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Nano S_β-Connectedness in Nano Topological Spaces

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ABSTRACT

The aim of this study is to introduce the notions of nano S_β-connected, nano S_β-hyperconnected and nano S_β-ultraconnected by using the all forms of nano S_β-open sets in nano topological spaces. Then, we study the relationship between them and show that if a nano topological space is nS_β-connected, then $\mathcal{W}$ is also nano connected space but not the converse. Also, we study each notion in terms of upper, lower and boundary approximations.

KEYWORDS: Nano S_β-open sets, nano S_β-connected, nano S_β-hyperconnected, nano S_β-ultraconnected.

الخلاصة

الهدف من هذه الدراسه هو ادخال مفاهيم الترابط النانوي من النمط- nS_β والترابط الفائق النانوي من النمط- nS_β والترابط المفرط النانوي من النمط- nS_β باستخدام كل انواع المجموعات النانوية المفتوحة من النمط- nS_β في الفضاءات التوبولوجية النانوية. ثم درسنا العلاقات بينهم وبيننا اذا كان الفضاء التوبولوجي مترابطا من النمط- nS_β فان $\mathcal{W}$ سيكون مترابطا نانويا والعكس غير صحيح. كذلك درسنا كل المفاهيم بواسطة التقاربيات الاعلى والادنى والحدودي.

INTRODUCTION

The concepts of nano topological space (briefly $\mathcal{NTS}$) introduced by Thivagar and Richard with respect to a subset X of $\mathcal{W}$ as the universe and also semi-open sets introduced by Thivagar and Richard [3], and nano β-open sets by Revathy and Ilango [1]. Later, by using nano semi-open sets, nano S_β-open and nano S_C-open sets were introduced by Pirbal and Ahmed [4-6]. Connectedness is one of the core concepts in topology. In the past, the notions of nano connected, hyperconnected and ultraconnected were introduced by Thivagar and Antoinette [2]. So, the aim of this paper is to study those notions in term of nano S_β-open sets in $\mathcal{NTS}$.

PRELIMINARIES

In this introductory section, we present some preliminaries, which will be used throughout the present work.

Definition 1. [7] Let $\mathcal{W} \neq \phi$ denote the finite universe and the equivalence relation R on the universe W called the indiscernibility relation. The pair $(\mathcal{W}, R)$ is called the approximation space. Let $X \subseteq \mathcal{W}$:

i. The lower approximation defined by $L_R(X) = \bigcup_{x \in W} \{R(x); R(x) \subseteq X\}$, x where $R(x)$ stands the equivalence class by x.
ii. The upper approximation defined by $U_R(X) = \bigcup_{x \in W} \{R(x); R(x) \cap X \neq \phi\}$.
iii. The boundary region defined by $B_R(X) = U_R(X) - L_R(X)$.

Definition 2. [4] Let $\mathcal{W}$ be the universe and R be an equivalence relation on W and $\tau_R(X) = \{\mathcal{W}, L_R(X), U_R(X), B_R(X), \phi\}$ where $X \subseteq \mathcal{W}$. Then $\tau_R(X)$ satisfies the followings axioms:

i. $\mathcal{W}, \phi \in \tau_R(X)$
ii. The union of members of $\tau_R(X)$ is in $\tau_R(X)$.
iii. The intersection of members of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ forms a topology on $\mathcal{W}$ and called the nano topology on W with respect to X.

Definition 3. [4] A nano semi-open set A of a $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ is said to be nano S_β-open set, if for each $x \in A$, there exist a nano β-closed set F

such that $x \in F \subseteq A$. The set of all nano S_β-open sets denoted by $nS_\beta O(\mathcal{W}, X)$.

Definition 4. [2] A $\mathcal{NTS}\,\mathcal{W}$ is called nano-ultraconnected if the intersection of any two non-empty nano-closed sets is non-empty.

Definition 5. [2] A $\mathcal{NTS}$ $\mathcal{W}$ is called nano s-space if every subset which contains a non-empty nano-open subset is nano-open.

Definition 6. [4] A $\mathcal{NTS}$ is called Nano extremally disconnected space, if the closure of any nano open subset is still an nano open subset.

Theorem 7. [4] Let A be a subset of a $(\mathcal{W}, \tau_R(X))$. If A is nano-clopen, then A is nS_β-clopen in $\mathcal{W}$.

Theorem 8. [4] Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$, then:

i. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi$, then $nS_\beta O(\mathcal{W}, X) = \{\mathcal{W}, \phi\}$.
ii. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi$, then $\tau_R(X) = nS_\beta O(\mathcal{W}, X)$.
iii. If $U_R(X) = L_R(X) \neq \{x\}, x \in \mathcal{W}$, then $nS_\beta O(\mathcal{W}, X) = \{\phi, \mathcal{W}\}$.
iv. If $U_R(X) = L_R(X) \neq \mathcal{W}$ and $U_R(X)$ contains more than one element of W, then the set of all nS_β-open sets in $\mathcal{W}$ are ϕ and those sets A for which $U_R(X) \subseteq A$.
v. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of $\mathcal{W}$, then the set of all nS_β-open sets in $\mathcal{W}$ are ϕ and those sets A for which $U_R(X) \subseteq A$.
vi. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi$, then ϕ, $L_R(X), B_R(X), L_R(X) \cup B, B_R(X) \cup B$ and any set containing $U_R(X)$ where $B \subseteq [U_R(X)]^c$ are the only nS_β-open sets in $\mathcal{W}$.

Definition 9. [5] Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$, then:

i. $nS_\beta\text{int}\ (A) = \cup \{G: G$ is nS_β-open and $G \subseteq A\}$.
ii. $nS_\beta cl(A) = \cap \{F: F$ is nS_β-closed and $A \subseteq F\}$.

Definition 10. [5] A function $f: (\mathcal{W}, \tau_R(X)) \to (V, \tau_{R'}(Y))$ is said to be nS_β- irresolute if $f^{-1}(G)$ is nS_β-open for every nS_β-open set G.

NANO S_β-CONNECTED

In this section, we study nano nS_β-connected spaces in $\mathcal{NTS}s$. Later, we study the relationship between nano connectedness and nano nS_β-connectedness and some properties.

Definition 11. Two non-empty subsets A and B in a $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ are said to be nS_β-separated if $A \cap nS_\beta cl(B) = B \cap nS_\beta cl(A) = \phi$.

Definition 12. The $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ is called nS_β-connected if there is no a nS_β-separation of $\mathcal{W}$. Otherwise, it is called nS_β-disconnected.

Definition 13. A subset A of a $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ is called nS_β-connected set if there is no a nS_β-separation of A.

Lemma 14. If A and B are nS_β-separated in $\mathcal{W}$ with $\phi \neq C \subseteq A$ and $\phi \neq D \subseteq B$, then C and D are also nS_β-separated sets in $\mathcal{W}$.
Proof. Obvious.

Theorem 15. A $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ is nS_β-connected space if and only if $\mathcal{W}$ can not expressed as the union of two disjoint non-empty nS_β-open sets in $\mathcal{W}$.

Proof. Let $\mathcal{W}$ be nS_β-connected. Assume that G and H are two disjoint non-empty nS_β-open subsets of U such that $= G \cup H$. Take $A = \mathcal{W} - G$ and $B = \mathcal{W} - H$. Then A and B are nS_β-closed in W. Thus $A \cap nS_\beta cl(B) = nS_\beta cl(A) \cap B = \phi$ and $\mathcal{W} = A \cup B$. Thus, $\mathcal{W}$ is not nS_β-connected, but this is a contradiction with the hypothesis. Thus, $\mathcal{W}$ cannot be expressed as the union of two disjoint non-empty nS_β-open subsets of $\mathcal{W}$.

Conversely, suppose that the condition holds. Let $\mathcal{W} = A \cup B,\ A, B \neq \phi\ and\ A \cap nS_\beta cl(B) = nS_\beta cl(A) \cap B = \phi$. Take $G = \mathcal{W} - nS_\beta cl(A)$ and $H = \mathcal{W} - nS_\beta cl(B)$. Then G and H are non-empty nS_β-open sets and $G \cup H = \left(\mathcal{W} - nS_\beta cl(A)\right) \cup \left(\mathcal{W} - nS_\beta cl(B)\right) = \mathcal{W} - (nS_\beta cl(A) \cap nS_\beta cl(B)) \subseteq W$. This implies that $\mathcal{W} = G \cup H$. Again $G \cap H = \left(\mathcal{W} - nS_\beta cl(A)\right) \cap \left(\mathcal{W} - nS_\beta cl(B)\right) = \mathcal{W} - (nS_\beta cl(A) \cup nS_\beta cl(B)) = \phi$, but this is a contradiction with the hypothesis. Therefore, $\mathcal{W}$ is nS_β-connected. □

Example 16. Let $\mathcal{W} = \{a, b, c\}$ with $\mathcal{W}/R = \{\{a\}, \{b, c\}\}$ and $X = \{a\}$, then $\tau_R(X) = \{\phi, \mathcal{W}, \{a\}\}$. Now, $nS_\beta O(\mathcal{W}, X) = \{\phi, \mathcal{W}\}$, then W cannot be expressed as a union of two disjoint

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non-empty nS_β-open sets in W. Hence, $\mathcal{W}$ is nS_β-connected space.

Theorem 17. For a $\mathcal{S}$ $(\mathcal{W}, \tau_R(X))$, the following statements are equivalent:

i. W is nS_β-connected.
ii. $\mathcal{W}$ and ϕ are the only nS_β-clopen subset of $\mathcal{W}$.
iii. $\mathcal{W}$ can not expressed as the union of two disjoint non-empty nS_β-open sets in $\mathcal{W}$.

Proof.
$(i \to ii)$ Suppose that $\mathcal{W}$ is nS_β-connected. Let A be a non-empty proper nS_β-clopen in $\mathcal{W}$, then $B = \mathcal{W} - A$ is also nS_β-clopen in $\mathcal{W}$. Therefore, $\mathcal{W} = A \cup B$ is a disjoint union of two non-empty nS_β-open sets, hence $\mathcal{W}$ is not nS_β-connected, which is a contradiction. Thus, $\mathcal{W}$ and ϕ are the only nS_β-clopen subset of $\mathcal{W}$.

$(ii \to iii)$ and $(iii \to i)$ Clearly follows from Theorem 15.

Theorem 18. If a $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ is nS_β-connected, then $\mathcal{W}$ is also nano connected space.

Proof. Let $\mathcal{W}$ be nS_β-connected. Then the only subsets of $\mathcal{W}$ which is nS_β-clopen are ϕ and $\mathcal{W}$. Suppose $\mathcal{W}$ is not nano connected. Then there exists a non-empty proper subset A of $\mathcal{W}$ which is nano-clopen in $\mathcal{W}$. By Theorem 7, A is also nS_β-clopen in $\mathcal{W}$. Hence A is a non-empty proper nano-clopen and also nS_β-clopen in $\mathcal{W}$, which is contradiction. Therefore, $\mathcal{W}$ is nano connected. □

The converse of Theorem 18, may not be true in general, as it shown by the following example.

Example 19. Let $\mathcal{W} = \{a, b, c, d\}$ with $\mathcal{W}/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, c\}$. Then $\tau_R(X) = \{\phi, \mathcal{W}, \{c\}, \{a, b, c\}, \{a, b\}\}$ and $nS_\beta O(X) = \{\phi, \mathcal{W}, \{c\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}\}$. Therefore, W is nano connected, but not nS_β-connected, since $\{c\}, \{a, b, d\} \in nS_\beta O(\mathcal{W}, X)$ and $\{c\} \cup \{a, b, d\} = \mathcal{W}$.

Proposition 20. Let $(\mathcal{W}, \tau_R(X))$ be an extremely disconnected $\mathcal{NTS}s$. If $\mathcal{W}$ is nano connected, then $\mathcal{W}$ is also nS_β-connected.

Proof. Follows form Theorem 8.

Theorem 21. Let $(\mathcal{W}, \tau_R(X))$ and $(\mathcal{W}, \tau_{R'}(Y))$ be two nano topologies on $\mathcal{W}$ with $\tau_{R'}(Y) \subseteq \tau_R(X)$. If the $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ is nS_β-connected, then $(\mathcal{W}, \tau_{R'}(Y))$ is also nS_β-connected.

Proof. Suppose that $\tau_R(X)$ is nS_β-connected but $\tau_{R'}(Y)$ is not. Then $\tau_{R'}(Y)$ contains a non-empty proper nS_β-clopen, since $\tau_{R'}(Y) \subseteq \tau_R(X)$, so also $\tau_R(X)$ contains nS_β-clopen, hence $\tau_R(X)$ is not nS_β-connected, which is contradiction. Therefore, $\tau_{R'}(Y)$ is also nS_β-connected. □

The converse of Theorem 21 is not true in general, as it shown by the following example.

Example 22. Let $\mathcal{W} = \{a, b, c, d\}$ with $\mathcal{W}/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, c\}$. Then $\tau_R(X) = \{\phi, \mathrm{W}, \{c\}, \{a, b, c\}, \{a, b\}\}$ and $nS_\beta O(\mathcal{W}, X) = \{\phi, \mathcal{W}, \{c\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}\}$. If $Y = \{c\}$, then $\tau_R(Y) = \{\phi, \mathcal{W}, \{c\}\} \subseteq \tau_R(X)$ and $nS_\beta O(\mathcal{W}, Y) = \{\phi, \mathcal{W}\}$. Thus, $nS_\beta O(\mathcal{W}, Y)$ is nS_β-connected space but $nS_\beta O(\mathcal{W}, X)$ is not.

Theorem 23. Let A be a nS_β-connected set of a $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ and G, H are nS_β-separated subsets of W such that $A \subseteq G \cup H$. Then either $A \subseteq G$ or $A \subseteq H$.

Proof. Since $A = (A \cap G) \cup (A \cap H)$, we have $(A \cap G) \cap nS_\beta cl(A \cap H) \subseteq G \cap nS_\beta cl(H) = \phi$. Similarly, we have $(A \cap H) \cap nS_\beta cl(A \cap G)) \subseteq H \cap nS_\beta cl(G) = \phi$. If $A \cap G$ and $A \cap H$ are non-empty, then A is not nS_β-connected, which is a contradiction. Therefore, either $A \cap G = \phi$ or $A \cap H = \phi$. Therefore, either $\subseteq G$ or $A \subseteq H$.

Theorem 24. If A is a nS_β-connected set of a $\mathcal{NTS}$ $(W, \tau_R(X))$ and $A \subseteq B \subseteq nS_\beta cl(A)$, then B is nS_β-connected.

Proof. Assume that B is not nS_β-connected. Then there exist nS_β-separated sets G and H such that $B = G \cup H$. Then G and H are non-empty and $G \cap nS_\beta cl(H) = \phi = nS_\beta cl(G) \cap H$, then by Theorem 24, we have either $A \subseteq G$ or $A \subseteq H$. So, we have two cases:

i. Suppose $A \subseteq G$. Then $nS_\beta cl(A) \subseteq nS_\beta cl(G)$ and $\cap$ $nS_\beta cl(A) = \phi$. By hypothesis, $G \cup H \subseteq$

$nS_\beta cl(A)$, that is $H = \phi$, but this is a contradiction to the fact that H is non-empty.

ii. Suppose $A \subseteq H$. Using similar argument of part (i), G is empty and this is a contradiction. Thus, B is nS_β-connected.

Corollary 25. Let A be a nS_β-connected subset of a $\mathcal{TS}$ $\mathcal{W}$. Then $nS_\beta cl(A)$ is nS_β-connected.
Proof. Obvious.

Proposition 26. Let A and B be subsets of a $(\mathcal{W}, \tau_R(X))$. If A and B are nS_β-connected sets and are not nS_β-separated in $\mathcal{W}$, then $A \cup B$ is nS_β-connected.
Proof. Suppose $A \cup B$ is not nS_β-connected. Then there exist nS_β-separated sets C, D in W such that $A \cup B = C \cup D$, then $A \subseteq C \cup D$. By Theorem 23, either $A \subseteq C$ or $A \subseteq D$. Again either $B \subseteq C$ or $B \subseteq D$. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$ and $D = \phi$, which is a contradiction. Therefore $A \subseteq C$ and $B \subseteq D$. Similarly, $A \subseteq D$ and $B \subseteq C$. Thus, we obtain $nS_\beta cl(A) \cap B \subseteq nS_\beta cl(C) \cap D = \phi$ and $nS_\beta cl(B) \cap A \subseteq nS_\beta cl(C) \cap D = \phi$. Hence A, B are nS_β-separated in W, which is a contradiction. Therefore, $A \cup B$ is nS_β-connected.

Proposition 27. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi$ in a $\mathcal{TS}$ $(\mathcal{W}, \tau_R(X))$, then $\mathcal{W}$ is nS_β-connected.
Proof. Since $nS_\beta O(\mathcal{W}, X) = \{\phi, \mathcal{W}\}$, then $\mathcal{W}$ is nS_β-connected.

Proposition 28. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi$ in a $\mathcal{TS}$ $(\mathcal{W}, \tau_R(X))$, then $\mathcal{W}$ is nS_β-disconnected space.
Proof. Since $\tau_R^{S_\beta}(X) = \{\phi, \mathcal{W}, L_R(X), B_R(X)\}$, but $L_R(X) \cap B_R(X) = \phi$ and $L_R(X) \cup B_R(X) = \mathcal{W}$, then $\mathcal{W}$ is nS_β-disconnected space. □

Proposition 29. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$. If $U_R(X) = L_R(X) = \{x\}$, $x \in \mathcal{W}$, then $\mathcal{W}$ is nS_β-connected.
Proof. Since $nS_\beta O(\mathcal{W}, X) = \{\phi, \mathcal{W}\}$, then U is nS_β-connected.

Proposition 30. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of $\mathcal{W}$, then $\mathcal{W}$ is nS_β-connected.
Proof. By Theorem 8, $U_R(X) \subseteq A \cap B$ for any non-empty nS_β-open sets A and B, hence $A \cap B \neq \phi$. Therefore, by Theorem 17 (ii), $\mathcal{W}$ is nS_β-connected.

Proposition 31. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$. If $U_R(X) = L_R(X) \neq \mathcal{W}$, and $U_R(X)$ contains more than one element of W, then $\mathcal{W}$ is nS_β-connected.
Proof. By Theorem 8, $U_R(X) \subseteq A \cap B$ for any non-empty nS_β-open sets A and B, hence $A \cap B \neq \phi$. Therefore, by Theorem 17 (ii), $\mathcal{W}$ is nS_β-connected.

Proposition 32. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi$, then $\mathcal{W}$ is nS_β-disconnected.
Proof. Since $L_R(X)$ is non-empty proper nS_β-clopen in $\mathcal{W}$. Therefore, $\mathcal{W}$ is nS_β-disconnected. □

Proposition 33. Let $f: (\mathcal{W}, \tau_R(X)) \to (V, \tau_{R'}(Y))$ be a surjective nS_β-irresolute function. If $\mathcal{W}$ is nS_β-connected, then $f(\mathcal{W}) = V$ is nS_β-connected.
Proof. Suppose that $\mathcal{W}$ is nS_β-connected. If A is a subset of V which is nS_β-clopen, then $f^{-1}(A)$ is nS_β-clopen in $\mathcal{W}$. Since $\mathcal{W}$ is nS_β-connected, so $f^{-1}(A)$ must be U or ϕ (i.e., $f^{-1}(A) = \mathcal{W}$ or $f^{-1}(A) = \phi$). Therefore $A = f(U) = V$ or $A = \phi$ and hence V is nS_β-connected.

Proposition 34. Every nS_β-connected space is extremally disconnected.
Proof. By Proposition 28, Proposition 30, Proposition 31 and Proposition 32, we have:

i. If $\tau_R(X) = \{\phi, \mathcal{W}\}$, then $ncl(\mathcal{W}) = \mathcal{W}$ and $ncl(\phi) = \phi$. Therefore, $\tau_R(X)$ is extremally disconnected.
ii. If $\tau_R(X) = \{\phi, \mathcal{W}, \{x\}\}$, then $ncl(\{x\}) = \mathcal{W}$. Therefore, $\tau_R(X)$ is extremally disconnected.
iii. If $\tau_R(X) = \{\phi, \mathcal{W}, U_R(X)\}$, then $ncl(U_R(X)) = \mathcal{W}$. Therefore, $\tau_R(X)$ is extremally disconnected.
iv. If $\tau_R(X) = \{\phi, \mathcal{W}, L_R(X)\}$, then $ncl(L_R(X)) = \mathcal{W}$. Therefore, $\tau_R(X)$ is extremally disconnected.

The converse of Proposition 34 is not true, as it is shown in the following example.

Example 35. Let $\mathcal{W} = \{a, b, c\}$ with $\mathcal{W}/R = \{\{a\}, \{b, c\}\}$ and $X = \{a, b\}$. Then $\tau_R(X) = \{\phi, U, \{a\}, \{b, c\}\}$. Then $\mathcal{W}$ is extremally disconnected but is not nS_β-connected, since $\{a\}$ is nS_β-clopen in $\mathcal{W}$.

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NANO S_β-HYPERCONNECTED

In this section, we define nano nS_β-hyperconnected space in $\mathcal{NTS}s$. Later, we study the relationship between nano hyperconnected and nano nS_β-connected.

Definition 36. A $\mathcal{NTS}$ $\left(\mathcal{W},\tau_R(X)\right)$ is said to be nS_β-hyperconnected if the intersection of any two non-empty nS_β-open sets is non-empty.

Proposition 37. In a $\mathcal{S}$ $\left(\mathcal{W},\tau_R(X)\right)$, if $\tau_R(X) = \{\mathcal{W},\phi\}$, then W is nS_β-hyperconnected space.

Proof. Let $\tau_R(X) = \{\mathcal{W},\phi\}$, then $nS_\beta O(\mathcal{W},X) = \{\mathcal{W},\phi\}$. Hence, U is nS_β-hyperconnected space. □

The converse of Proposition 37 may not to be true in general, as it is shown by the following example.

Example 38. Let $\mathcal{W} = \{a,b,c\}$ with $\mathcal{W}/R = \{\{a\},\{b,c\}\}$ and $X = \{a\}$, then $\tau_R(X) = \{\phi,\mathcal{W},\{a\}\}$ but $nS_\beta O(\mathcal{W},X) = \{\phi,\mathcal{W}\}$.

Proposition 39. If $U_R(X) = W$ and $L_R(X) = \phi$ in a $\left(\mathcal{W},\tau_R(X)\right)$, then $\mathcal{W}$ is nS_β-hyperconnected.

Proof. Since $\tau_R(X) = \{\mathcal{W},\phi\}$. Therefore, W is nS_β-hyperconnected.

Remark 40. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$ when $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi$, then $\mathcal{W}$ is not nS_β-hyperconnected, since, $\phi, W, L_R(X)$ and $B_R(X)$ are the only nS_β-open sets in W, and $L_R(X) \cap B_R(X) = \phi$. Hence, $\mathcal{W}$ is not nS_β-hyperconnected.

Proposition 41. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$. If $U_R(X) = L_R(X) = \{x\}$, $x \in \mathcal{W}$, then $\mathcal{W}$ is nS_β-hyperconnected.

Proof. By Theorem 9, $nS_\beta O(\mathcal{W},X) = \{\phi,\mathcal{W}\}$. Therefore, $\mathcal{W}$ is nS_β-hyperconneted.

Proposition 42. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$. If $U_R(X) = L_R(X) \neq \mathcal{W}$ and $U_R(X)$ contains more than one element of $\mathcal{W}$, then $\mathcal{W}$ is nS_β-hyperconnected.

Proof. By Theorem 8, ϕ and those sets A for which $U_R(X) \subseteq A$ are the only nS_β-open sets in $\mathcal{W}$. Let A and B be any two non-empty nS_β-open set, then $A \cap B \neq \phi$, since $U_R(X)$ is a subset of A and B. Hence, $\mathcal{W}$ is nS_β-hyperconneted.

Proposition 43. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of $\mathcal{W}$, then $\mathcal{W}$ is nS_β-hyperconnected.

Proof. The proof is similar to the proof of above proposition.

Remark 44. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$. If $U_R(X) \neq \mathcal{W}$, $L_R(X) \neq \phi$ and $U_R(X) \neq L_R(X)$, then $\mathcal{W}$ is not nS_β-hyperconnected. Since $B_R(X) \cap L_R(X) = \phi$.

Proposition 45. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$. Then the following are equivalent:

i. $\mathcal{W}$ is nS_β-hyperconnected.
ii. $\mathcal{W}$ is nS_β-connected.

Proof. Obvious.

NANO S_β-ULTRACONNECTED

In this section, we define nano nS_β-ultraconnected space in $\mathcal{NTS}s$. Later, we study the relationship between nano ultraconnected and nano nS_β-ultraconnected.

Definition 46. A $\mathcal{NTS}$ $\mathcal{W}$ is called nS_β-ultraconnected if the intersection of any two non-empty nS_β-closed sets is non-empty.

Proposition 47. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi$ in a $\mathcal{NTS}$ $\left(\mathcal{W},\tau_R(X)\right)$, then $\mathcal{W}$ is nS_β-ultraconnected.

Proof. Since $nS_\beta O(\mathcal{W},X) = \{\mathcal{W},\phi\}$, Hence, $\mathcal{W}$ is nS_β-ultraconnected space.

Remark 48. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi$, then $\mathcal{W}$ is not S_β-ultraconnected. Since $\tau_R^{S_\beta}(X) = \{\phi,\mathcal{W},L_R(X),B_R(X)\} = \left[\tau_R^{S_\beta}(X)\right]^c$ $L_R(X) \cap B_R(X) = \phi$, hence $\mathcal{W}$ is not nS_β-ultraconnected space.

Proposition 49. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$. If $U_R(X) = L_R(X) = \{x\}$, $x \in \mathcal{W}$, then $\mathcal{W}$ is nS_β-ultraconnected.

Proof. Since $nS_\beta O(\mathcal{W},X) = \{\mathcal{W},\phi\}$, Hence, $\mathcal{W}$ is nS_β-ultraconnected space. □

Remark 50. Let $\left(\mathcal{W},\tau_R(X)\right)$ be a $\mathcal{NTS}$, then:

i. If $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of $\mathcal{W}$, then $\mathcal{W}$ may not be nS_β-ultraconnected.
ii. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of $\mathcal{W}$, then $\mathcal{W}$ may not be nS_β-ultraconnected.

As shown in the following example.

Example 51.
i. Let $\mathcal{W} = \{a, b, c, d\}$ with $\mathcal{W}/R = \{\{a, b\}, \{c, d\}\}$ and $X = \{a, b\}$. Then $\tau_R(X) = \{\phi, \mathcal{W}, \{a, b\}\}$. Thus, $nS_\beta C(\mathcal{W}, X) = \{\phi, \mathcal{W}, \{c, d\}, \{d\}, \{c\}\}$. Since $\{d\} \cap \{c\} = \phi$, hence $\mathcal{W}$ is not nS_β-ultraconnected.
ii. Let $\mathcal{W} = \{a, b, c, d\}$ with $\mathcal{W}/R = \{\{a, b\}, \{c, d\}\}$ and $Y = \{a\}$, then $\tau_R(Y) = \{\phi, \mathcal{W}, \{a, b\}\}$. Then $nS_\beta C(\mathcal{W}, Y) = \{\phi, \mathcal{W}, \{c, d\}, \{d\}, \{c\}\}$. Since $\{d\} \cap \{c\} = \phi$, hence $\mathcal{W}$ is not nS_β-ultraconnected.

Proposition 52. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$ where $[U_R(X)]^c$ is singleton subset of W:
i. If $U_R(X) = L_R(X) \neq \mathcal{W}$ and $U_R(X)$ contains more than one element of W, then $\mathcal{W}$ is nS_β-ultraconnected.
ii. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of $\mathcal{W}$, then $\mathcal{W}$ is nS_β-ultraconnected.

Proof. In both cases, since $nS_\beta O(\mathcal{W}, X) = \{\phi, \mathcal{W}, U_R(X)\}$, it follows that W is nS_β-ultraconnected.

Remark 53. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$. If $\mathcal{W}_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi$, then $\mathcal{W}$ is not is nS_β-ultraconnected, since $L_R(X)$ and $B_R(X)$ are disjoint nS_β-clopen subset in $\mathcal{W}$.

Proposition 54. Every nS_β-ultraconnected space is nano ultraconnected.
Proof. Obvious.

The converse of Proposition 54 is not true in general, as it is shown in the following example.

Example 55. Let $\mathcal{W} = \{a, b, c, d\}$ with $\mathcal{W}/R = \{\{a, b\}, \{c, d\}\}$ and $X = \{a, b\}$. Then $[\tau_R(X)]^c = \{\phi, U, \{c, d\}\}$ and $nS_\beta C(\mathcal{W}, X) = \{\phi, U, \{c, d\}, \{d\}, \{c\}\}$. Since $\{d\} \cap \{c\} = \phi$, hence $\mathcal{W}$ is not nS_β-ultraconnected but nano ultraconnected.

Definition 56. A $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$ is called nS_β^s-space if every subset which contains a non-empty nS_β-open subset is nS_β-open.

Proposition 57. If $U_R(X) = \mathcal{W}$ and $L_R(X) = \phi$ in a $(\mathcal{W}, \tau_R(X))$, then $\mathcal{W}$ is nS_β^s-space.
Proof. Since $nS_\beta O(\mathcal{W}, X) = \{\mathcal{W}, \phi\}$. Hence, $\mathcal{W}$ is nS_β^s-space.

Proposition 58. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$. If $U_R(X) = L_R(X) = \{x\}$, $x \in \mathcal{W}$, then $\mathcal{W}$ is nS_β^s-space.
Proof. Since $nS_\beta O(\mathcal{W}, X) = \{\mathcal{W}, \phi\}$. Hence, $\mathcal{W}$ is nS_β^s-space.

Remark 59. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$, then $\mathcal{W}$ is not nS_β^s-space in the following cases:
i. If $U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi$ in a $(\mathcal{W}, \tau_R(X))$, then W is not nS_β^s-space. Since by Theorem 8, $\tau_R^{S_\beta}(X) = \{\phi, \mathcal{W}, L_R(X), B_R(X)\}$ and it is clear W contains more than two points, since $L_R(X)$ and $B_R(X)$ are nS_β-clopen subsets of U, then there exist a point $x \in B_R(X)$ such that $L_R(X) \subseteq L_R(X) \cup \{x\}$, but $L_R(X) \cup \{x\}$ is not nS_β-open set in W. Or there exist a point $x \in L_R(X)$ such that $B_R(X) \subseteq B_R(X) \cup \{x\}$, but $B_R(X) \cup \{x\}$ is not nS_β-open set in W. Therefore, $\mathcal{W}$ is not nS_β^s-space.
ii. If $U_R(X) \neq L_R(X)$ where $U_R(X) \neq \mathcal{W}$ and $L_R(X) \neq \phi$. Since $L_R(X) \cap B_R(X) = \phi$ and $L_R(X) \subseteq \{x\} \cup L_R(X)$, where $x \in B_R(X)$ and $\{x\} \cup L_R(X)$ is not nS_β-open. Therefore, $\mathcal{W}$ is not nS_β^s-space.

Proposition 60. Let $(\mathcal{W}, \tau_R(X))$ be a $\mathcal{NTS}$.
i. If $U_R(X) = L_R(X) \neq \mathcal{W}$ and $U_R(X)$ contains more than one element of $\mathcal{W}$, then W is nS_β^s-space.
ii. If $U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of W, then $\mathcal{W}$ is nS_β^s-space.

Proof.
i. By Theorem 8, ϕ and those sets A for which $U_R(X) \subseteq A$ are the only nS_β-open sets in $\mathcal{W}$. Let A is any nS_β-open and B be any subset of $\mathcal{W}$ which $A \subseteq B$. Since A is nS_β-open, then

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$U_R(X) \subseteq A \subseteq B$. Hence B is also nS_β-open. Therefore, $\mathcal{W}$ is nS_β^s-space.

ii. The proof is similar to part (i).

Proposition 61. Every nano s-space is nS_β^s-space in a $\mathcal{TS}$ $\mathcal{W}$.

Proof. Obvious.

The converse of Proposition 61 is not true in general, as it is shown in the following example.

Example 62. Let $\mathcal{W} = \{a, b, c, d\}$ with $\mathcal{W}/R = \{\{a, b\}, \{c, d\}\}$ and $X = \{a, b\}$. Then $\tau_R(X) = \{\phi, \mathcal{W}, \{a, b\}\}$, it is clear $\mathcal{W}$ is nS_β^s-space but not nano s-space.

CONCLUSIONS

In this paper, we introduced the notions of nano S_β-connected, nano S_β-hyperconnected and nano S_β-ultraconnected by using the all forms of nano S_β-open sets in $\mathcal{NTS}s$. Then, we studied the relations among them and we showed that if a $\mathcal{NTS}$ is nS_β-connected, then $\mathcal{W}$ is also nano connected space but not the converse. Our results (the most important) as conclusion shown in the following table. By considering the results in Table 1, several results may be derived. The "1" means the case holds and "0" means the case does not hold and the result is closed. Additionally, the "0*" means the case holds under a condition and the result is not closed. Now, we see that if "$U_R(X) = \mathcal{W}$ and $L_R(X) = \phi$" or "$U_R(X) = L_R(X) = \{x\}$, $x \in \mathcal{W}$" in a $\mathcal{NTS}$ $(\mathcal{W}, \tau_R(X))$, then the space is nano S_β-connected, nano S_β-hyperconnected and nano S_β-ultraconnected. But if "$U_R(X) \neq \mathcal{W}$, $L_R(X) \neq \phi$ and $U_R(X) \neq L_R(X)$" or "$U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi$", then it is not nano S_β-connected, nano S_β-hyperconnected and nano S_β-ultraconnected.

Table 1. Relation among nS_β-connected, nS_β-hyperconnected, nS_β-ultraconnected and $nS_{S_\beta}^S$-spaces.

Family of nS_β-open sets in term of upper and lower approximations if:	nS_β-connected	nS_β-hyperconnected	nS_β-ultraconnected	$nS_{S_\beta}^S$-space
$U_R(X) = \mathcal{W}$ and $L_R(X) = \phi$	1	1	1	1
$U_R(X) = \mathcal{W}$ and $L_R(X) \neq \phi$	0	0	0	0
$U_R(X) = L_R(X) = \{x\}$, $x \in \mathcal{W}$	1	1	1	1
$U_R(X) = L_R(X) \neq \mathcal{W}$ and $U_R(X)$ contains more than one element of U.	1	1	0*	1
$U_R(X) \neq \mathcal{W}$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of $\mathcal{W}$.	1	1	0*	1
$U_R(X) \neq \mathcal{W}$, $L_R(X) \neq \phi$ and $U_R(X) \neq L_R(X)$	0	0	0	0

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