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Classical Continuous Constraint Boundary Optimal Control Vector Problem for Triple Nonlinear Parabolic System

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ABSTRACT

In this paper, our purpose is to study the classical continuous constraints boundary optimal triple control vector problem dominating nonlinear triple parabolic boundary value problem. The existence theorem for a classical continuous triple optimal control vector CCCBOTCV is stated and proved under suitable assumptions. Mathematical formulation of the adjoint triple boundary value problem associated with the nonlinear triple parabolic boundary value problem is discovered. The Fréchet derivative of the Hamiltonian derived. Under proper assumptions, both theorems are granted; the necessary conditions for optimality and the sufficient conditions for optimality of the classical continuous constraints boundary optimal triple control vector problem are stated and prove.

KEYWORDS: Classical constraints boundary optimal triple control, nonlinear triple parabolic boundary value problem, necessary and sufficient optimality conditions.

الخلاصة

في هذا البحث هدفنا هو دراسة مسالة متجه السيطرة التقليدية المستمرة الثلاثية الحدودية الأمثلية المقيدة لمسائل القيم الحدودية الثلاثية غير الخطية المكافئة، تم ذكر نص وبرهان مبرهنة وجود سيطرة امثلية ثلاثية حدودية مقيدة بوجود شروط مناسبة. تم ايجاد الصياغة الرياضية لمسالة القيم الحدودية الثلاثية المصاحبة لمسالة القيم الحدودية غير الخطية المكافئة ومن ثم ايجاد مشتقه فريشيه لدالة الهاملتون. بوجود شروط مناسبة، تم ذكر نص وبرهان مبرهنتي الشروط الضرورية والكافية لوجود لمساله متجه السيطرة الثلاثية الحدودية الأمثلية المقيدة التقليدية المستمرة.

INTRODUCTION

Optimal control problems (OCPs) play an important role in many practical applications, such as in medicine [1], aircraft [2], economics [3], robotics [4], weather conditions [5] and many other scientific fields. They are two types of OCPs; the classical and the relax type. The first type was studied mostly in the last century, while the second was studied in the beginning of this century. Each one of these two types is dominated either by nonlinear ordinary differential equations (ODEs) [6] or by nonlinear PDEs (NLPDEs) [7]. The classical continuous constraints boundary optimal control problem (CCCBOTCP) dominated by nonlinear parabolic or elliptic or hyperbolic PDEs are studied in [8-10] respectively (resp.). Later, the study of the CCCBOTCPs dominated by each one of these types of PDEs are generalized in [11-13] to deal with CCCBOTCPs dominated by couple NLPDEs (CNLPDES) of these types respectively, and then the studies for the couple nonlinear elliptic and hyperbolic PDEs types are generalized also to deal with CCCBOTCPs dominated by triple NLPDEs of these two indicated types respectively [14, 15]. All of the studies mentioned have motivated us to consider generalization, the study of the CCCBOTCP dominated by CNLPDEs of parabolic type to study the classical continuous constraints boundary optimal triple control vector problem (CCCBOTCVP) dominating by nonlinear triple parabolic boundary value problem (NLTPBVP). According to this generalization, the mathematical model for the dominating equations is needed to be found, as well as the cost function, the spaces of definition for the control and the state vectors, which all of them are needed to be generalized.

In this paper, the CCCBOTCVP dominated by the NLTPBVP is proposed. Section2 deals with problem description, and some mathematical concepts, In Section 3 the statement and proof of the existence theorem of a classical continuous triple optimal control vector (CCCBOTCV) under suitable Assumptions are studied. The mathematical formulation for the adjoint triple boundary value problem (ATBVP) associated with TNLPBVP is investigated. The Fréchet derivative (FD) of the Hamiltonian"(Ham) is derived. Both theorems the necessary conditions (NCOs) for optimality (OP) and the sufficient conditions (SCOs) for OP of the considered CCCBOTCP are stated and proved under suitable Assumptions.

Problem Description

Let $I = (0,T)$, with $T < \infty$, $\Omega \subset \mathbb{R}^2$ be an open and bounded region with Lipschitz boundary $\Gamma = \partial\Omega$, $Q = \Omega \times I$, $\Sigma = \Gamma \times I$. Consider the following CCCBOTCP which is consisted of the triple state equations (TSVEs) describe by the following TNLPPDEs:

$$y_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial y_1}{\partial x_j}) + b_1 y_1 - b_4 y_2 - b_5 y_3 = f_1(x,t,y_1), \text{ in Q,} \quad (1)$$

$$y_{2t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}\frac{\partial y_2}{\partial x_j}) + b_2 y_2 + b_6 y_3 + b_4 y_1 = f_2(x,t,y_2), \text{ in Q,} \quad (2)$$

$$y_{3t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(c_{ij}\frac{\partial y_3}{\partial x_j}) + b_3 y_3 + b_5 y_1 - b_6 y_2 = f_3(x,t,y_3), \text{ in Q,} \quad (3)$$

$$\frac{\partial y_1}{\partial n_a} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial y_1}{\partial x_j}\cos(n_1,x_j) = u_1(x,t), \quad (4)$$

$$y_1(x,0) = y_1^0(x) \text{ , in} \quad (5)$$

$$\frac{\partial y_2}{\partial n_b} = \sum_{i,j=1}^{n} b_{ij}\frac{\partial y_2}{\partial x_j}\cos(n_2,x_j) = u_2(x,t), \quad (6)$$

$$y_2(x,0) = y_2^0(x) \text{ , in } \Omega \quad (7)$$

$$\frac{\partial y_3}{\partial n_a} = \sum_{i,j=1}^{n} c_{ij}\frac{\partial y_3}{\partial x_j}\cos(n_1,x_j) = u_3(x,t) \quad (8)$$

$$y_3(x,0) = y_3^0(x), \text{ in } \Omega \quad (9)$$

where $(f_1,f_2,f_3) \in (L^2(\mathrm{Q}))^3$ is given, $(x_1,x_2) \in \Omega$, $a_{lij}(x,t)$, $b_l(x,t) \in C^\infty(Q)$, n_ℓ, (for $\ell = 1,2,3$) is a unit vector normal outer on the boundary Σ, (n_ℓ, x_j) is the angle between n_ℓ and the $x_j - axis$, $\vec{u} = (u_1,u_2,u_3) \in (L^2(\Sigma))^3$ is a CCCBTCV and $\vec{y} = \vec{y}_{\vec{u}} = (y_{1u_1}, y_{2u_2}, y_{3u_2}) \in (H^2(\Omega))^3$ is the triple state vector solution(TSVS) corresponding to the CCCBTCV.

The set of admissible CCCBTCV (ACCBTCV) is:

$$\vec{W}_A = \{\vec{u} \in \vec{W} | \vec{u} \in \vec{U} \text{ a.e. in } \Sigma, G_1(\vec{w}) = 0, G_2(\vec{w}) \leq 0\}$$

$\vec{U} = U_1 \times U_2 \times U_3 \subset \mathbb{R}^3$ is convex set, and $\vec{W} = (L^2(\Sigma))^3$,

The cost function (CF) is

$$G_0(\vec{u}) = \int_Q [g_{01}(x,t,y_1) + g_{02}(x,t,y_2)]dxdt + \int_\Sigma [h_{01}(x,t,u_1) + h_{02}(x,t,u_2)]d\sigma \quad (10)$$

The state vector constraints (SVCs) are

$$G_1(\vec{u}) = \int_Q [g_{11}(x,t,y_1) + g_{12}(x,t,y_2)]dxdt + \int_\Sigma [h_{11}(x,t,u_1) + h_{12}(x,t,u_2)]d\sigma = 0 \quad (11)$$

$$G_2(\vec{u}) = \int_Q^- [g_{21}(x,t,y_1) + g_{22}(x,t,y_2)]dxdt + \int_\Sigma [h_{21}(x,t,u_1) + h_{22}(x,t,u_2)]d\sigma \leq 0 \quad (12)$$

Let $\vec{V} = V_1 \times V_2 \times V_3 = \mathrm{V} \times \mathrm{V} \times \mathrm{V} = \{\vec{v}: \vec{v} = (v_1(\mathrm{x}), v_2(\mathrm{x}), v_3(\mathrm{x}),) \in (H^1(\Omega))^3\}$, the weak form (WFO) of the TSVEs (1-9) when $\vec{y} \in (H^1(\Omega))^3$ is given by:

$$\langle y_{1t}, v_1\rangle + a_1(t,y_1,v_1) + (b_1(t)y_1,v_1)_\Omega - (b_4(t)y_2,v_1)_\Omega - (b_5(t)y_3,v_1)_\Omega = (f_1(y_1),v_1)_\Omega + (u_1,v_1)_\Gamma, \forall v_1 \in V \quad (13a)$$

$$(y_1^0, v_1)_\Omega = (y_1(0), v_1)_\Omega \quad (14)$$

$$\langle y_{2t}, v_2\rangle + a_2(t,y_2,v_2) + (b_2(t)y_2,v_2)_\Omega + (b_6(t)y_3,v_2)_\Omega + (b_4(t)y_1,v_2)_\Omega = (f_2(y_2),v_2)_\Omega + (u_2,v_2)_\Gamma, \forall v_2 \in V \quad (15)$$

$$(y_2^0, v_2)_\Omega = (y_2(0), v_2)_\Omega \quad (16)$$

$$\langle y_{3t}, v_3\rangle + a_3(t,y_3,v_3) + (b_3(t)y_3,v_3)_\Omega + (b_5(t)y_1,v_3)_{\Omega -} (b_6(t)y_2,v_3)_\Omega = (f_3(y_3),v_3)_\Omega + (u_3,v_3)_\Gamma, \forall v_3 \in V \quad (17)$$

$$(y_3^0, v_2)_\Omega = (y_3(0), v_3)_\Omega \quad (18)$$

Where $a_l(t,y_l,v_l)=\int_\Omega \sum_{i,j=1}^{n} a_{lij}\frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}dx$ for $l=1,2,3$

Assumptions (A):

(i) f_i is of a Carathéodory type (C-T) on $Q\times\mathbb{R}$, satisfies $|f_i(x,t,y_i)|\le\eta_i(x,t)+c_i|y_i|$

Where$(x,t)\in Q$,$y_i,u_i\in\mathbb{R}$, $c_i>0$ and $\eta_i\in L^2(Q,\mathbb{R})$, $\forall i=1,2,3$.

(ii) f_i is Lipschitz w.r.t y_i,(, $\forall i=1,2,3$) i.e.:

$|f_i(x,t,y_i)-f_i(x,t,\hat{y}_i)|\le L_i|y_i-\hat{y}_i|$.

Where$(x,t)\in Q$, $y_i,\hat{y}_i\in\mathbb{R}$ and $L_i>0$.

(iii) $|a_i(t,y_i,v_i)|\le\alpha_i\|y_i\|_1\|v_i\|_1$,

$|(b_i(t)y_i,v_i)_\Omega|\le\beta_i\|y_i\|_0\|v_i\|_0$,

$a_i(t,y_i,y_i)\ge\bar{\alpha}_i\|y_i\|_1^2$,

$(b_i(t)y_i,y_i)_\Omega\ge\bar{\beta}_i\|y_i\|_0^2$,

$|(b_4(t)y_2,v_1)_\Omega|\le\epsilon_1\|y_2\|_0\|v_1\|_0$,

$|(b_4(t)y_1,v_2)_\Omega|\le\epsilon_2\|y_1\|_0\|v_2\|_0$,

$|(b_5(t)y_3,v_1)_\Omega|\le\epsilon_3\|y_3\|_0\|v_1\|_0$,

$|(b_5(t)y_1,v_3)_\Omega|\le\epsilon_4\|y_1\|_0\|v_5\|_0$,

$|(b_6(t)y_3,v_2)_\Omega|\le\epsilon_5\|y_3\|_0\|v_2\|_0$,

$|(b_6(t)y_2,v_3)_\Omega|\le\epsilon_6\|y_2\|_0\|v_3\|_0$,

$c(t,\vec{y},\vec{y})=\sum_{i=1}^{3}[a_i(t,y_i,y_i)+(b_i(t)y_i,y_i)_\Omega]$, with $c(t,\vec{y},\vec{y})\ge\bar{\alpha}\|\vec{y}\|_1^2$

here α_i, $\bar{\alpha}_i$, β_i, $\bar{\beta}_i$ ($\forall i=1,2,3$), ϵ_i ($\forall i=1,2,3,4,5,6$) and $\bar{\alpha}$ are real positive constants.

Theorem 1 [16]**:** With assumptions (A), for each "fixed" $\vec{u}\in\left(L^2(\Sigma)\right)^3$, the WFO ((13)-(15)) has a unique TSVS $\vec{y}=(y_1,y_2,y_3)$ s.t. $\vec{y}\in$,$\vec{y}_t=\left(y_{1t},y_{2t,}y_{3t,}\right)\in\left(L^2(\mathrm{I},\mathrm{V})\right)^3$.

Assumptions (B):

Consider g_{li} and h_{li} (for each $l=0,1,2,3$ and $i=1,2,3$) is of C -T on $(Q\times\mathbb{R})$ and on$(\Sigma\times\mathbb{R})$ respectively, and satisfies the following sub quadratic condition with respect to y_i and u_i

$|g_{li}(x,t,y_i)|\le\gamma_{li}(x,t)+c_{li}(y_i)^2$,

$|h_{li}(x,t,u_i)|\le\delta_{li}(x,t)+d_{li}(u_i)^2$

Where$y_i,u_i\in\mathbb{R}$ with $\gamma_{li}\in L^1(Q)$, $\delta_{li}\in L^1(\Sigma)$

Lemma 1[16]**:**

If assumptions (B) are held, the functional $G_l(\vec{u})$ is continuous on $(L^2(\Sigma))^3$, $\forall\, l=0,1,2$.

Theorem 2 [16]**:**

Beside the assumptions (A) and (B). If $\vec{U}$ is compact, $\overrightarrow{W_A}\ne\emptyset$, $G_0(\vec{u})$ is convex. with respect to $\vec{u}$ for fixed $(x,t,\vec{y})$. Then there exists a CCCBOTCV.

Assumptions (C):

If f_{iy_i}, $g_{li_{y_i}}$, $h_{li_{u_i}}$, ($l=0,1,2\&i=1,2,3$) are of C-T on $(Q\times\mathbb{R})$, $(Q\times\mathbb{R})$,$(\Sigma\times\mathbb{R})$ respectively, and $|f_{iy_i}(x,t,y_i)|\le\acute{L}_i(x,t)$,

$\left|g_{li_{y_i}}(x,t,y_i)\right|\le\bar{\gamma}_{li}(x,t)+c_{li}|y_i|$,

$\left|h_{li_{u_i}}(x,t,u_i)\right|\le\bar{\delta}_{li}(x,t)+d_{li}|u_i|$,

Where$(x,t)\in Q$, $y_i,u_i\in\mathbb{R}$, $\bar{\gamma}_{li}(x,t),\acute{L}_i(x,t)\in L^2(Q)$ and $\bar{\delta}_{li}(x,t)\in L^2(\Sigma)$.

RESULTS

Existence of the CCCBOTCV and the FD

This section deals with the existence of the CCCBOTCV and the derivation of the FD under some suitable Assumptions after the ATBVP is defined.

Theorem 3:

In addition to assumptions. (A) and (B), if $\vec{U}$ in the $\overrightarrow{W_A}$ is compact, $\overrightarrow{W_A}\ne\emptyset$. If for each $i=1,2,3$,$G_1(\vec{u})$is independent of u_i , $G_0(\vec{u})$ and $G_2(\vec{u})$ are convex w.r.t u_i, for fixed (x,t,y_i). Then there exists a CCCBOTCV for the considered problem.

Proof:

From the assumptions. on $\vec{U}$ and $G_l(\vec{u})$, for $l=0,1,2$, using Lemma 1 and theorem 2, one can get that there exists a CCCBOTCV.

Theorem 4: Dropping index l in g_l ,h_l and G_l, The Ham H is defined by

$H(x,t,y_i,z_i,u_i)=\sum_{i=1}^{3}(z_if_i(x,t,y_i)+g_i(x,t,y_i)+h_i(x,t,u_i))$

And the ATBVP $z_i=z_{u_i}$ (where $y_i=y_{u_i}$)equation satisfy (in Q):

$$-z_{1t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial z_1}{\partial x_j})+b_1z_1-b_4z_2-b_5z_3=z_1f_{1y_1}(x,t,y_1)+g_{y_1}(x,t,y_1), \quad (19)$$

$$-z_{2t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(b_{ij}\frac{\partial z_2}{\partial x_j})+b_2z_2+b_6z_3+b_4z_1=z_2f_{2y_2}(x,t,y_2)+g_{y_2}(x,t,y_2), \quad (1720)$$

$$-z_{3t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(c_{ij}\frac{\partial z_3}{\partial x_j})+b_2z_2+b_5z_1-b_6z_2=z_3f_{3y_3}(x,t,y_3)+g_{y_3}(x,t,y_3) \quad (18)$$

$$z_i(T)=0,\ \text{in}\ \Omega\ \forall i=1,2,3 \quad (1921)$$

$\frac{\partial z_i}{\partial n} = 0$, on Σ $\forall i = 1,2,3$ (22)

Then the FD of G is

$$\acute{G}(\vec{u})\overrightarrow{\Delta u} = \int_\Sigma \begin{pmatrix} z_1 + h_{1u_1} \\ z_2 + h_{2u_2} \\ z_2 + h_{2u_2} \end{pmatrix} \cdot \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_2 \end{pmatrix} d\sigma$$

Proof:

The WFO of the ATBVP is considered by:

$$-\langle z_{1t}, v_1\rangle + a_1(t, z_1, v_1) + (b_1(t)z_1, v_1)_\Omega - (b_4(t)z_2, v_1)_\Omega - (b_5(t)z_3, v_1)_\Omega = \left(z_1 f_{1y_1}, v_1\right)_\Omega + \left(g_{1y_1}, v_1\right)_\Omega \quad (23)$$

$$-\langle z_{2t}, v_2\rangle + a_2(t, z_2, v_2) + (b_2(t)z_2, v_2)_\Omega + (b_6(t)z_2, v_2)_\Omega + (b_4(t)z_1, v_2)_\Omega = \left(z_2 f_{2y_2}, v_2\right)_\Omega + \left(g_{2y_2}, v_2\right)_\Omega \quad (24)$$

$$-\langle z_{3t}, v_3\rangle + a_3(t, z_3, v_3) + (b_3(t)z_3, v_3)_\Omega + (b_5(t)z_1, v_3)_\Omega + (b_6(t)z_2, v_3)_\Omega = \left(z_3 f_{3y_3}, v_3\right)_\Omega + \left(g_{3y_3}, v_3\right)_\Omega \quad (25)$$

Substituting $v_i = \Delta y_i$, $\forall i = 1,2,3$ in ((21)- (23)) respectively, finally integrating both sides w.r.t from 0 to T, then using integration by parts (IBP) for the 1st obtained term in each equation, finally adding these three equations, to get:

$$\int_0^T \langle\overrightarrow{\Delta y}_t, \vec{z}\rangle dt + \int_0^T [a_1(t, z_1, \Delta y_1) + (b_1(t)z_1, \Delta y_1)_\Omega - (b_4(t)z_2, \Delta y_1)_\Omega - (b_5(t)z_3, \Delta y_1)_\Omega]dt + \int_0^T [a_2(t, z_2, \Delta y_2) + (b_2(t)z_2, \Delta y_2)_\Omega + (b_6(t)z_3, \Delta y_3)_\Omega + (b_4(t)z_1, \Delta y_2)_\Omega + a_3(t, z_3, \Delta y_3)]dt + \int_0^T [(b_3(t)z_3, \Delta y_3)_\Omega + (b_5(t)z_1, \Delta y_3)_\Omega]dt = \int_0^T \left(z_1 f_{1y_1} + g_{1y_1}, \Delta y_1\right)_\Omega dt + \int_0^T \left(z_2 f_{2y_2} + g_{2y_2}, \Delta y_2\right)_\Omega dt + \int_0^T (z_3 f_{3y_3} + g_{3y_3}, \Delta y_3)_\Omega dt \quad (26)$$

Now, substituting $y_i = \Delta y_i$ and $v_i = z_i$ in ((13)-(15)) respectively, $\forall i = 1,2,3$, integrating both sides from 0 to T then adding three obtained equations to get.

$$\int_0^T \langle\overrightarrow{\Delta y}_t, \vec{z}\rangle dt + \int_0^T [a_1(t, \Delta y_1, z_1) + (b_1(t)\Delta y_1, z_1)_\Omega - (b_4(t)\Delta y_2, z_1)_\Omega - (b_5(t)\Delta y_3, z_1)_\Omega + a_2(t, \Delta y_2, z_2) + (b_2(t)\Delta y_2, z_2)_\Omega + (b_6(t)\Delta y_3, z_2)_\Omega + (b_4(t)\Delta y_1, z_2)_\Omega + a_3(t, \Delta y_3, z_3) + (b_3(t)\Delta y_3, z_3)_\Omega + (b_5(t)\Delta y_1, z_3)_\Omega - (b_6(t)\Delta y_2, z_3)_\Omega]dt = \int_0^T (f_1(y_1 + \Delta y_1) - f_1(y_1), z_1)_\Omega dt + \int_0^T (f_2(y_2 + \Delta y_2) - f_2(y_2), z_2)_\Omega dt + \int_0^T (\Delta u_1, z_1)_\Gamma dt + \int_0^T (\Delta u_2, z_2)_\Gamma dt + \int_0^T (f_3(y_3 + \Delta y_3) - f_3(y_3), z_3)_\Omega dt + \int_0^T (\Delta u_3, z_3)_\Gamma dt \quad (27)$$

Now, from the assumptions(A-i), the FD of f_i exists for each $i = 1,2,3$, then from theorem 2- a [16], and the inequality of Minkowski, adding the obtained results, to get:

$$\sum_{i=1}^3 \int_0^T (f_i(x, t, y_i + \Delta y_i) - f_i(x, t, y_i), z_i)_\Omega dt = \sum_{i=1}^3 (\int_0^T (f_{iy_i}\Delta y_i, z_i)dt + \varepsilon_1(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma \quad (28)$$

Where $\sum_{i=1}^3 \varepsilon_{i1}(\Delta y_i) = \varepsilon_1(\overrightarrow{\Delta u}) \longrightarrow 0$, as $\left\|\overrightarrow{\Delta u}\right\|_\Sigma \longrightarrow 0$

By using (26) in R.H.S. of (25), to get:

$$\int_0^T \langle\overrightarrow{\Delta y}_t, \vec{z}\rangle dt + \int_0^T [a_1(t, \Delta y_1, z_1) + (b_1(t)\Delta y_1, z_1)_\Omega - (b_4(t)\Delta y_2, z_1)_\Omega - (b_5(t)\Delta y_3, z_1)_\Omega + a_2(t, \Delta y_2, v_2) + (b_2(t)\Delta y_2, v_2)_\Omega + (b_6(t)\Delta y_3, z_2)_\Omega + (b_4(t)\Delta y_1, z_2)_\Omega + a_3(t, \Delta y_3, z_3) + (b_3(t)\Delta y_3, z_3)_\Omega + (b_5(t)\Delta y_1, z_3)_\Omega - (b_6(t)\Delta y_2, z_3)_\Omega]dt = \int_0^T (f_{1y_1}\Delta y_1, z_1)_\Omega dt + \int_0^T (f_{2y_2}\Delta y_2, z_2)_\Omega dt + \int_0^T \left(f_{3y_3}\Delta y_3, z_3\right)_\Omega dt + \int_0^T (\Delta u_1, z_1)_\Gamma dt + \int_0^T (\Delta u_2, z_2)_\Gamma dt + \int_0^T (\Delta u_3, z_3)_\Gamma dt + \varepsilon_1(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma \quad (29)$$

Now, by subtracting (24) from (27).one get:

$$\int_0^T \left(g_{1y_1}, \Delta y_1\right)_\Omega dt + \int_0^T \left(g_{2y_2}, \Delta y_2\right)_\Omega dt + \int_0^T \left(g_{3y_3}, \Delta y_3\right)_\Omega dt = \int_0^T (\Delta u_1, z_1)_\Gamma\, dt + \int_0^T (\Delta u_2, z_2)_\Gamma dt + \int_0^T (\Delta u_3, z_3)_\Gamma dt + \varepsilon_1(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma \quad (28)$$

Now, let $G_A(\vec{u}) = \int_Q k_1(x, t, y_1, y_2, y_3)dxdt$, $G_B(\vec{u}) = \int_\Sigma k_2(x, t, u_1, u_2, u_3)d\sigma$

Where $k_1(x, t, y_1, y_2, y_3) = g_1(x, t, y_1) + g_2(x, t, y_2) + g_3(x, t, y_3)$, and $k_2(x, t, u_1, u_2) = h_1(x, t, u_1) + h_2(x, t, u_2) + h_3(x, t, u_3)$,

From the definition of the FD and the result of Theorem (2-(a)) [16] and from the assumptions on

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g_i $(\forall i = 1,2,3)$, and then using the inequality of Minkowski once obtains:

$$G(\vec{u}+\overrightarrow{\Delta u}) - G(\vec{u}) = \int_Q (g_{1y_1}\Delta y_1 + g_{2y_2}\Delta y_2 + g_{3y_2}\Delta y_3) + \int_\Sigma (h_{1u_1}\Delta u_1 + h_{2u_2}\Delta u_2 + h_{3u_2}\Delta u_3) d\sigma + \varepsilon_4(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_\Sigma \quad (29)$$

Using (28) in (29), give

$$G(\vec{u}+\overrightarrow{\Delta u}) - G(\vec{u}) = \int_\Sigma (\Delta u_1, z_1)\, d\sigma + \int_\Sigma (\Delta u_2, z_2)\, d\sigma + \int_\Sigma (\Delta u_3, z_3)\, d\sigma + \int_\Sigma (h_{1u_1}\Delta u_1 + h_{2u_2}\Delta u_2 + h_{3u_3}\Delta u_3) d\sigma + \varepsilon_5(\overrightarrow{\Delta u})\|\overrightarrow{\Delta u}\|_\Sigma$$

where, $\varepsilon_1(\overrightarrow{\Delta u}) + \varepsilon_4(\overrightarrow{\Delta u}) = \varepsilon_5(\overrightarrow{\Delta u}) \longrightarrow 0$, as $\|\overrightarrow{\Delta u}\|_\Sigma \longrightarrow 0$

From the FD of G , we get that

$$(\acute{G}(\vec{u}), \overrightarrow{\Delta u}) = \int_\Sigma \begin{pmatrix} z_1 + h_{1u_1} \\ z_2 + h_{2u_2} \\ z_3 + h_{3u_3} \end{pmatrix} \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix} d\sigma. \blacksquare$$

The NCOs and the SCOs

The NCOs and the SCOs theorems under suitable assumptions are considered in this section.

Theorem 5: The NCOs

(i) In addition to the assumptions (A), (B), and (C), if $\vec{u} \in \vec{W}_A$ is a CCBOTCV, then there exist multipliers $\lambda_l \in \mathbb{R}, l = 0,1,2$ with $\lambda_0 \ge 0$, $\lambda_2 \ge 0$, $\sum_{l=0}^{2}|\lambda_l| = 1$, s.t.

$$\sum_{l=0}^{2} \lambda_l \acute{G}_l(\vec{u})\overrightarrow{\Delta u}(\vec{\acute{u}} - \vec{u}) \ge 0, \ \forall \vec{\acute{u}} \in \vec{W}, \quad \overrightarrow{\Delta u} = (\vec{\acute{u}} - \vec{u}) \quad (30)$$

$$\lambda_2 G_2(\vec{u}) = 0 \quad (31)$$

(ii)The inequality (30) is equivalent to the minimum WFO (MWFO)

$$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = \min_{\vec{\acute{u}} \in \vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{\acute{u}}$$

a.e on Σ (32)

Proof:

From lemma (1), the functionals $G_l(\vec{u})$ & $\acute{G}_l(\vec{u})$ (for each $l = 0,1,2$) are continuous with respect to $(\vec{\acute{u}} - \vec{u})$ and liner with respect to $(\vec{\acute{u}} - \vec{u})$, then $G_l(\vec{u})$ is ρ-differential at each $\vec{u} \in \vec{W}$, $\forall \rho$, then by the Kuhn-Tucker-Lagrange theorem, there exist multipliers $\lambda_l \in \mathbb{R}, l = 0,1,2$, with $\lambda_0, \lambda_2 \ge 0$, $\sum_{l=0}^{2}|\lambda_l| = 1$ s.t(30)&(31) hold, and by utilizing the result of the theorem 4, then (30) gives

$$\sum_{l=0}^{2}\sum_{i=1}^{3}\int_\Sigma \lambda_l (z_{li} + h_{li_{u_i}})(\acute{u}_i - u_i) d\sigma \ge 0 \quad (33)$$

Let $z_i = \sum_{l=0}^{3}\lambda_l z_{li}$, $h_{iu_i} = \sum_{l=0}^{3}\lambda_l h_{li_{u_i}}$ $\forall i = 1,2,3$ and $l = 0,1,2$.

Now, let $\{\vec{u_k}\}$ be a dense sequence in $\vec{W}_{\vec{A}}$, and let $S \subset \Sigma$ be a meab set with μ is Lebesgue measure on Σ , s.t :

$$\vec{u}(x,t) = \begin{cases} \vec{u}_k(x,t), & \text{if } (x,t) \in S \\ \vec{u}(x,t), & \text{if } (x,t) \notin S \end{cases}$$

Therefore (33) becomes

$$\int_S H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\,(\vec{u}_k - \vec{u}) \ge 0$$

Since $\mu(\Sigma_k) = 0, \forall k$, then the inequality holds in $\Sigma - \Sigma_k$, and since $\mu(\cup_k \Sigma_k) = 0$, thus it holds in $\Sigma / \cup_k \Sigma_k$. But $\{\vec{u}_k\}$ is a dense sequence in $\vec{W}$, then there is $\vec{\acute{u}} \in \vec{W}$, s.t

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{\acute{u}} - \vec{u}) \ge 0$, a.e. in $\Sigma \Rightarrow$

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = \min_{\vec{\acute{u}} \in \vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{\acute{u}}$, a.e. on Σ

Conversely, let

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = \min_{\vec{\acute{u}} \in \vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{\acute{u}}$, a.e. on $\Sigma \Rightarrow H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{\acute{u}} - \vec{u}) \ge 0, \forall \vec{\acute{u}} \in \vec{W}$, a.e. on $\Sigma \Rightarrow \int_\Sigma H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\Delta u}\, d\sigma \ge 0, \forall \vec{\acute{u}} \in \vec{W}$.

Theorem 6: The SCOs

In addition to the assumptions (A), (B) and (C), suppose for each $i = 1,2,3$, f_i , g_{1i} are affine with respect to y_i for each $(x,t) \in Q$ and h_{1i} is affine with respect to u_i for each $(x,t) \in \Sigma$, g_{0i} , g_{2i}(h_{0i} , h_{2i})are convex with respect to y_i for each $(x,t) \in Q$ (u_i for each $(x,t) \in \Sigma$). Then NCOs in theorem 5 with $\lambda_0 > 0$ are also sufficient.

Proof:

Suppose $\vec{u}$ is satisfied the Kuhn-Tucker-Lagrange conditions, and $\vec{u} \in \vec{W}_A$, i.e.

$\int_\Sigma H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\Delta u}\, d\sigma \ge 0, \forall \vec{\acute{u}} \in \vec{W}$

$\lambda_2 G_2(\vec{u}) = 0$

Let $(\vec{u}) = \sum_{l=0}^{2}\lambda_l G_l(\vec{u})$, then from theorem 4

$$\acute{G}(\vec{u}) \cdot \overrightarrow{\Delta u} = \sum_{l=0}^{3}\lambda_l \acute{G}_l(\vec{u}) \cdot \overrightarrow{\Delta u} = \lambda_0 \int_\Sigma \sum_{i=1}^{3}(z_{0i} + h_{0iu_i})\Delta u_i\, d\sigma + \lambda_1 \int_\Sigma \sum_{i=1}^{3}(z_{1i} + h_{1iu_i})\Delta u_i\, d\sigma + \lambda_2 \int_\Sigma \sum_{i=1}^{3}(z_{2i} + h_{2iu_i})\Delta u_i\, d\sigma$$

Now, consider the three functions in the R.H.S. of TSVEs ((1)-(3)) are affine with respect to y_1, y_2, y_3 respectively, for each $(x,t) \in Q$, i.e.
$f_i(x,t,y_i) = f_{i1}(x,t)y_1 + f_{i2}(x,t), \forall i = 1,2,3$
Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{\bar{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ be two given CCBTCV, then $\vec{y} = (y_{u_1}, y_{u_2}, y_{u_3}) = (y_1, y_2, y_3)$ and $\vec{\bar{y}} = (\bar{y}_{\bar{u}_1}, \bar{y}_{\bar{u}_2}, \bar{y}_{\bar{u}_3}) = (\bar{y}_1, \bar{y}_2, \bar{y}_3)$ (by Theorem (1) are their corresponding TSVs, i.e. for the first components$y_1$and$\bar{y}_1$,we have

$$y_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial y_1}{\partial x_j}) + b_1 y_1 - b_4 y_2 - b_5 y_3 = f_{11}(x,t)y_1 + f_{12}(x,t)$$
$$y_1(x,0) = y_1^0(x) \text{ , in } \Omega$$
$$\frac{\partial y_1}{\partial n_a} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial y_1}{\partial x_j}\cos(n_1, x_j) = u_1(x,t), \text{ on } \Sigma$$
$$\bar{y}_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial \bar{y}_1}{\partial x_j}) + b_1\bar{y}_1 - b_4\bar{y}_2 - b_5\bar{y}_3 = f_{11}(x,t)\bar{y}_1 + f_{12}(x,t)$$
$$\bar{y}_1(x,0) = y_1^0(x)$$
$$\frac{\partial \bar{y}_1}{\partial n_a} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial \bar{y}_1}{\partial x_j}\cos(n_1, x_j) = \bar{u}_1(x,t), \text{ on}\Sigma$$

By MBS the TSEs ((1)-(9)) by $\theta \in [0,1]$, and then MBS of these equalities $(1-\theta)$ after substituting $\vec{\bar{y}}$ instead of $\vec{y}$, one has

$$(\theta y_1 + (1-\theta)\bar{y}_1)_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial(\theta y_1 + (1-\theta)\bar{y}_1)}{\partial x_j}) + b_1(\theta y_1 + (1-\theta)\bar{y}_1) - b_4(\theta y_2 + (1-\theta)\bar{y}_2) - b_5(\theta y_3 + (1-\theta)\bar{y}_3) = f_{11}(x,t)(\theta y_1 + (1-\theta)\bar{y}_1) + f_{12}(x,t) \quad (34a)$$
$$\theta y_1(x,0) + (1-\theta)\bar{y}_1(x,0) = y_1^0(x) \quad (35)$$
$$\frac{\partial\partial(\theta y_1 + (1-\theta)\bar{y}_1)}{\partial n_a} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial(\theta y_1 + (1-\theta)\bar{y}_1)}{\partial n}\cos(n_1, x_j) = \theta u_1(x,t) + (1-\theta)\bar{u}_1 \quad (36)$$
$$(\theta y_2 + (1-\theta)\bar{y}_2)_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}\frac{\partial(\theta y_2 + (1-\theta)\bar{y}_2)}{\partial x_j}) + b_2(\theta y_2 + (1-\theta)\bar{y}_2) + b_6(\theta y_3 + (1-\theta)\bar{y}_3) + b_4(\theta y_1 + (1-\theta)\bar{y}_1) = f_{21}(x,t)(\theta y_2 + (1-\theta)\bar{y}_2) + f_{22}(x,t) \quad (37)$$
$$\theta y_2(x,0) + (1-\theta)\bar{y}_2(x,0) = y_2^0(x) \quad (38)$$
$$\frac{\partial(\theta y_2 + (1-\theta)\bar{y}_2}{\partial n_a} = \sum_{i,j=1}^{n} b_{ij}\frac{\partial(\theta y_2 + (1-\theta)\bar{y}_2)}{\partial n}\cos(n_2, x_j) = \theta u_2(x,t) + (1-\theta)\bar{u}_2 , \quad (39)$$
$$(\theta y_3 + (1-\theta)\bar{y}_3)_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(c_{ij}\frac{\partial(\theta y_3 + (1-\theta)\bar{y}_3)}{\partial x_j}) + b_3(\theta y_3 + (1-\theta)\bar{y}_3) + b_5(\theta y_1 + (1-\theta)\bar{y}_1) - b_6(\theta y_2 + (1-\theta)\bar{y}_2) = \quad (36a40)$$
$$f_{31}(x,t)(\theta y_3 + (1-\theta)\bar{y}_3) + f_{32}(x,t)$$
$$\theta y_3(x,0) + (1-\theta)\bar{y}_3(x,0) = y_3^0(x) \quad (36b41)$$
$$\frac{\partial\partial(\theta y_3 + (1-\theta)\bar{y}_3)}{\partial n_a} = \sum_{i,j=1}^{n} c_{ij}\frac{\partial(\theta y_3 + (1-\theta)\bar{y}_3)}{\partial n}\cos(n_3, x_j) = \theta u_3(x,t) + (1-\theta)\bar{u}_3 \quad (36c42)$$

From equations ((34)-(36)), we conclude that the TSVs,$\vec{\tilde{y}} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$, $\vec{\tilde{y}} = \theta\vec{y} + (1-\theta)\vec{\bar{y}}$ is the corresponding CCBTCV $\vec{\tilde{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, with $\vec{\tilde{u}} = \theta\vec{u} + (1-\theta)\vec{\bar{u}}$,i.e.

$$\tilde{y}_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial \tilde{y}_1}{\partial x_j}) + b_1\tilde{y}_1 - b_4\tilde{y}_2 - b_5\tilde{y}_3 = f_{11}(x,t)\tilde{y}_1 + f_{12}(x,t)$$
$$\tilde{y}_1(x,0) = y_1^0(x)$$
$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial \tilde{y}_1}{\partial n} = \tilde{u}_1,$$
$$\tilde{y}_{2t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}\frac{\partial \tilde{y}_2}{\partial x_j}) + b_2\tilde{y}_2 + b_6\tilde{y}_3 + b_4\tilde{y}_1 = f_{21}(x,t)\tilde{y}_2 + f_{22}(x,t)$$
$$\tilde{y}_2(x,0) = y_2^0(x)$$
$$\sum_{i,j=1}^{n} b_{ij}\frac{\partial \tilde{y}_2}{\partial n} = \tilde{u}_2$$
$$\tilde{y}_{3t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(c_{ij}\frac{\partial \tilde{y}_3}{\partial x_j}) + b_3\tilde{y}_3 + b_5\tilde{y}_1 - b_6\tilde{y}_2 = f_{31}(x,t)\tilde{y}_3 + f_{32}(x,t)$$
$$\tilde{y}_3(x,0) = y_3^0(x)$$
$$\sum_{i,j=1}^{n} c_{ij}\frac{\partial \tilde{y}_3}{\partial n} = \tilde{u}_3$$

Hence the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is convex-linear (con-l) with respect to $(\vec{y}, \vec{u})$ for each(x,t) .
Also, since for each $i = 1,2,3$, $g_{1i}(x,t,y_i)$ ($h_{1i}(x,t,u_i)$) is affine w.r.t y_i $\forall(x,t) \in Q$ (is affine with respect to u_i $\forall(x,t) \in \Sigma$), i.e.
$g_{1i}(x,t,y_i) = I_{1i}(x,t)y_i + I_{2i}(x,t)$,
$h_{1i}(x,t,u_i) = I_{1i}(x,t)u_i + I_{3i}(x,t)$
Since $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is con-l, then

$$G_1(\theta\vec{u} + (1-\theta)\vec{\bar{u}}) = \sum_{i=1}^{3}\{ \int_Q [I_{1i}(x,t)_{\theta y_i + (1-\theta)\bar{y}_i)} + (I_{2i}(x,t)] \, dxdt + \sum_{i=1}^{3}\{ \int_\Sigma [I_{1i}(x,t)_{(\theta u_i + (1-\theta)\bar{u}_i)} + I_{3i}(x,t)] \, d\sigma$$
$$\Rightarrow G_1(\theta\vec{u} + (1-\theta)\vec{\bar{u}}) = \theta\sum_{i=1}^{3}\int_Q [I_{1i}(x,t)y_i + I_{2i}(x,t)]dxdt + (1-\theta)\sum_{i=1}^{3}\int_Q [I_{1i}(x,t)\bar{y}_i + I_{2i}(x,t)]dxdt + \theta\sum_{i=1}^{3}\int_\Sigma [I_{1i}(x,t)u_i + I_{3i}(x,t)]d\sigma + (1-\theta)\sum_{i=1}^{3}\int_\Sigma [I_{1i}(x,t)\bar{u}_i + I_{3i}(x,t)]d\sigma$$
$$\Rightarrow G_1(\theta\vec{u} + (1-\theta)\vec{\bar{u}}) = \theta G_1(\vec{u}) + (1-\theta)G_1(\vec{\bar{u}})$$

$\therefore G_1(\vec{u})$ is con-l with respect to $(\vec{y}, \vec{u})$, $(\forall(x,t) \in \overline{Q})$.

Also, from Lemma (1), the integrals $\sum_{i=1}^{2}\int_Q g_{0i}\,dxdt \& \sum_{i=1}^{2}\int_Q g_{2i}\,dxdt$ ($\sum_{i=1}^{2}\int_\Sigma h_{0i}\,d\sigma \& \sum_{i=1}^{2}\int_\Sigma h_{2i}\,d\sigma$) are convex with respect to y_i $\forall(x,t)\in Q$ (with respect to u_i $\forall(x,t)\in\Sigma$), then $G_0(\vec{u})$and $G_2(\vec{u})$ are convex with respect to $(\vec{y},\vec{u})$, $(\forall(x,t)\in\overline{Q})$, i.e. $G(\vec{u})$ is convex with respect to $(\vec{y},\vec{u})$, $(\forall(x,t)\in\overline{Q})$.

On the other hand, since $\overrightarrow{W}_{\vec{A}}$ is convex, and the FD of $G_l(\vec{u})$, $(l=0,1,2,3)$ exists and is continuous for each $\vec{u}\in\overrightarrow{W}$ (by Theorem (4)), then $\acute{G}(\vec{u})\overrightarrow{\Delta u}\geq 0$, which means $G(\vec{u})$ has a minimum at $\vec{u}$, i.e.

$\sum_{l=1}^{2}\lambda_l G_l(\vec{u})\leq\sum_{l=1}^{2}\lambda_l G_l(\vec{w})\ (\vec{w})$

Let $\vec{w}\in\overrightarrow{W}_A$, with $\lambda_2\geq 0$, then from (31), once get

$\lambda_0 G_0(\vec{u})\leq\lambda_0 G_0(\vec{w})$, $\forall\vec{w}\in\overrightarrow{W}\Rightarrow$ $G_0(\vec{u})\leq G_0(\vec{w})$, $\forall\vec{w}\in\overrightarrow{W}$, since $(\lambda_0>0)$

Hence $\vec{u}$ is a CCCBOTCV.

CONCLUSIONS

In this article, the classical continuous constraint boundary optimal control vector problem dominated by the triple nonlinear parabolic boundary value problem is studied. The existence theorem of a classical continuous constraint boundary optimal control vector is stated and proved under suitable assumptions. Mathematical formulation of the adjiont triple boundary value problem associated with the triple nonlinear parabolic boundary value problem is investigated. The Fréchet derivative of the Hamiltonian is derived. Both theorems of necessary conditions and sufficient condition for the optimality of the classical continuous constraint boundary optimal control vector problem are stated and proved under suitable assumptions.

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