega+(b_{11}(t)y_2-\hat{y}_2,y_4-\hat{y}_4)_\Omega-(b_{15}(t)y_3-\hat{y}_3,y_4-\hat{y}_4)_\Omega=(f_4(y_4),y_4-\hat{y}_4)_\Omega-(f_4(\hat{y}_4),y_4-\hat{y}_4)_\Omega \quad (62)$$

Adding Eqs. ((59) - (62)) and using Lemma (2.1), finally using assumptions (A-ii and iii) to obtain:

$$\frac{1}{2}\frac{d}{dt}\|\vec{y}-\vec{\hat{y}}\|_0^2+\bar{\alpha}\|\vec{y}-\vec{\hat{y}}\|_1^2 \le \int_\Omega [L_1|y_1-\hat{y}_1|^2+L_2|y_2-\hat{y}_2|^2+L_3|y_3-\hat{y}_3|^2+L_4|y_4-\hat{y}_4|^2]dx \quad (63)$$

Since the second term of L.H.S. of Eq. (63) is positive, and by IBS from 0 to t, to get:

$\|(\vec{y}-\vec{\hat{y}})(t)\|_0^2 \le \int_0^t 2L\|\vec{y}-\vec{\hat{y}}\|_0^2 dt,$ $L=\max\{L_1,L_2,L_3,L_4\}$.

Using the BGI, to get:

$$\|(\vec{y}-\vec{\hat{y}})(t)\|_0^2 \le 0e^{\int_0^t 2Ldt}=0, \forall t\in I$$

Now, integrating both sides BS of (63) on [0, T], using the IC and the above result:

$$\int_0^T \frac{d}{dt}\|\vec{y}-\vec{\hat{y}}\|_0^2 dt+2\bar{\alpha}\int_0^T\|\vec{y}-\vec{\hat{y}}\|_1^2 dt \le L\int_0^T\|\vec{y}-\vec{\hat{y}}\|_0^2 dt \le 0$$

$$\Rightarrow \int_0^T\|\vec{y}-\vec{\hat{y}}\|_1^2 dt=0 \Rightarrow \vec{y}=\vec{\hat{y}}.$$

Existence of a QCCBOCV

In this section, the continuity of the LIO between the QSVS of the WF for the QNLPBVP and the corresponding QCCBCV is proved. The existence theorem of a QCCBOCV is stated and demonstrated under suitable assumptions.

Theorem (4.1):

a. Consider assumptions (A) are hold, $\vec{y}$ and $\vec{y}+\Delta\vec{y}$ are the QSVS corresponding to the QCCBCV $\vec{u}$ and $\vec{u}+\Delta\vec{u}$ resp., with $\vec{u}$ and $\Delta\vec{u}$ are bounded in $\boldsymbol{L^2(\Sigma)}$, then:

$$\|\Delta\vec{y}\|_{L^\infty(I,L^2(\Omega))} \le k\|\Delta\vec{u}\|_\Sigma,$$
$$\|\Delta\vec{y}\|_{L^2(Q)} \le k\|\Delta\vec{u}\|_\Sigma, \|\Delta\vec{y}\|_{L^2(I,V)} \le k\|\Delta\vec{u}\|_\Sigma$$

b. with assumptions (A), the LIO $\vec{u}\to\vec{y}_{\vec{u}}$ from $\boldsymbol{L^2(\Sigma)}$ into $\boldsymbol{L^\infty(I,L^2(\Omega))}$ or into $\boldsymbol{L^2(Q)}$, or into $\boldsymbol{L^2(I,V)}$ is cont.

Proof:

a. Let $\vec{u},\vec{\hat{u}}\in \boldsymbol{L^2(\Sigma)}$ and let $\Delta\vec{u}=\vec{\hat{u}}-\vec{u}$, hence by theorem (3.1), there exists a unique QSVS $\vec{y}$ and $\vec{\hat{y}}$ of Eqs. ((8) - (11)), satisfies the following equations:

$$(\hat{y}_{1t},v_1)+a_1(t,\hat{y}_1,v_1)+(b_1(t)\hat{y}_1,v_1)_\Omega-(b_5(t)\hat{y}_2,v_1)_\Omega+(b_6(t)\hat{y}_3,v_1)_\Omega+(b_7(t)\hat{y}_4,v_1)_\Omega=(f_1(x,t,\hat{y}_1),v_1)_\Omega+(\hat{u}_1,v_1)_\Gamma \quad (64a)$$

$$(\hat{y}_1(0),v_1)_\Omega=(\hat{y}_1^0,v_1)_\Omega \quad (64b)$$

$$(\hat{y}_{2t},v_2)+a_2(t,\hat{y}_2,v_2)+(b_2(t)\hat{y}_2,v_2)_\Omega+(b_5(t)\hat{y}_1,v_2)_\Omega-(b_9(t)\hat{y}_3,v_2)_\Omega-(b_{11}(t)\hat{y}_4,v_2)_\Omega=(f_2(x,t,\hat{y}_2),v_2)_\Omega+(\hat{u}_2,v_2)_\Gamma \quad (65a)$$

$$(\hat{y}_2(0),v_2)_\Omega=(\hat{y}_2^0,v_2)_\Omega \quad (65b)$$

$$(\hat{y}_{3t},v_3)+a_3(t,\hat{y}_3,v_3)+(b_3(t)\hat{y}_3,v_3)_\Omega+(b_9(t)\hat{y}_2,v_3)_\Omega-(b_6(t)\hat{y}_1,v_3)_\Omega+(b_{15}(t)\hat{y}_4,v_3)_\Omega=(f_3(x,t,\hat{y}_3),v_3)_\Omega+(\hat{u}_3,v_3)_\Gamma \quad (66a)$$

$$(\hat{y}_3(0),v_3)_\Omega=(\hat{y}_3^0,v_3)_\Omega \quad (66b)$$

$$(\hat{y}_{4t}, v_4) + a_4(t, \hat{y}_4, v_4) + (b_4(t)\hat{y}_4, v_4)_\Omega - (b_7(t)\hat{y}_1, v_4)_\Omega + (b_{11}(t)\hat{y}_2, v_4)_\Omega - (b_{15}(t)\hat{y}_3, v_4)_\Omega = (f_4(x,t,\hat{y}_4), v_4)_\Omega + (\hat{u}_4, v_4)_\Gamma \quad (67a)$$

$$(\hat{y}_4(0), v_4)_\Omega = (\hat{y}_4^0, v_4)_\Omega \quad (67b)$$

Subtracting the WF from Eqs. ((64-67) a and b), then setting $\Delta y_r = \hat{y}_r - y_r$, $\forall r = 1,2,3,4$, to get:

$$(\Delta y_{1t}, v_1) + a_1(t, \Delta y_1, v_1) + (b_1(t)\Delta y_1, v_1)_\Omega - (b_5(t)\Delta y_2, v_1)_\Omega + (b_6(t)\Delta y_3, v_1)_\Omega + (b_7(t)\Delta y_4, v_1)_\Omega = (f_1(y_1 + \Delta y_1), v_1)_\Omega - (f_1(y_1), v_1)_\Omega + (\Delta u_1, v_1)_\Gamma \quad (68a)$$

$$(\Delta y_1(0), v_1)_\Omega = 0 \quad (68b)$$

$$(\Delta y_{2t}, v_2) + a_2(t, \Delta y_2, v_2) + (b_2(t)\Delta y_2, v_2)_\Omega + (b_5(t)\Delta y_1, v_2)_\Omega - (b_9(t)\Delta y_3, v_2)_\Omega - (b_{11}(t)\Delta y_4, v_2)_\Omega = (f_2(y_2 + \Delta y_2), v_2)_\Omega - (f_2(y_2), v_2)_\Omega + (\Delta u_2, v_2)_\Gamma \quad (69a)$$

$$(\Delta y_2(0), v_2)_\Omega = 0 \quad (69b)$$

$$(\Delta y_{3t}, v_3) + a_3(t, \Delta y_3, v_3) + (b_3(t)\Delta y_3, v_3)_\Omega + (b_9(t)\Delta y_2, v_3)_\Omega - (b_6(t)\Delta y_1, v_3)_\Omega + (b_{15}(t)\Delta y_4, v_3)_\Omega = (f_3(y_3 + \Delta y_3), v_3)_\Omega - (f_3(y_3), v_3)_\Omega + (\Delta u_3, v_3)_\Gamma \quad (70a)$$

$$(\Delta y_3(0), v_3)_\Omega = 0 \quad (70b)$$

$$(\Delta y_{4t}, v_4) + a_4(t, \Delta y_4, v_4) + (b_4(t)\Delta y_4, v_4)_\Omega - (b_7(t)\Delta y_1, v_4)_\Omega + (b_{11}(t)\Delta y_2, v_4)_\Omega - (b_{15}(t)\Delta y_3, v_4)_\Omega = (f_4(y_4 + \Delta y_4), v_4)_\Omega - (f_4(y_4), v_4)_\Omega + (\Delta u_4, v_4)_\Gamma \quad (71a)$$

$$(\Delta y_4(0), v_4)_\Omega = 0 \quad (71b)$$

By utilizing $v_r = \Delta y_r$, $\forall r = 1,2,3,4$, into Eqs. ((68a) - (71a)) respectively. Adding the obtained four equations together, then using Lemma (2.1) for the first term in L.H.S., and finally using assumption (A-iii), to get:

$$\frac{1}{2}\frac{d}{dt}\|\Delta\vec{y}\|_0^2 + \bar{\alpha}\|\Delta\vec{y}\|_1^2 \le |(f_1(y_1 + \Delta y_1) - f_1(y_1), \Delta y_1)| + |(f_2(y_2 + \Delta y_2) - f_2(y_2), \Delta y_2)| + |(f_3(y_3 + \Delta y_3) - f_3(y_3), \Delta y_3)| + |(f_4(y_4 + \Delta y_4) - f_4(y_4), \Delta y_4)| + |(\Delta u_1, \Delta y_1)_\Gamma| + |(\Delta u_2, \Delta y_2)_\Gamma| + |(\Delta u_3, \Delta y_3)_\Gamma| + |(\Delta u_4, \Delta y_4)_\Gamma| \quad (72)$$

Since the second term in the L.H.S. of (72) is positive, then integrating both sies for t from 0 to t, using assumption (A-ii) for the first two terms in the R.H.S. of the above equation, to obtain:

$$\int_0^t \frac{d}{dt}\|\Delta\vec{y}\|_0^2\, dt \le \int_0^t \|\Delta u_1\|_\Gamma^2 dt + 2L_1\int_0^t \|\Delta y_1\|_0^2 dt + \int_0^t \|\Delta y_1\|_\Gamma^2 dt + \int_0^t \|\Delta u_2\|_\Gamma^2 dt + 2L_2\int_0^t \|\Delta y_2\|_0^2 dt + \int_0^t \|\Delta y_2\|_\Gamma^2 dt + \int_0^t \|\Delta u_3\|_\Gamma^2 dt + 2L_3\int_0^t \|\Delta y_3\|_0^2 dt + + \int_0^t \|\Delta y_3\|_\Gamma^2 dt + \int_0^t \|\Delta u_4\|_\Gamma^2 dt + 2L_4\int_0^t \|\Delta y_4\|_0^2 dt + \int_0^t \|\Delta y_4\|_\Gamma^2 dt$$

$$\Rightarrow \|\Delta\vec{y}\|_0^2 \le \|\Delta u_1\|_\Sigma^2 + \|\Delta u_2\|_\Sigma^2 + \|\Delta u_3\|_\Sigma^2 + \|\Delta u_4\|_\Sigma^2 + s_1\int_0^t \|\Delta\vec{y}\|_0^2 dt + \int_0^t \|\Delta\vec{y}\|_\Gamma^2 dt$$

where $s_1 = \max(2L_1 2L_2, 2L_3, 2L_4)$
Now, by using the Trace Theorem, to get:

$$\|\Delta\vec{y}\|_0^2 \le \|\Delta\vec{u}\|_\Sigma^2 + s_3\int_0^t \|\Delta\vec{y}\|_0^2 dt,$$

where, $s_3 = s_1 + s_2$
Using the Bellman Gromwell inequality:

$$\|\Delta\vec{y}\|_0^2 \le \|\Delta\vec{u}\|_\Sigma^2 e^{\int_0^t s_3\, dt} \le K^2\|\Delta\vec{u}\|_\Sigma^2, \ K > 0 \text{ for each } t \in [0, T]$$

$$\Longrightarrow \|\Delta\vec{y}\|_{L^\infty(I, L^2(\Omega))} \le K\|\Delta\vec{u}\|_\Sigma$$

Since,

$$\|\Delta\vec{y}\|_{L^2(Q)}^2 = \int_0^t \|\Delta\vec{y}\|_{L^2(\Omega)}^2 dt \le TK^2\|\Delta\vec{u}\|_\Sigma^2$$

$$\Longrightarrow \|\Delta\vec{y}\|_{L^2(Q)} \le K\|\Delta\vec{u}\|_\Sigma \text{ , where } TK^2 = K^2$$

Using, the same way which was used in the above steps for R.H.S. of (72), then the integrating both sides for t from 0 to T, to get:

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$\int_0^T \frac{d}{dt}\|\Delta\vec{y}\|_0^2\,dt + 2\bar{\alpha}\int_0^T \|\Delta\vec{y}\|_1^2\,dt \leq \|\Delta\vec{u}\|_\Sigma^2 + s_3\int_0^t \|\Delta\vec{y}\|_0^2\,dt$

Then $\int_0^T \|\Delta\vec{y}\|_1^2\,dt \leq (1+s_3K^2)/2\bar{\alpha}\|\Delta\vec{u}\|_\Sigma^2 = K^2\|\Delta\vec{u}\|_\Sigma^2 \Rightarrow \|\Delta\vec{y}\|_{L^2(I,V)} \leq K\|\Delta\vec{u}\|_\Sigma$

b. Let $\Delta\vec{u} = \vec{\hat{u}} - \vec{u}$ and $\Delta\vec{y} = \vec{\hat{y}} - \vec{y}$ where $\vec{\hat{y}}$and $\vec{y}$ are the corresponding QSVS to the QCCBCV $\vec{\hat{u}}$and $\vec{u}$, then using part (a) of this theorem, to get:

$$\|\vec{\hat{y}} - \vec{y}\|_{L^\infty(I,L^2(\Omega))} \leq K\|\vec{\hat{u}} - \vec{u}\|_\Sigma .$$

This means the LIO $\vec{u} \to \vec{y}$ is continuous from $\boldsymbol{L^2(\Sigma)}$into $\boldsymbol{L^\infty\left(I, L^2(\Omega)\right)}$ the other results are obtained similarly.

Assumption (B)

Consider g_{0r}, $h_{0r}, r = 1,2,3,4$ is of Ca-T on $(Q\times\mathbb{R})$ and on $(\Sigma\times\mathbb{R})$ respe and satisfy the following condition with respect toy_r, u_r

$|\, g_{0r}(x,t,y_r)\,| \leq \gamma_{0r}(x,t) + c_{0r}(y_r)^2$,

$|\, h_{0r}(x,t,u_r)\,| \leq \delta_{0r}(x,t) + d_{0r}(u_r)^2$,

where $y_r, u_r \in \mathbb{R}$, $\gamma_{0r} \in L^1(Q)$, $\delta_{0r} \in L^1(\Sigma)$

Lemma (3.3.1): With assumptions (B), the functional $G_0(\vec{u})$ is cont on $\boldsymbol{L^2(\Sigma)}$,

Proof: From assumptions (B), we get that

$\|\, g_{0r}(x,t,y_r)\| \leq \gamma_{0r}(x,t) + c_{0r}\|y_r\|^2$,

$\|\, h_{0r}(x,t,y_r)\| \leq \delta_{0r}(x,t) + d_{0r}\|u_r\|^2$

(where$\|.\|$ is the usual Euclidean norm). Using Proposition (2.1), we have

$\int_Q g_{0r}(x,t,y_r)\,dxdt$, $\int_\Sigma h_{0r}(x,t,u_r)\,d\sigma$ are cont on $L^2(Q)$,$L^2(\Sigma)$ respe $\forall r = 1,2,3,4$.so, $G_0(\vec{u})$ is cont on $\boldsymbol{L^2(\Sigma)}$.

Theorem (4.2):

In addition to Assumptions (A) and (B). If $\vec{U}$ in $\vec{W}_{\vec{A}}$ is convex and compact ,$\vec{W}_{\vec{A}} \neq \emptyset$. If $\forall r = 1,2,3,4$, $G_0(\vec{u})$is convex with respect to $\vec{u}$ for fixed $(x,t,\vec{y})$. Then there exists a QCCBOVC**.**

Proof: Since $\vec{U}$ is convex and compact then $\vec{W}_A$ is weakly com. Since $\vec{W}_{\vec{A}} \neq \emptyset$. Then there exists a minimizing sequence $\{\vec{u}_k\} \in \vec{W}_A$,$\forall K$ s.t.$\lim_{n\to\infty} G_0(\vec{u}_k) = inf_{\vec{u}\in\vec{W}_A} G_0(\vec{u})$. Then by Alauglu's theorem there exists a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ s.t. $\vec{u}_k \xrightarrow{w} \vec{u}$ in $\boldsymbol{L^2(\Sigma)}$ and $\|\vec{u}_k\|_\Sigma \leq c, \forall k$. Then by theorem (3.1), there exists a eqauence of QSVS $\{\vec{y}_k\}$, corresponding to the Seq of the CCBOQCV $\{\vec{u}_k\}$ and that $\|\vec{y}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ and$\|\vec{y}_k\|_{L^2(I,V)}$ are bounded. Then by Alauglu's theorem there exists a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ s.t.

$\vec{y}_k = \vec{y}_{\vec{u}_k} \xrightarrow{w} \vec{y} = \vec{y}_{\vec{u}}$ in $\boldsymbol{L^\infty(I, L^2(\Omega))}$, $\boldsymbol{L^2(Q)}$, and $\boldsymbol{L^2(I,V)}$

And again from theorem (3.1) we got $\|\vec{y}_{kt}\|_{L^2(I,V^*)}$ is bounded and since $\boldsymbol{L^2(I,V)} \subset \boldsymbol{L^2(Q)} \cong (\boldsymbol{L^2(Q)})^* \subset \boldsymbol{L^2(I,V^*)}$. Hence by the FCTH there exists a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ s.t. $\vec{y}_k \xrightarrow{S} \vec{y}$ in $\boldsymbol{L^2(Q)}$.

Now, since $\vec{y}_k$ is the QSVS corresponding to the $\vec{u}_k$, then:

$$(y_{1kt}, v_1) + a_1(t, y_{1k}, v_1) + (b_1(t)y_{1k}, v_1)_\Omega - (b_5(t)y_{2k}, v_1)_\Omega + (b_6(t)y_{3k}, v_1)_\Omega + (b_7(t)y_{4k}, v_1)_\Omega = (f_1(x,t,y_{1k}), v_1)_\Omega + (u_{1k}, v_1)_\Gamma \quad (73)$$

$$(y_{2kt}, v_2) + a_2(t, y_{2k}, v_2) + (b_2(t)y_{2k}, v_2)_\Omega + (b_5(t)y_{1k}, v_2)_\Omega - (b_9(t)y_{3k}, v_2)_\Omega - (b_{11}(t)y_{4k}, v_2)_\Omega = (f_2(x,t,y_{2k}), v_2)_\Omega + (u_{2k}, v_2)_\Gamma \quad (74)$$

$$(y_{3kt}, v_3) + a_3(t, y_{3k}, v_3) + (b_3(t)y_{3k}, v_3)_\Omega + (b_9(t)y_{2k}, v_3)_\Omega - (b_6(t)y_{1k}, v_3)_\Omega + (b_{15}(t)y_{4k}, v_3)_\Omega = (f_3(x,t,y_{3k}), v_3)_\Omega + (u_{3k}, v_3)_\Gamma \quad (75)$$

$$(y_{4kt}, v_4) + a_4(t, y_{4k}, v_4) + (b_4(t)y_{4k}, v_4)_\Omega - (b_7(t)y_{1k}, v_4)_\Omega + (b_{11}(t)y_{2k}, v_4)_\Omega - (b_{15}(t)y_{3k}, v_4)_\Omega = (f_4(x,t,y_{4k}), v_4)_\Omega + (u_{4k}, v_4)_\Gamma \quad (76)$$

(75)

Then let $\varphi_r \in c^1[0,T]$, s.t. $\varphi_r(T) = 0, \forall r = 1,2,3,4$, multiplying both sides of Eqs. ((73)-(76)) by $\varphi_1(t)$, $\varphi_2(T)$, $\varphi_3(T)$ and $\varphi_4(T)$ resp., then integrating both sides of each obtained equation with respect to t from 0 to T. Finally, integrating by parts for the first terms in L.H.S., to get:

$$-\int_0^T (y_{1k},v_1)\varphi_1'(t)\,dt + \int_0^T [a_1(t,y_{1k},v_1) + (b_1(t)y_{1k},v_1)_\Omega - (b_5(t)y_{2k},v_1)_\Omega + (b_6(t)y_{3k},v_1)_\Omega + (b_7(t)y_{4k},v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(x,t,y_{1k}),v_1)_\Omega\varphi_1(t)dt + \int_0^T (u_{1k},v_1)_\Gamma\varphi_1(t)dt + (y_{1k}(0),v_1)_\Omega\varphi_1(0) \quad (77)$$

$$-\int_0^T (y_{2k},v_2)\,\varphi_2'(t)dt + \int_0^T [a_2(t,y_{2k},v_2) + (b_2(t)y_{2k},v_2)_\Omega + (b_5(t)y_{1k},v_2)_\Omega - (b_9(t)y_{3k},v_2)_\Omega - (b_{11}(t)y_{4k},v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(x,t,y_{2k}),v_2)_\Omega\varphi_2(t)dt + \int_0^T (u_{2k},v_2)_\Gamma\varphi_2(t)dt + (y_{2k}(0),v_2)_\Omega\varphi_2(0) \quad (78)$$

$$-\int_0^T (y_{3k},v_3)\varphi_3'(t)\,dt + \int_0^T [a_3(t,y_{3k},v_3) + (b_3(t)y_{3k},v_3)_\Omega + (b_9(t)y_{2k},v_3)_\Omega - (b_6(t)y_{1k},v_3)_\Omega + (b_{15}(t)y_{4k},v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3(x,t,y_{3k}),v_3)_\Omega\varphi_3(t)dt + \int_0^T (u_{3k},v_3)_\Gamma\varphi_3(t)dt + (y_{3k}(0),v_3)_\Omega\varphi_3(0) \quad (79)$$

$$-\int_0^T (y_{4k},v_4)\varphi_4'(t)dt + \int_0^T [a_4(t,y_{4k},v_4) + (b_4(t)y_{4k},v_4)_\Omega - (b_7(t)y_{1k},v_4)_\Omega + (b_{11}(t)y_{2k},v_4)_\Omega - (b_{15}(t)y_{3k},v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4(x,t,y_{4k}),v_4)_\Omega\varphi_4(t)dt + \int_0^T (u_{4k},v_4)_\Gamma\varphi_4(t)dt + (y_{4k}(0),v_4)_\Omega\varphi_4(0) \quad (80)$$

Since $\vec{y}_k \xrightarrow{w} \vec{y}$ in $\boldsymbol{L^2(Q)}$ and $\boldsymbol{L^2(I,V)}$, $\vec{y}_k(0) \xrightarrow{s} \vec{y}(0)$ in $\boldsymbol{L^2(\Omega)}$. Then using similar steps to those which were used in the proof of theorem (3.1), we can prove that the L.H.S. of Eqs. ((77)-(80)) convergence. For the R.H.S. since $\vec{u}_k \xrightarrow{w} \vec{u}$ in $\boldsymbol{L^2(\Sigma)}$, then:

$$\int_\Gamma (u_{rk},v_r)\,d\gamma dt \longrightarrow \int_\Gamma (u_r,v_r)\,d\gamma dt\,.$$

On the other side since f_r is of a Ca-T and so as the proof of theorem 3.1, we can show that

$$\int_0^T (f_r(y_{rk}),v_{rk})_\Omega\varphi_r(t)dt \to \int_0^T (f_r(y_r),v_r)_\Omega\varphi_r(t)dt\,, \forall r = 1,2,3,4$$

Finally, by the previous convergences, gets:

$$-\int_0^T (y_1,v_1)\varphi_1'(t)\,dt + \int_0^T [a_1(t,y_1,v_1) + (b_1(t)y_1,v_1)_\Omega - (b_5(t)y_2,v_1)_\Omega + (b_6(t)y_3,v_1)_\Omega + (b_7(t)y_4,v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(x,t,y_1),v_1)_\Omega\varphi_1(t)dt + \int_0^T (u_1,v_1)_\Gamma\varphi_1(t)dt + (y_1(0),v_1)_\Omega\varphi_1(0) \quad (81)$$

$$-\int_0^T (y_2,v_2)\,\varphi_2'(t)dt + \int_0^T [a_2(t,y_2,v_2) + (b_2(t)y_2,v_2)_\Omega + (b_5(t)y_1,v_2)_\Omega - (b_9(t)y_3,v_2)_\Omega - (b_{11}(t)y_4,v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(x,t,y_2),v_2)_\Omega\varphi_2(t)dt + \int_0^T (u_2,v_2)_\Gamma\varphi_2(t)dt + (y_2(0),v_2)_\Omega\,\varphi_2(0) \quad (82)$$

$$-\int_0^T (y_3,v_3)\varphi_3'(t)\,dt + \int_0^T [a_3(t,y_3,v_3) + (b_3(t)y_3,v_3)_\Omega + (b_9(t)y_2,v_3)_\Omega - (b_6(t)y_1,v_3)_\Omega + (b_{15}(t)y_4,v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3(x,t,y_3),v_3)_\Omega\varphi_3(t)dt + \int_0^T (u_3,v_3)_\Gamma\varphi_3(t)dt + (y_3(0),v_3)_\Omega\varphi_3(0) \quad (83)$$

$$-\int_0^T (y_4,v_4)\varphi_4'(t)dt + \int_0^T [a_4(t,y_4,v_4) + (b_4(t)y_4,v_4)_\Omega - (b_7(t)y_1,v_4)_\Omega + (b_{11}(t)y_2,v_4)_\Omega - (b_{15}(t)y_3,v_4)_\Omega]\varphi_4(t)dt = \quad (84)$$

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$$\int_0^T (f_4(x,t,y_4),v_4)_\Omega \varphi_4(t)dt + \int_0^T (u_4,v_4)_\Gamma \varphi_4(t)dt + (y_4(0),v_4)_\Omega \varphi_4(0)$$

Now, one has the following two cases:

Case1: Choose $\varphi_r \in D[0,T]$, i.e., $\varphi_r(T) = \varphi_r(0) = 0$,$\forall r = 1,2,3,4$, Now, by using integrating both sides for the first terms in the L.H.S. of ((80) – (83)), to get:

$$\int_0^T (y_{1t},v_1)\,\varphi_1(t)dt + \int_0^T [\,a_1(t,y_1,v_1) + (b_1(t)y_1,v_1)_\Omega - (b_5(t)y_2,v_1)_\Omega + (b_6(t)y_3,v_1)_\Omega + (b_7(t)y_4,v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(x,t,y_1),v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1,v_1)_\Gamma \varphi_1(t)dt \quad (85)$$

$$\int_0^T (y_{2t},v_2)\,\varphi_2(t)dt + \int_0^T [\,a_2(t,y_2,v_2) + (b_2(t)y_2,v_2)_\Omega + (b_5(t)y_1,v_2)_\Omega - (b_9(t)y_3,v_2)_\Omega - (b_{11}(t)y_4,v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(x,t,y_2),v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2,v_2)_\Gamma \varphi_2(t)dt \quad (86)$$

$$\int_0^T (y_{3t},v_3)\varphi_3(t)\,dt + \int_0^T [\,a_3(t,y_3,v_3) + (b_3(t)y_3,v_3)_\Omega + (b_9(t)y_2,v_3)_\Omega - (b_6(t)y_1,v_3)_\Omega + (b_{15}(t)y_4,v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3(x,t,y_3),v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3,v_3)_\Gamma \varphi_3(t)dt \quad (87)$$

$$\int_0^T (y_{4t},v_4)\varphi_4(t)dt + \int_0^T [\,a_4(t,y_4,v_4) + (b_4(t)y_4,v_4)_\Omega - (b_7(t)y_1,v_4)_\Omega + (b_{11}(t)y_2,v_4)_\Omega - (b_{15}(t)y_3,v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4(x,t,y_4),v_4)_\Omega \varphi_4(t)dt + \int_0^T (u_4,v_4)_\Gamma \varphi_4(t)dt \quad (88)$$

This gives that the QSVS $\vec{y} = \vec{y}_{\vec{u}}$ satisfy the WF ((8a)-(11a)).

Case 2: Choose $\varphi_r \in C^1[0,T]$, i.e., $\varphi_r(T) = 0$ and $\varphi_r(0) \neq 0, \forall r = 1,2,3,4$. Using IBP the first term in the L.H.S. of (84), to obtain:

$$-\int_0^T (y_1,v_1)\varphi_1'(t)\,dt + \int_0^T [\,a_1(t,y_1,v_1) + (b_1(t)y_1,v_1)_\Omega - (b_5(t)y_2,v_1)_\Omega + (b_6(t)y_3,v_1)_\Omega + (b_7(t)y_4,v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(x,t,y_1),v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1,v_1)_\Gamma \varphi_1(t)dt + (y_1^0,v_1)_\Omega \varphi_1(0) \quad (89)$$

Subtracting Eq. (89) from Eq. (81) to get : $(y_1^0,v_1)_\Omega \varphi_1(0) = (y_1(0),v_1)_\Omega \varphi_1(0), \forall \varphi_1 \in [0,T] \Longrightarrow y_1^0 = y_1(0)$.

Same manner can be utilized to get that the ICs ((8b) - (11b)) are held.

Since $\forall r = 1,2,3,4$, $g_{0r}(x,t,y_r)$ is cont with respect to y_r. Then from assumptions (B) and Lemma (2.1), one has

$$\int_Q g_{0r}(x,t,y_{rk})dx\,dt \to \int_Q g_{0r}(x,t,y_r)dx\,dt \quad (90)$$

From the hypotheses on h_{0r}, $h_{0r}(x,t,u_r)$ is WLSC with respect to u_r, then with using Eq. (90), to get:

$$\int_Q g_{0r}(x,t,y_r)dx\,dt + \int_\Sigma h_{0r}(x,t,u_r)d\sigma \le \lim_{k\to\infty} inf \int_\Sigma h_{0r}(x,t,u_{rk})d\sigma + \int_Q g_{0r}(x,t,y_r)dx\,dt$$
$$= \lim_{k\to\infty} inf \int_\Sigma h_{0r}(x,t,u_{rk})d\sigma + \lim_{k\to\infty} inf \int_Q g_{0r}(x,t,y_{rk})dx\,dt + \lim_{k\to\infty} inf \int_Q (g_{0r}(x,t,y_r) - g_{0r}(x,t,y_{rk}))dx\,dt$$
$$= \lim_{k\to\infty} inf \int_\Sigma h_{0r}(x,t,u_{rk})d\sigma + \lim_{k\to\infty} inf \int_Q g_{0r}(x,t,y_{rk})dx\,dt \Longrightarrow$$
$$G_0(\vec{u}) \le \lim_{k\to\infty} inf\, G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = inf_{\vec{u}\in\vec{W}_A} G_0(\vec{u}) \Longrightarrow$$
$$G_0(\vec{u}) = min_{\vec{u}\in\vec{W}_A} G_0(\vec{u})$$

Thus $\vec{u}$ is a QCCBOVC

CONCLUSIONS

In this paper, the Galerkin method, along with the first compactness theorem, is successfully employed to prove the existence theorem of a unique QSVS for the WF of the QNLPBVP under suitable assumptions when the QCCBCV is fixed. The continuity of the Lipschitz operator between the QSVS of the WF for the QNLPBVP and the QCCBOCV is also demonstrated. Furthermore, the existence theorem of a QCCBOCV governed by the QNLPBVP is developed and proven under appropriate assumptions.

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