Classical Continuous Constraint Boundary Optimal Control Vector Problem for Triple Nonlinear Parabolic System

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ABSTRACT
In this paper, our purpose is to study the classical continuous constraints boundary optimal triple control vector problem dominating nonlinear triple parabolic boundary value problem. The existence theorem for a classical continuous triple optimal control vector CCCBOTCVP is stated and proved under suitable assumptions. Mathematical formulation of the adjoint triple boundary value problem associated with the nonlinear triple parabolic boundary value problem is discovered. The Fréchet derivative of the Hamiltonian derived. Under proper assumptions, both theorems are granted; the necessary conditions for optimality and the sufficient conditions for optimality of the classical continuous constraints boundary optimal triple control vector problem are stated and prove.

KEYWORDS: Classical constraints boundary optimal triple control, nonlinear triple parabolic boundary value problem, necessary and sufficient optimality conditions.

INTRODUCTION
Optimal control problems (OCPs) play an important role in many practical applications, such as in medicine [1], aircraft [2], economics [3], robotics [4], weather conditions [5] and many other scientific fields. They are two types of OCPs; the classical and the relax type. The first type was studied mostly in the last century, while the second was studied in the beginning of this century. Each one of these two types is dominated either by nonlinear ordinary differential equations (ODEs) [6] or by nonlinear PDEs (NLPDEs) [7]. The classical continuous constraints boundary optimal control problem (CCCBOTCP) dominated by nonlinear parabolic or elliptic or hyperbolic PDEs are studied in [8-10] respectively (resp.). Later, the study of the CCCBOTCPs dominated by each one of these types of PDEs are generalized in [11-13] to deal with CCCBOTCPS dominated by couple NLPDEs (CNLPDES) of these types respectively, and then the studies for the couple nonlinear elliptic and hyperbolic PDEs types are generalized also to deal with CCCBOTCPS dominated by triple NLPDEs of these two indicated types respectively [14, 15]. All of the studies mentioned have motivated us to consider generalization, the study of the CCCBOTCP dominated by CNLPDEs of parabolic type to study the classical continuous constraints boundary optimal triple control vector problem (CCCBOTCVP) dominating by nonlinear triple parabolic boundary value problem (NLTPBV). According to this generalization, the mathematical model for the dominating equations is needed to be found, as well as the cost function, the spaces of definition for the control and the
state vectors, which all of them are needed to be generalized.

In this paper, the CCCBOTCVP dominated by the NLTBPBVP is proposed. Section 2 deals with problem description, and some mathematical concepts, In Section 3 the statement and proof of the existence theorem of a classical continuous triple optimal control vector (CCCBOTCVP) under suitable Assumptions are studied. The mathematical formulation for the adjoint triple boundary value problem (ATBVP) associated with TNLPPBVP is investigated. The Fréchet derivative (FD) of the Hamiltonian"(Ham) is derived. Both theorems the necessary conditions (NCOs) for optimality (OP) and the sufficient conditions (SCOs) for OP of the considered CCCBOTCVP are stated and proved under suitable Assumptions.

**Problem Description**

Let $l = (0, T)$, with $T < \infty$, $\Omega \subset \mathbb{R}^2$ be an open and bounded region with Lipschitz boundary $\Gamma = \partial \Omega$, $Q = \Omega \times I$, $\Sigma = \Gamma \times l$. Consider the following CCCBOTCVP which is composed of the triple state equations (TSVEs) describe by the following TNLPPDEs:

\[
y_{1t} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_{1j}}{\partial x_j} + b_1 y_1 - b_4 y_2 - b_5 y_3 = f_1(x,t,y_1), \quad \text{in } Q, \tag{1}
\]

\[
y_{2t} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_{2j}}{\partial x_j} + b_2 y_2 + b_6 y_3 + b_4 y_1 = f_2(x,t,y_2), \quad \text{in } Q, \tag{2}
\]

\[
y_{3t} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_{3j}}{\partial x_j} + b_3 y_3 + b_5 y_1 - b_6 y_2 = f_3(x,t,y_3), \quad \text{in } Q, \tag{3}
\]

\[
\frac{\partial y_{1j}}{\partial t} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_{1j}}{\partial x_j} \cos(n_1, x_j) = u_1(x,t), \tag{4}
\]

\[
\frac{\partial y_{2j}}{\partial t} = \sum_{i,j=1}^{n} b_{ij} \frac{\partial y_{2j}}{\partial x_j} \cos(n_2, x_j) = u_2(x,t), \tag{5}
\]

\[
\frac{\partial y_{3j}}{\partial t} = \sum_{i,j=1}^{n} c_{ij} \frac{\partial y_{3j}}{\partial x_j} \cos(n_3, x_j) = u_3(x,t), \tag{6}
\]

where $(f_1, f_2, f_3) \in \left( L^2(Q) \right)^3$ is given, $(x_1, x_2) \in \Omega$, $a_{ij}(x,t), b_i(x,t) \in C^\infty(Q)$, $n_\ell$ (for $\ell = 1, 2, 3$) is a unit vector normal outer on the boundary $\Sigma$, $(n_\ell, x_j)$ is the angle between $n_\ell$ and the $x_j - axis$, $\vec{u} = (u_1, u_2, u_3) \in \left( L^2(\Sigma) \right)^3$ is a CCCBOTCV and $\vec{y} = \vec{y}_{\vec{u}} = (y_{1u_1}, y_{2u_2}, y_{3u_3}) \in \left( H^2(\Omega) \right)^3$ is the triple state vector solution(TSVS) corresponding to the CCCBOTCV.

The set of admissible CCCBOTC (ACCBTCV) is:

\[
W_A = \{ \vec{u} \in \vec{W} | \vec{u} \in \vec{U} \ a.e. \text{ in } \Sigma, G_1(\vec{w}) = 0, G_2(\vec{w}) \leq 0 \}
\]

$\vec{U} = U_1 \times U_2 \times U_3 \subset \mathbb{R}^3$ is convex set, and $\vec{W} = \left( L^2(\Sigma) \right)^3$.

The cost function (CF) is

\[
G_0(\vec{u}) = \int_Q \left[ g_{01}(x,t,y_1) + g_{02}(x,t,y_2) \right] dx dt + \int_\Sigma [h_{01}(x,t,u_1) + h_{02}(x,t,u_2)] d\sigma \tag{10}
\]

The state vector constraints (SVCs) are

\[
G_1(\vec{u}) = \left\{ \int_Q [g_{11}(x,t,y_1) + g_{12}(x,t,y_2)] dx dt + \int_\Sigma [h_{11}(x,t,u_1) + h_{12}(x,t,u_2)] d\sigma = 0 \right\} \tag{11}
\]

\[
G_2(\vec{u}) = \left\{ \int_Q [g_{21}(x,t,y_1) + g_{22}(x,t,y_2)] dx dt + \int_\Sigma [h_{21}(x,t,u_1) + h_{22}(x,t,u_2)] d\sigma \leq 0 \right\} \tag{12}
\]

Let $\vec{V} = V_1 \times V_2 \times V_3 = V \times V \times V = \{ \vec{v}, \vec{\bar{v}} | (v_1(x), v_2(x), v_3(x)) \in \left( L^2(\Omega) \right)^3 \}$, the weak form (WFO) of the TSVEs (1-9) when $\vec{y} \in \left( L^2(\Omega) \right)^3$ is given by:

\[
\begin{align}
(y_{1t}, v_1) + & a_1(t,y_1,v_1) + (b_1(t)y_1,v_1)_\Omega - (b_2(t)y_2,v_1)_\Omega - (b_3(t)y_3,v_1)_\Omega = \\
& (f_1(v_1),v_1)_\Omega + (u_1,v_1)_\Gamma, \forall v_1 \in V \tag{13a}
\end{align}
\]

\[
\begin{align}
(y_{2t}, v_2) + & a_2(t,y_2,v_2) + (b_2(t)y_2,v_2)_\Omega + (b_6(t)y_3,v_2)_\Omega + (b_5(t)y_3,v_2)_\Omega = \\
& (f_2(v_2),v_2)_\Omega + (u_2,v_2)_\Gamma, \forall v_2 \in V \tag{14}
\end{align}
\]

\[
\begin{align}
(y_{3t}, v_3) + & a_3(t,y_3,v_3) + (b_3(t)y_3,v_3)_\Omega + (b_5(t)y_1,v_3)_\Omega - (b_6(t)y_2,v_3)_\Omega = \\
& (f_3(v_3),v_3)_\Omega + (u_3,v_3)_\Gamma, \forall v_3 \in V \tag{15}
\end{align}
\]

\[
\begin{align}
(y_0^0, v_2)_\Omega = & (y_2(0),v_2)_\Omega \tag{16}
\end{align}
\]

\[
\begin{align}
(y_0^0, v_0)_\Omega = & (y_0(0),v_0)_\Omega \tag{17}
\end{align}
\]
Where \( a_i(t, y_i, v_i) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial y_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} \, dx \) for \( l = 1, 2, 3 \)

**Assumptions (A):**

(i) \( f_i \) is of a Carathéodory type (C-T) on \( Q \times \mathbb{R} \), satisfies
\[ |f_i(x, t, y_i)| \leq c_i(z_i(x, t)) \]
Where \((x, t) \in Q, y_i, v_i \in \mathbb{R}, c_i > 0 \) and \( \eta_t \in L^2(Q, \mathbb{R}), \forall i = 1, 2, 3. \)

(ii) \( f_i \) is Lipschitz w.r.t. \( y_i, (w_i = 1, 2, 3) \) i.e.:
\[ |f_i(x, t, y_i) - f_i(x, t, \tilde{y}_i)| \leq L_1 (y_i - \tilde{y}_i). \]
Where \((x, t) \in Q, y_i, \tilde{y}_i \in \mathbb{R} \) and \( L_1 > 0. \)

(iii) \( a_i(t, y_i, v_i) \leq \alpha_i \| y_i \| _0 \| v_i \| _0 \),
\[ (b_i(t) y_i, v_i) = \beta_i \| y_i \| _0 \| v_i \| _0. \]
\[ |b_i(t) y_i v_i| \leq |y_i|^2 |v_i|^2, \]
\[ (b_i(t) y_i v_i) \geq \beta_i |y_i|^2 |v_i|^2, \]
\[ |b_i(t) y_i v_i| \leq |y_i|^2 |v_i|^2 \]
\[ |b_i(t) y_i v_i| \leq |y_i|^2 |v_i|^2 \]
\[ |b_i(t) y_i v_i| \leq |y_i|^2 |v_i|^2 \]
\[ c(t, y_i, \tilde{y}_i) = \sum_{i=1}^3 \| a_i(t, y_i, \tilde{y}_i) \| + \| b_i(t) y_i v_i \|, \]
with \( c(t, y_i, \tilde{y}_i) \geq \alpha_i \| y_i \| _0 \)
here \( \alpha_i, \beta_i, \beta_i (\forall i = 1, 2, 3), \epsilon_i (\forall i = 1, 2, 3, 4, 5, 6) \) and \( \tilde{a} \) are real positive constants.

**Theorem 1 [16]:** With assumptions (A), for each “fixed” \( \tilde{u} \in \left( L^2(\Sigma) \right)^3 \), the WFO (13)-(15)) has a unique TSVE \( \tilde{y} = (y_1, y_2, y_3) \) s.t. \( \tilde{y} \in \tilde{y} = (y_1, y_2, y_3, 3) \) is \( (L^2(\Sigma)) \).

**Assumptions (B):**

Consider \( g_i(t, h_i) \) for each \( l = 0, 1, 2, 3 \) is of C -T on \( Q \times \mathbb{R} \) and on \( \Sigma \times \mathbb{R} \) respectively, and satisfies the following sub quadratic condition with respect to \( y_i \) and \( u_i \)
\[ |g_i(t, x, y_i)| \leq \gamma_i(h_i(x, t)) + c_i(y_i)^2, \]
\[ |h_i(t, x, u_i)| \leq \delta_i(h_i(x, t)) + d_i(u_i)^2 \]
Where \( y_i, u_i \in \mathbb{R} \) with \( y_i \in L^2(Q), \delta_i \in L^1(\Sigma) \)

**Lemma 1[16]:** If the assumptions (B) are held, the functional \( G_i(u_i) \) is continuous on \( (L^2(\Sigma))^3, \forall l = 0, 1, 2 \).

**Theorem 2 [16]:**

Beside the assumptions (A) and (B). If \( \tilde{u} \) is compact, \( \tilde{u} \neq \emptyset \), \( G_i(u_i) \) is convex with respect to \( u_i \) for fixed \((x, t, \tilde{y}) \). Then there exists a CCCBOTCV.

**Assumptions (C):**

If \( f_i(y_i), g_{ii, y_i}, h_{ii, u_i}, (l = 0, 1, 2, \& i = 1, 2, 3) \) are of C-T on \( (Q \times \mathbb{R}), (Q \times \mathbb{R})(\Sigma \times \mathbb{R}) \) respectively, and \( |f_i(y_i, x, t, y_i)| \leq \bar{L}_i(x, t), \)
\[ g_{ii, y_i}(x, t, y_i) \leq \bar{y}_i(x, t) + c_i|y_i|, \]
\[ h_{ii, u_i}(x, t, u_i) \leq \delta_i(x, t) + d_i|u_i|, \]
Where \((x, t) \in Q, \gamma_i, \tilde{y}_i, \bar{y}_i \in \mathbb{R}, \gamma_i(x, t), \bar{y}_i(x, t) \in L^2(Q) \) and \( \delta_i(x, t) \in L^2(\Sigma). \)

**RESULTS**

**Existence of the CCCBOTCV and the FD**

This section deals with the existence of the CCCBOTCV and the derivation of the FD under some suitable Assumptions after the ATBVP is defined.

**Theorem 3:**

In addition to assumptions. (A) and (B), if \( \tilde{u} \) in the \( \tilde{W}_A \) is compact, \( \tilde{W}_A \neq \emptyset \). If for each \( i = 1, 2, 3, G_i(u_i) \) is independent of \( u_i \), \( G_i(u_i) \) and \( G_i(u_i) \) are convex w.r.t. \( u_i \), for fixed \((x, t, y_i) \). Then there exists a CCCBOTCV for the considered problem.

**Proof:**

From the assumptions. on \( \tilde{u} \) and \( G_i(u_i) \), for \( l = 0, 1, 2 \), using Lemma 1 and theorem 2, one can get that there exists a CCCBOTCV.

**Theorem 4:** Dropping index \( l \) in \( g_i, h_i \) and \( G_i \), The Ham \( H \) is defined by
\[ H(x, t, y_i, z_i, u_i) = \sum_{i=1}^3 (z_i f_i(x, t, y_i) + g_i(x, t, y_i, h_i(x, t, u_i))) \]
And the ATBVP \( z_i = z_{ui} \) (where \( yi = y_{ui} \)) equation satisfy (in Q):
\[ -z_{1t} - \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial z_i}{\partial x_j}) + b_1 z_1 - b_4 z_2 - b_3 z_3 = z_1 f_1(x, t, y_1) + g_1(x, t, y_1), \]
\[ -z_{2t} - \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} (b_{ij} \frac{\partial z}{\partial x_j}) + b_2 z_2 + b_5 z_3 + b_4 z_1 = z_2 f_2(x, t, y_2) + g_2(x, t, y_2), \]
\[ -z_{3t} - \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} (c_{ij} \frac{\partial z}{\partial x_j}) + b_2 z_2 + b_5 z_3 + b_4 z_1 = z_3 f_3(x, t, y_3) + g_3(x, t, y_3), \]
\[ z_i(\Omega) = 0, \text{ in } \Omega \forall i = 1, 2, 3. \]
\[
\frac{\partial z_i}{\partial n} = 0 , \quad \text{on } \Sigma \quad \forall i = 1,2,3 \quad \text{(22)}
\]

Then the FD of \( G \) is

\[
\hat{G}(\mathbf{u}) = \int_{\Sigma} \begin{pmatrix} z_1 + h_{1u1} \\ z_2 + h_{2u2} \end{pmatrix} \cdot \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} \, d\sigma
\]

**Proof:**

The WFO of the ATBVP is considered by:

\[-(z_{1t}, v_1) + a_1(t, z_1, v_1) + (b_1(t)z_1, v_1)_\Omega - (b_4(t)z_2, v_1)_\Omega - (b_5(t)z_3, v_2)_\Omega = (z_1f_{1y1}, v_1)_\Omega + (g_{1y1}, v_1)_\Omega \quad \text{(23)}
\]

\[-(z_{2t}, v_2) + a_2(t, z_2, v_2) + (b_2(t)z_2, v_2)_\Omega + (b_6(t)z_2, v_2)_\Omega = (z_2f_{2y2}, v_2)_\Omega + (g_{2y2}, v_2)_\Omega \quad \text{(24)}
\]

\[-(z_{3t}, v_3) + a_3(t, z_3, v_3) + (b_3(t)z_3, v_3)_\Omega + (b_5(t)z_1, v_3)_\Omega + (b_6(t)z_2, v_3)_\Omega = (z_3f_{3y3}, v_3)_\Omega + (g_{3y3}, v_3)_\Omega \quad \text{(25)}
\]

Substituting \( v_i = \Delta y_i, \quad \forall i = 1,2,3 \) in ((21)- (23)) respectively, finally integrating both sides w.r.t from 0 to T, then using integration by parts (IBP) for the 1st obtained term in each equation, finally adding these three equations, to get:

\[
\int_0^T \left( \Delta y_t, \Delta z \right) dt + \int_0^T \left[ a_1(t, z_1, \Delta y_1) + (b_1(t)z_1, \Delta y_1)_\Omega - (b_4(t)z_2, \Delta y_1)_\Omega - (b_5(t)z_3, \Delta y_1)_\Omega \right] dt + \\
\int_0^T \left[ a_2(t, z_2, \Delta y_2) + (b_2(t)z_2, \Delta y_2)_\Omega + (b_6(t)z_3, \Delta y_2)_\Omega \right] dt + \\
\int_0^T \left[ a_3(t, z_3, \Delta y_3) + (b_3(t)z_3, \Delta y_3)_\Omega + (b_5(t)z_1, \Delta y_3)_\Omega + (b_6(t)z_2, \Delta y_3)_\Omega \right] dt = \\
\int_0^T \left( f_1(t, y_1, \Delta y_1), z_1 \right)_\Omega + (g_{1y1}, \Delta y_1)_\Omega \quad \text{(26)}
\]

Now, substituting \( y_i = \Delta y_i \) and \( v_i = z_i \) in ((13)-(15)) respectively, \( \forall i = 1,2,3 \), integrating both sides from 0 to T then adding three obtained equations to get:

\[
\int_0^T \left( \Delta y_t, \Delta z \right) dt + \int_0^T \left[ a_1(t, \Delta y_1, \Delta z_1) + (b_1(t)\Delta y_1, \Delta y_1)_\Omega - (b_4(t)\Delta y_2, \Delta y_1)_\Omega - (b_5(t)\Delta y_3, \Delta y_1)_\Omega + a_2(t, \Delta y_2, \Delta z_2) + (b_2(t)\Delta y_2, \Delta y_2)_\Omega + (b_6(t)\Delta y_3, \Delta y_2)_\Omega + (b_4(t)\Delta y_2, \Delta z_2)_\Omega + a_3(t, \Delta y_3, \Delta z_3) + (b_3(t)\Delta y_3, \Delta y_3)_\Omega + (b_5(t)\Delta y_1, \Delta y_3)_\Omega - (b_6(t)\Delta y_2, \Delta y_3)_\Omega \right] dt + \\
\int_0^T \left( f_1(t, y_1, \Delta y_1), z_1 \right)_\Omega + (g_{1y1}, \Delta y_1)_\Omega = (b_3(t)\Delta y_3, z_3)_\Omega + (b_5(t)\Delta y_1, z_3)_\Omega - (b_6(t)\Delta y_2, z_3)_\Omega \quad \text{(27)}
\]

By using (26) in R.H.S. of (25), to get:

\[
\int_0^T \left( \Delta y_t, \Delta z \right) dt + \int_0^T \left[ a_1(t, \Delta y_1, \Delta z_1) + (b_1(t)\Delta y_1, \Delta y_1)_\Omega - (b_4(t)\Delta y_2, \Delta y_1)_\Omega - (b_5(t)\Delta y_3, \Delta y_1)_\Omega + a_2(t, \Delta y_2, \Delta z_2) + (b_2(t)\Delta y_2, \Delta y_2)_\Omega + (b_6(t)\Delta y_3, \Delta z_2)_\Omega + a_3(t, \Delta y_3, \Delta z_3) + (b_3(t)\Delta y_3, \Delta y_3)_\Omega + (b_5(t)\Delta y_3, z_3)_\Omega - (b_6(t)\Delta y_2, z_3)_\Omega \right] dt = \\
\int_0^T \left( f_1(t, y_1, \Delta y_1), z_1 \right)_\Omega + (g_{1y1}, \Delta y_1)_\Omega + (f_{2y2}, \Delta y_2, z_2)_\Omega dt + \\
\int_0^T \left( f_{3y3}, \Delta y_3, z_3 \right)_\Omega dt + \int_0^T \left( \Delta u_1, \Delta z_1 \right)_\Omega dt \quad \text{(29)}
\]

Now, by subtracting (24) from (27), one get:

\[
\int_0^T \left( g_{1y1}, \Delta y_1 \right)_\Omega dt + \int_0^T \left( g_{2y2}, \Delta y_2 \right)_\Omega dt = \\
\int_0^T \left( g_{3y3}, \Delta y_3 \right)_\Omega dt = \\
\int_0^T \left( \Delta u_1, \Delta z_1 \right)_\Omega dt + \int_0^T \left( \Delta u_2, \Delta z_2 \right)_\Omega dt + \\
\int_0^T \left( \Delta u_3, z_3 \right)_\Omega dt + \varepsilon_1(\Delta u) \| \Delta u \|_\Sigma \quad \text{(28)}
\]

Now, let \( G_a(\mathbf{u}) = \int_0^T k_1(x, t, y_1, y_2, y_3) \, dx \, dt \),

\[
G_b(\mathbf{u}) = \int_{\Sigma} k_2(x, t, u_1, u_2, u_3) \, d\sigma
\]

Where \( k_1(x, t, y_1, y_2, y_3) = g_1(x, t, y_1) + g_2(x, t, y_2) + g_3(x, t, y_3) \), and

\[
k_2(x, t, u_1, u_2, u_3) = h_1(x, t, u_1) + h_2(x, t, u_2) + h_3(x, t, u_3)
\]

From the definition of the FD and the result of Theorem (2-(a)) [16] and from the assumptions on
\( g_i \) (\( \forall i = 1,2,3 \)), and then using the inequality of Minkowski one obtains:
\[
G(\tilde{u} + \tilde{\tilde{u}}) - G(\tilde{u}) = \int_{\Sigma} (g_{1y_1} \Delta y_1 + g_{2y_2} \Delta y_2 + g_{3y_3} \Delta y_3) + \int_{\Sigma} (h_{1u_1} \Delta u_1 + h_{2u_2} \Delta u_2 + h_{3u_3} \Delta u_3) d\sigma + \epsilon_{\gamma} (\tilde{\tilde{u}}) \parallel \Delta u \parallel_{\Sigma} \tag{29}
\]
Using (28) in (29), give
\[
G(\tilde{u} + \tilde{\tilde{u}}) - G(\tilde{u}) = \int_{\Sigma} (\Delta u_{11}, z_1) d\sigma + \int_{\Sigma} (\Delta u_{22}, z_2) d\sigma + \int_{\Sigma} (\Delta u_{33}, z_3) d\sigma
\]
where, \( \epsilon_{\gamma} (\tilde{\tilde{u}}) = \epsilon_{\gamma} (\Delta u) \rightarrow 0 \), as
\[
\parallel \Delta u \parallel_{\Sigma} \rightarrow 0
\]
From the FD of \( G \), we get that
\[
(\hat{G} (\tilde{u}), \tilde{\Delta u}) = \int_{\Sigma} \left( \frac{\partial \Delta u_1}{\partial u_1}, \frac{\partial \Delta u_2}{\partial u_2}, \frac{\partial \Delta u_3}{\partial u_3} \right) d\sigma.
\]

The NCOs and the SCOs

The NCOs and the SCOs theorems under suitable assumptions are considered in this section.

**Theorem 5: The NCOs**

(i) In addition to the assumptions (A), (B), and (C), if \( \tilde{u} \in \tilde{W}_A \) is a CCBOTCV, then there exist multipliers \( \lambda_i \in \mathbb{R}, l = 0,1,2 \) with \( \lambda_0 \geq 0, \lambda_2 \geq 0 \), \( \sum_{l=0}^{2} \lambda_l = 1 \), s.t.
\[
\sum_{l=0}^{2} \lambda_l \hat{G}_l (\tilde{u}) \Delta u_l (\tilde{u} - \tilde{\tilde{u}}) \geq 0, \forall \tilde{\tilde{u}} \in \tilde{W}, \Delta u = (\tilde{u} - \tilde{\tilde{u}}) \tag{30}
\]
\[
\lambda_2 G_2 (\tilde{u}) = 0 \tag{31}
\]
(ii) The inequality (30) is equivalent to the minimum WFO (MWFO)
\[
H_{\tilde{u}} (x,y,z,\tilde{u}) \tilde{\tilde{u}} = \min_{\tilde{\tilde{u}} \in \tilde{W}} H_{\tilde{u}} (x,y,z,\tilde{u}) \tilde{\tilde{u}} \tag{32}
\]

de on \( \Sigma \)

**Proof:**

From lemma (1), the functionals \( G_l (\tilde{u}) \) & \( \hat{G}_l (\tilde{u}) \) (for each \( l = 0,1,2 \)) are continuous with respect to \( (\tilde{u} - \tilde{\tilde{u}}) \) and liner with respect to \( (\tilde{u} - \tilde{\tilde{u}}) \), then \( G_l (\tilde{u}) \) is \( \rho \)-differential at each \( \tilde{u} \in \tilde{W}, \forall \rho \), then by the Kuhn-Tucker-Lagrange theorem, there exist multipliers \( \lambda_l \in \mathbb{R}, l = 0,1,2 \), with \( \lambda_0, \lambda_2 \geq 0 \), \( \sum_{l=0}^{2} \lambda_l = 1 \), s.t.(30)&(31) hold, and by utilizing the result of the theorem 4, then (30) gives
\[
\sum_{l=0}^{2} \lambda_l (z_{l0} + h_{l0u_l}) (\tilde{u} - \tilde{\tilde{u}}) d\sigma \geq 0
\]
Let \( z_i = \sum_{l=0}^{2} \lambda_l z_{l0} + h_{l0u_l}, i = 0,1,2 \) and \( l = 0,1,2 \)
Now, let \( \tilde{u}_k \) be a dense sequence in \( \tilde{W} \), and let \( S \in \Sigma \) be a meab set with \( \mu \) is Lebesgue measure on \( \Sigma \), s.t.:
\[
\tilde{\tilde{u}} (x,t) = (\tilde{u}_k (x,t), \text{ if } (x,t) \in S)
\]
\[
(\tilde{u}_k (x,t), \text{ if } (x,t) \notin S)
\]

Therefore (33) becomes
\[
\int_{\Sigma} H_{\tilde{u}} (x,y,z,\tilde{u}) (\tilde{u} - \tilde{\tilde{u}}) d\sigma \geq 0
\]
Since \( \mu (\Sigma_k) = 0, \forall k \), then the inequality holds in \( \Sigma - \Sigma_k \), and since \( \mu (U_s, \Sigma_k) = 0 \), thus it holds in \( \Sigma/ \Sigma_k \Sigma_k \). But \( \tilde{u}_k \) is a dense sequence in \( \tilde{W} \), then there is \( \tilde{u} \in \tilde{W}, \text{s.t.} \)
\[
H_{\tilde{u}} (x,y,z,\tilde{u}) (\tilde{u} - \tilde{\tilde{u}}) \geq 0, \forall \text{ in } \Sigma \Rightarrow \tilde{\tilde{u}} = \min \{ H_{\tilde{u}} (x,y,z,\tilde{u}) \tilde{\tilde{u}} \} \tilde{\tilde{u}} = \tilde{\tilde{u}} \tilde{\tilde{u}} \text{. a.e. on } \Sigma
\]

Conversely, let
\[
H_{\tilde{u}} (x,y,z,\tilde{u}) (\tilde{\tilde{u}} - \tilde{\tilde{u}}) \geq 0\text{. a.e. in } \Sigma
\]
\[
\Sigma \Rightarrow \tilde{\tilde{u}} = \min \{ H_{\tilde{u}} (x,y,z,\tilde{u}) \tilde{\tilde{u}} \} \tilde{\tilde{u}} = \tilde{\tilde{u}} \tilde{\tilde{u}} \text{. a.e. on } \Sigma
\]

**Theorem 6: The SCOs**

In addition to the assumptions (A), (B), and (C), suppose for each \( i = 1,2,3 \), \( f_i \) & \( g_{1i} \) are affine with respect to \( y_i \) for each \( (x,t) \in Q \) and \( h_{1i} \) is affine with respect to \( u_i \) for each \( (x,t) \in \Sigma, g_{0i}, g_{2i}, h_{0i}, h_{2i} \) are convex with respect to \( y_i \) for each \( (x,t) \in Q (u_i \text{ for each } (x,t) \in \Sigma) \). Then NCOs in theorem 5 with \( \lambda_0 > 0 \) are also sufficient.

**Proof:**

Suppose \( \tilde{u} \) is satisfied the Kuhn-Tucker-Lagrange conditions, and \( \tilde{u} \in \tilde{W}_A \), i.e.
\[
\int_{\Sigma} H_{\tilde{u}} (x,y,z,\tilde{u}) (\tilde{\tilde{u}} - \tilde{\tilde{u}}) d\sigma \geq 0, \forall \tilde{\tilde{u}} \in \tilde{W}, \lambda_2 G_2 (\tilde{u}) = 0
\]
Let \( \hat{G} (\tilde{u}) = \lambda_0 \hat{G}_1 (\tilde{u}) \), then from theorem 4
\[
\hat{G} (\tilde{u}) \cdot \tilde{\Delta u} = \lambda_0 \int_{\Sigma} (z_{i0} + h_{i0u_i}) \Delta u_i d\sigma + \lambda_1 \int_{\Sigma} \int_{\Sigma} (z_{1i} + h_{1iu_i}) \Delta u_i d\sigma + \lambda_2 \int_{\Sigma} \int_{\Sigma} (z_{2i} + h_{2iu_i}) \Delta u_i d\sigma
\]
Now, consider the three functions in the R.H.S. of TSEVs ((1)-(3)) are affine with respect to $y_1, y_2, y_3$ respectively, for each $(x, t) \in Q$, i.e. $f_i(x, t, y_i) = f_{i1}(x, t) + f_{i2}(x, t)$, $\forall i = 1, 2, 3$. Let $\bar{u} = (u_1, u_2, u_3)$ and $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3)$ be two given CCBTCVs, then $\bar{y} = (y_{u_1}, y_{u_2}, y_{u_3}) = (y_1, y_2, y_3)$ and $\bar{\bar{y}} = (\bar{y}_{\bar{u}_1}, \bar{y}_{\bar{u}_2}, \bar{y}_{\bar{u}_3}) = (\bar{y}_1, \bar{y}_2, \bar{y}_3)$ (by Theorem (1) they are their corresponding TSEVs, i.e. for the i-th component $y_i$, we have

$$y_i(t) = \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial y_i}{\partial x_j} (a_{ij} \frac{\partial y_i}{\partial x_j} + b_1 y_i - b_4 y_2 - b_5 y_3 = f_{i1}(x, t) y_1 + f_{i2}(x, t)$$

$$y_i(x, 0) = y_0^i(x)$$

Then, MBS of these equalities $(1 - \theta)$ after substituting $\bar{y}$ instead of $y$, one has

$$(\theta y_1 + (1 - \theta) \bar{y}_1) t -$$

$$\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial \theta y_i(1 + \theta) \bar{y}_1}{\partial x_j} =$$

$$b_1 y_1 + (1 - \theta) \bar{y}_1 - b_4 y_2 - b_5 y_3 = f_{i1}(x, t) y_1 + f_{i2}(x, t)$$

$$\theta y_i(x, 0) (1 - \theta) \bar{y}_1(x, 0) = y_0^i(x)$$

From equations ((34)-(36)), we conclude that the TSEVs $\bar{y}$ is $\bar{\theta} \bar{y} + (1 - \theta) \bar{y}$ is the corresponding CCBCTV $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$, with $\bar{u} = \theta \bar{u} + (1 - \theta) \bar{u}$, i.e.

$$\bar{y}_1(t) - \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial y_i}{\partial x_j} + b_1 y_1 - b_4 y_2 - b_5 y_3 = f_{i1}(x, t) y_1 + f_{i2}(x, t)$$

$$\bar{y}_1(x, 0) = y_0^i(x)$$

Hence the operator $\bar{u} \mapsto \bar{y}$ is convex-linear (con-l) with respect to $(\bar{y}, \bar{u})$ for each $(x, t)$. Also, since for each $i = 1, 2, 3$, $g_{i1}(x, t, y_i)$ is affine w.r.t $y_i \in Q$ (is affine with respect to $ui \in Q$, i.e. $g_{i1}(x, t, y_i) = l_{i1}(x, t) y_i + l_{i2}(x, t)$, $h_{i1}(x, t, u_i) = l_{i1}(x, t) u_i + l_{i3}(x, t)$)

Since $\bar{u} \mapsto \bar{y}$ is con-l, then

$$G_i(\theta \bar{u} + (1 - \theta) \bar{u}) =$$

$$\sum_{i=1}^{3} \int Q [ l_{i1}(x, t) y_i + l_{i2}(x, t) ] d\sigma dt +$$

$$\theta G_i \left( \bar{u} + (1 - \theta) \bar{u} \right) \mapsto : G_1(\bar{u}) \text{ is con-l with respect to } (\bar{y}, \bar{u}) \text{, } (\forall x, t \in Q)$$
Also, from Lemma (1), the integrals 
\[ \Sigma_{i=1}^{2} \int_{Q} g_{0i} dxdt + \Sigma_{i=1}^{2} \int_{t} g_{2i} dxdt \] 
are convex with respect to \( y_i \) \((i=0,1,2,3) \) and continuous for each \( \vec{u} \in \vec{W} \) (by Theorem (4)), then \( G_0(\vec{u}) \) and \( G_2(\vec{u}) \) are convex with respect to \((\vec{y}, \vec{u})\), \((\vec{y}, \vec{u}) \in \vec{Q})\), i.e., \( G(\vec{u}) \) is convex with respect to \((\vec{y}, \vec{u})\), \((\vec{y}, \vec{u}) \in \vec{Q})\).

On the other hand, since \( \vec{W}_A \) is convex, and the FD of \( G_1(\vec{u}) \), \((l = 0,1,2,3) \) exists and is continuous for each \( \vec{u} \in \vec{W} \) (by Theorem (4)), then \( \hat{G}(\vec{u})\Delta \hat{u} \geq 0 \), which means \( G(\vec{u}) \) has a minimum at \( \vec{u} \), i.e., \[ \sum_{i=1}^{2} \lambda_i \hat{G}_i(\vec{u}) \leq \sum_{i=1}^{2} \lambda_i \hat{G}_i(\vec{w}) \] \( \vec{w} \in \vec{W}_A \), with \( \lambda_2 \geq 0 \), then from (31), once get \( \lambda_0 G_0(\vec{u}) \leq \lambda_0 G_0(\vec{w}) \), \( \forall \vec{w} \in \vec{W} \Rightarrow G_0(\vec{u}) \leq G_0(\vec{w}) \), \( \forall \vec{w} \in \vec{W} \), since \( \lambda_0 > 0 \).

Hence \( \vec{u} \) is a CCCBOTCV.

CONCLUSIONS

In this article, the classical continuous constraint boundary optimal control vector problem dominated by the triple nonlinear parabolic boundary value problem is studied. The existence theorem of a classical continuous constraint boundary optimal control vector is stated and proved under suitable assumptions. Mathematical formulation of the adjoint triple boundary value problem associated with the triple nonlinear parabolic boundary value problem is investigated. The Fréchet derivative of the Hamiltonian is derived. Both theorems of necessary conditions and sufficient condition for the optimality of the classical continuous constraint boundary optimal control vector problem are stated and proved under suitable assumptions.

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