Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 34, Issue 1, 2023 DOI: http://doi.org/10.23851/mjs.v34i1.1241

Research Article **Open Access**

Classical Continuous Boundary Optimal Control Vector Problem for Triple Nonlinear Parabolic System

Jamil A. Ali Al-Hawasy*, Yasameen H. Rashid

Department of Mathematics, College of Science, Mustansiriyah University, 10052 Baghdad, IRAQ

*Correspondent contact: jhawassy17@uomustansiriyah.edu.iq

Article Info

Received
15/10/2022

Accepted
04/12/2022

Published
30/03/2023

ABSTRACT

In this paper, our purpose is to study the classical continuous boundary optimal triple control vector problem (CCBOTCVP) dominating by nonlinear triple parabolic boundary value problem (NLTPBVP). Under suitable assumptions and with given classical continuous boundary triple control vector (CCBTCV), the existence theorem for a unique state triple vector solution (STVS) of the weak form W.F for the NLTPBVP is stated and demonstrated via the Method of Galerkin (MGa), and the first compactness theorem. Furthermore, the continuity operator between the STVS of the WFO for the NLTPBVP and the corresponding CCBTCV is stated and demonstrated. The continuity of the Lipschitz (LIP.) operator between the STVS of the WFo for the QNLPBVP and the corresponding CCBTCV is proved. The existence of a CCBOTCV is stated and demonstrated under suitable conditions.

KEYWORDS: Classical boundary optimal triple control; Lipschitz continuity; nonlinear triple parabolic boundary value problem; method of Galerkin.

الخلاصة

في هذا البحث هدفنا هو دراسة مسالة متجه السيطرة الثلاثية الحدودية الأمثلية التقليدية المستمرة لمسائل القيم الحدودية الثلاثية غير الخطية المكافئة، بوجود شروط مناسبة. تم ذكر نص وبرهان مبرهنة وجود ووحدانية الحل لمتجه الحالة الثلاثي المستمر للصيغة الضعيفة لمسائل القيم الحدودية الثلاثية الغير خطية المكافئة عندمايكون متجه السيطرة الحدودية الثلاثية التقليدية المستمرة معلوماً، بواسطة طريقة كاليركن والمبرهنة المرصوصة الاولى. تم برهان عامل الاستمرارية بين متجه الحالة الثلاثي المستمر للصيغة الضعيفة لمسالة القيم الحدودية الثلاثية المكافئة ومتجه السيطرة الثلاثية الحدودية التقليدية المستمرة. تم برهان عامل الاستمرارية بين متجه الحالة الثلاثي المستمر للصيغة الضعيفة لمسالة القيم الحدودية الثلاثية المكافئة وبين متجه السيطرة الثلاثية الحدودية التقليدية المستمرة. أيضاً تم برهان مبرهنة وجود متجه ثلاثي لسيطرة أمثلية حدودية تقليدية مستمرة لهذه المسألة بوجود شروط مناسبة.

INTRODUCTION

Optimal control problems (OCPs) play an important role in many practical applications, such as in medicine [1], aircraft [2], economics [3], robotics [4], weather conditions [5] and many other scientific fields. They are two types of OCPs; the classical and the relax type, each one of these two types is dominated either by nonlinear ODEs [6] or by nonlinear PDEs (NLPDEs) [7]. The classical continuous optimal boundary control problem (CCOBCP) dominated by nonlinear parabolic or elliptic or hyperbolic PDEs is studied in [8-10] respectively (resp.). Later, the study of the CCOBCPs dominated by the three types of PDEs is generalized in [11-13] to deal with CCOBCPs dominating by couple nonlinear PDEs (CNLPDES) of these types resp., and then the studies of the second and the third types are generalized also to deal with continuous classical optimal control problems (CCOCPs) dominated by triple and NLPDEs of the elliptic and the hyperbolic types [14, 15].

All of the above-mentioned studies encouraged us to think about generalizing the study of the CCOCP dominated by CNLPDEs of parabolic type to a CCOCP dominated by TNLPBVP. According to this idea for the generalization, the mathematical model for the dominating equation is needed to be found, as well as the cost function, the spaces of definition for the control and the state vectors, which all of them are needed to be generalized. The study of the CCBOTCVP dominated by the NLTPBVP which is proposed in this paper starts with the state and proof of the existence theorem of the STVS of the W.F for the NLTPBVP using the MGa with the first compactness theorem, under suitable conditions and when the CCBTCV is known. The continuity of the Lip. operator between the STVS of the W.F for the QNLPBVP and the corresponding CCBTCV is proved. The existence theorem of a CCBOCV is stated and demonstrated under suitable conditions.

PROBLEM DESCRIPTION:

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded region with Lipschitz (Lip)boundary $\Gamma = \partial\Omega, \mathrm{x} = (x_{1,}x_{2,})$, $Q = \Omega \times I$, $I = [0,T]]$, $\Sigma = \Gamma \times I$.

The CCOCP consists of the TSEs which are given by the following TNLPPDEs:

$y_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{1ij}(x,t)\frac{\partial y_1}{\partial x_j}) + b_1 y_1 - b_4 y_2 - b_5 y_3 = f_1(x,t,y_1)$, in Q (1)

$y_{2t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{2ij}(x,t)\frac{\partial y_2}{\partial x_j}) + b_2 y_2 + b_6 y_3 + b_4 y_1 = f_2(x,t,y_2)$, in Q (2)

$y_{3t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{3ij}(x,t)\frac{\partial y3}{\partial x_j}) + b_3 y_3 + b_5 y_1 - b_6 y_2 = f_3(x,t,y_2)$, in Q (3)

With the following BCs and ICs

$\frac{\partial y_1}{\partial n_1} = \sum_{i,j=1}^{2} a_{1ij}(x,t)\frac{\partial y_1}{\partial x_j}\cos(n_1,x_j) = u_1(x,t)$ on Σ, (4)

$\frac{\partial y_2}{\partial n_2} = \sum_{i,j=1}^{2} a_{2ij}(x,t)\frac{\partial y_2}{\partial x_j}\cos(n_2,x_j) = u_2(x,t)$ on Σ (5)

$\frac{\partial y_3}{\partial n_3} = \sum_{i,j=1}^{2} a_{3ij}(x,t)\frac{\partial y_2}{\partial x_j}\cos(n_3,x_j) = u_3(x,t)$, on Σ (6)

$y_1(x,0) = y_1^0(x)$, in Ω (7)

$y_2(x,0) = y_2^0(x)$, in Ω (8)

$y_3(x,0) = y_3^0(x)$, in (9)

Where $(f_1,f_2,f_3) \in (L^2(\mathrm{Q}))^3$ is a vector of a given function $(x_1,x_2) \in \Omega$, $a_{lij}(x,t)$, $b_l(x,t) \in C^\infty(Q)$, n_ℓ, (for $\ell = 1,2,3$) is a unit vector normal outer on the boundary Σ , (n_ℓ, x_j) is the angle between n_ℓ and the $x_j - axis$, $\vec{u} = (u_1,u_2,u_3) \in (L^2(\Sigma))^3$ is a CCBCV and $\vec{y} = \vec{y}_{\vec{u}} = (y_{1u_1}, y_{2u_2}, y_{3u_2}) \in (H^2(\Omega))^3$ is the TSVS corresponding to the CCBCV.

The admissible set of the CCBCV is defined by

$\vec{W}_A = \left\{\vec{u} \in (L^2(\Sigma))^3 \,\middle|\, \vec{u} \in \vec{U} \text{ a.e. in } \Sigma\right\}$,

$\vec{u} = (u_1,u_3,u_3)$, $\vec{U} = U_1 \times U_2 \times U_3 \subset \mathbb{R}^3$

The cost function (CF.) is

$G_0(\vec{u}) = \sum_{l=1}^{3}[\int_Q g_{0l}(x,t,y_l)dxdt + \int_\Sigma h_{0l}(x,t,u_l)d\sigma]$ (10)

Let $\vec{V} = V_1 \times V_2 \times V_3 = \mathrm{V} \times \mathrm{V} \times \mathrm{V} = \{\vec{v}: \vec{v} = (v_1(x), v_2(x), v_3(x)) \in (H^1(\Omega))^3\}$.

The W.F of the TSEs (1-9) when $\vec{y} \in (H_0^1(\Omega))^3$ is given $(\forall v_1, v_2, v_3 \in V)$by

$\langle y_{1t}, v_1\rangle + a_1(t,y_1,v_1) + (b_1(t)y_1,v_1)_\Omega - (b_4(t)y_2,v_1)_\Omega - (b_5(t)y_3,v_1)_\Omega = (f_1(y_1),v_1)_\Omega + (u_1,v_1)_\Gamma$, (11a)

$(y_1^0, v_1)_\Omega = (y_1(0), v_1)_\Omega$ (11b)

$\langle y_{2t}, v_2\rangle + a_2(t,y_2,v_2) + (b_2(t)y_2,v_2)_\Omega + (b_6(t)y_3,v_2)_\Omega + (b_4(t)y_1,v_2)_\Omega = (f_2(y_2),v_2)_\Omega + (u_2,v_2)_\Gamma$, (12a)

$(y_2^0, v_2)_\Omega = (y_2(0), v_2)_\Omega$ (12b)

$\langle y_{3t}, v_3\rangle + a_3(t,y_3,v_3) + (b_3(t)y_3,v_3)_\Omega + (b_5(t)y_1,v_3)_{\Omega-}(b_6(t)y_2,v_3)_\Omega = (f_3(y_3),v_3)_\Omega + (u_3,v_3)_\Gamma$, (13a)

$(y_3^0, v_2)_\Omega = (y_3(0), v_3)_\Omega$ (13b)

Where $a_l(t,y_l,v_l) = \int_\Omega \sum_{i,j=1}^{n} a_{lij}\frac{\partial y_l}{\partial x_i}\frac{\partial v_l}{\partial x_j}dx$ for $l = 1,2,3$

***Assumptions (A)*:** for $i = 1,2,3$

(i) f_i is of a Carathéodory type (C-T) on $Q \times \mathbb{R}$, satisfies: $|f_i(x,t,y_i)| \le \eta_i(x,t) + c_i|y_i|$ where $y_i, u_i \in \mathbb{R}$, $c_i > 0$ and $\eta_i \in L^2(Q,\mathbb{R})$.

(ii) f_i is LIP. w.r.t. y_i, i.e.:

$|f_i(x,t,y_i) - f_i(x,t,\hat{y}_i)| \le L_i|y_i - \hat{y}_i|$

where $y_i, \hat{y}_i \in \mathbb{R}$ and $L_i > 0$

(iii) $|a_i(t,y_i,v_i)| \le \alpha_i\|y_i\|_1\|v_i\|_1$,

$|(b_i(t)y_i,v_i)_\Omega| \le \beta_i\|y_i\|_0\|v_i\|_0$, $a_i(t,y_i,y_i) \ge \bar{\alpha}_i\|y_i\|_1^2$,

$(b_i(t)y_i,y_i)_\Omega \ge \bar{\beta}_i\|y_i\|_0^2$, $|(b_4(t)y_2,v_1)_\Omega| \le \epsilon_1\|y_2\|_0\|v_1\|_0$,

$|(b_4(t)y_1,v_2)_\Omega| \le \epsilon_2\|y_1\|_0\|v_2\|_0$,

$|(b_5(t)y_3,v_1)_\Omega| \le \epsilon_3\|y_3\|_0\|v_1\|_0$,

$|(b_5(t)y_1,v_3)_\Omega| \le \epsilon_4\|y_1\|_0\|v_5\|_0$,

$|(b_6(t)y_3,v_2)_\Omega| \le \epsilon_5\|y_3\|_0\|v_2\|_0$,

$|(b_6(t)y_2,v_3)_\Omega| \le \epsilon_6\|y_2\|_0\|v_3\|_0$,

$c(t,\vec{y},\vec{y}) = \sum_{i=1}^{3}[a_i(t,y_i,y_i) + (b_i(t)y_i,y_i)_\Omega]$, with $c(t,\vec{y},\vec{y}) \ge \bar{\alpha}\|\vec{y}\|_1^2$.

Where $\|v\|_0$,and $\|v\|_1$ are denote to the norms in the spaces $\mathrm{L}^2(\Omega)$, $H^1(\Omega)$ resp. and $\|\vec{v}\|_1^2 = \sum_{i=1}^{3}\|v_i\|_1^2$, $\alpha_i, \bar{\alpha}_i, \beta_i, \bar{\beta}_i$ ($\forall\, i = 1,2,3$), ϵ_i ($\forall\, i = 1,2,3,4,5,6$) and $\bar{\alpha}$ are real positive constants.

Assumptions (B):

Consider g_{kl} and h_{kl} (for each $k = 0,1,2,3$ and $l = 1,2,3$) is of C-T on $(Q \times \mathbb{R})$ and on$(\Sigma \times \mathbb{R})$ resp.

ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 34, Issue 1, 2023 DOI: http://doi.org/10.23851/mjs.v34i1.1241

and satisfy the following (with $\gamma_{kl} \in L^1(Q)$, $\delta_{kl} \in L^1(\Sigma)$):
$|g_{kl}(x,t,y_l)| \leq \gamma_{kl}(x,t) + c_{kl}(y_l)^2$,
$|h_{kl}(x,t,u_l)| \leq \delta_{kl}(x,t) + d_{kl}(u_l)^2$.

MAIN RESULTS

Solvability of the TSEs

Theorem 1: With Assumptions (A), for fixed CCBCV $\vec{u} \in \left(L^2(\Sigma)\right)^3$, the WFO of the ((11)-(13) has a unique solution $\vec{y} = (y_1, y_2, y_3)$ s.t. $\vec{y} \in$, $\vec{y}_t = \left(y_{1t}, y_{2t,}y_{3t,}\right) \in \left(L^2(\mathrm{I},\mathrm{V})\right)^3$.

Proof:

Let $\vec{V}_n \subset \vec{V}$ be the set of continuous and piecewise affine functions in Ω, and $\{\vec{v}_1, \vec{v}_2, \vec{v}_{3,} \dots, \vec{v}_n\}$ be basis of $\vec{V}_n$, where $n = 3N$ (N is the dimension of each V), then the TSVS $\vec{y}$ of ((2.10)-(2.12)) is approximated for each n by $\vec{y}_n = \left(y_{1n}, y_{2n,}y_{2n,}\right)$, s.t.

$y_{ln} = \sum_{j=1}^{n} c_{ij}(t) v_{ij}(x) \qquad \forall l = 1,2,3$ (14)

where $c_{ij}(t)$ is an unknown function of t $\forall l = 1,2,3$, and $j = 1,2,\dots\dots,n$.

The MGa is used to approximate the WFO ((2.11) - (2.13)) w.r.t the space variable, and to get

$\langle y_{1nt}, v_1\rangle + a_1(t, y_{1n}, v_1) + (b_1(t)y_{1n}, v_1)_\Omega - (b_4(t)y_{2n}, v_1)_{\Omega -} (b_5(t)y_{3n}, v_1)_\Omega = (f_1(y_{1n}), v_1)_\Omega + (u_1, v_1)_\Gamma$ (15a)

$(y_{1n}^0, v_1)_\Omega = (y_1^0, v_1)_\Omega$, (15b)

$\langle y_{2nt}, v_2\rangle + a_2(t, y_{2n}, v_2) + (b_2(t)y_{2n}, v_2)_\Omega + (b_6(t)y_{3n}, v_2)_\Omega + (b_4(t)y_{1n}, v_2)_\Omega = (f_2(y_{2n}), v_2)_\Omega + (u_2, v_2)_\Gamma$ (16a)

$(y_{2n}^0, v_2)_\Omega = (y_2^0, v_2)_\Omega$, (16b)

$\langle y_{3nt}, v_3\rangle + a_3(t, y_{3n}, v_3) + (b_3(t)y_{3n}, v_3)_\Omega + (b_5(t)y_{1n}, v_3)_\Omega + (b_6(t)y_{2n}, v_3)_\Omega = (f_3(y_{3n}), v_3)_\Omega + (u_3, v_3)_\Gamma$ (17a)

$(y_{3n}^0, v_3)_\Omega = (y_3^0, v_3)_\Omega$, (17b)

where $y_{in}^0 = y_{in}(x,0) \in V_n \subset V \subset L^2(\Omega)$ is the projection (pro.) of y_i^0 for the norm $\|.\|_0$ i.e., $(y_{ln}^0, v_l)_\Omega = (y_l^0, v_l)_\Omega \Leftrightarrow \left\|y_{ln}^0 - y_l^0\right\|_0 \leq \left\|y_l^0 - l\right\|_0$, $\forall v_l \in V_n$,$\forall l = 1,2,3$

Utilizing (14) in ((15) -(17)), setting $v_l = v_{lj}$,$\forall l = 1,2,3$, the following system, which has a unique solution $\vec{y}_n$ is obtained:

$A_1 C_1'(t) + D_1 C_1(t) - E_1 C_2(t) - F_1 C_1(t) = b_1\,(\bar{V}_1^T(x) C_1(t))$ (18a)

$A_1 C_1(0) = b_1^0$ (18b)

$A_2 C_2'(t) + D_2 C_2(t) + E_2 C_3(t) + F_2 C_1(t) = b_2\,(\bar{V}_2^T(x) C_2(t))$ (19a)

$A_2 C_2(0) = b_2^0$ (19b)

$A_3 C_3'(t) + D_3 C_3(t) + E_3 C_1(t) - F_3 C_2(t) = b_3\,(\bar{V}_3^T(x) C_3(t))$ (20a)

$A_3 C_3(0) = b_3^0$ (20b)

where $A_l = \left(a_{lij}\right)_{n\times n}$, $a_{lij} = (v_{lj}, v_{li})_\Omega$, $D_l = \left(d_{lij}\right)_{n\times n}$, $d_{lij} = [a_l(t, v_{lj}, v_{li}) + (b_l(t)v_{lj}, v_{li})_\Omega]$, $E_1 = \left(e_{ij}\right)_{n\times n}$, $e_{ij} = (b_4(t)v_{2j}, v_{1i})_\Omega$, $F_1 = \left(f_{ij}\right)_{n\times n}$, $f_{ij} = (b_5(t)v_{3j}, v_{1i})_\Omega$, $C_l(t) = (c_{lj}(t))_{n\times 1}$, $C_l'(t) = (c_{lj}'(t))_{n\times 1}$, $C_l(0) = (c_{lj}(0))_{n\times 1}$, $b_l = (b_{li})_{n\times 1}$, $b_{li} = (f_l(\bar{V}_l^T C_l(t)), v_{li})_\Omega + (u_l, v_{li})_\Gamma$,$\bar{V}_l = (v_l)_{n\times 1}$, $b_l^0 = (b_{li}^0)$,$b_{li}^0 = (y_l^0, v_{li})_\Omega$, $E_2 = \left(h_{ij}\right)_{n\times n}$, $h_{ij} = (b_4(t)v_{1i}, v_{2i})_\Omega$, $F_2 = \left(k_{ij}\right)_{n\times n}$, $k_{ij} = (b_6(t)v_{3i}, v_{2i})_\Omega$, $E_3 = \left(n_{ij}\right)_{n\times n}$, $n_{ij} = (b_5(t)v_{1i}, v_{3i})_\Omega$, $F_3 = \left(z_{ij}\right)_{n\times n}$, $z_{ij} = (b_6((t)v_{2i}, v_{3i})_\Omega$,for $l = 1,2,3$, and $i,j = 1,2,\dots,n$.

***The norm $\left\|\vec{y}_n^0\right\|_0$ is bounded*:** since for $l = 1,2,3$, $y_l^0 = y_l^0(x) \in L^2(\Omega)$, then there exists $\{v_{ln}^0\}$, with $v_{ln}^0 \in V_n$, such that $v_{ln}^0 \to y_l^0$ strongly (ST) in $L^2(\Omega)$, and since
$\left\|y_{ln}^0 - y_l^0\right\|_0 \leq \left\|y_l^0 - v_l\right\|_0$, $\forall v_l \in V$ then $\forall n$
$\left\|y_{ln}^0 - y_l^0\right\|_0 \leq \left\|y_l^0 - v_{ln}^0\right\|_0$, $\forall v_{ln}^0 \in V_n \subset V$,
thus $y_{ln}^0 \longrightarrow y_l^0$ is St in $L^2(\Omega)$ and $\left\|y_{ln}^0\right\|_0 \leq b_l$

***The norm $\|\vec{y}_n(t)\|_{L^\infty\left(I, L^2(\Omega)\right)}$ and $\|\vec{y}_n(t)\|_Q$ are bounded*:**

Setting $v_l = y_{ln}$ in ((15a) - (17a)), I.B.S of each obtaining equation on $[0,T]$, adding the resulting equations, finally with Assumption (A-iii), one has

$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n\rangle dt + \int_0^T c(t, \vec{y}_n, \vec{y}_n) dt = \int_0^T (f_1(y_{1n}), y_{1n})_\Omega dt + \int_0^T (f_2(y_{2n}), y_{2n})_\Omega\, dt + \int_0^T (f_3(y_{3n}), y_{3n})_\Omega\, dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (u_2, y_{2n})_\Gamma dt + \int_0^T (u_3, y_{3n})_\Gamma dt$ (21)

Since $\vec{y}_{nt} \in (L^2(I,V^*))^3 = (L^2(I,V))^3$ and $\vec{y}_n \in (L^2(I,V))^3$ in the 1^{st} term of the L.H.S. of (21), hence for this term we can use Lemma 1.2 in [16] and since the 2^{nd} term is positive, taking $T = t \in [0,T]$, finally using Assum. (A-i) for the 1^{st} three terms in the R.H.S. and the Cauchy-Schwarz inequality (C-S-I) for the three reminder terms, we get

$$\int_0^t \frac{d}{dt}\|\vec{y}_n(t)\|_0^2\,dt \le \int_0^t\int_\Omega(\eta_1^2+|y_{1n}|^2)\,dxdt + 2\int_0^t\int_\Omega c_1|y_{1n}|^2\,dxdt + \int_0^t\int_\Gamma(|u_1|^2+|y_{1n}|^2)d\gamma dt + \int_0^t\int_\Gamma(|u_1|^2+|y_{1n}|^2)d\gamma dt + \int_0^t\int_\Omega(\eta_2^2+|y_{2n}|^2)\,dxdt + 2\int_0^t\int_\Omega c_2|y_{2n}|^2\,dxdt + \int_0^t\int_\Gamma(|u_2|^2+|y_{2n}|^2)d\gamma dt + \int_0^t\int_\Omega(\eta_3^2+|y_{3n}|^2)\,dxdt + 2\int_0^t\int_\Omega c_3|y_{3n}|^2\,dxdt + \int_0^t\int_\Gamma(|u_3|^2+|y_{3n}|^2)d\gamma dt$$

Let $c_4=\max((2c_1+1),(2c_2+1),(2c_3+1))$, then the above inequality becomes

$$\|\vec{y}_n(t)\|_0^2-\|\vec{y}_n(0)\|_0^2 \le \|\eta_1\|_Q^2+\|\eta_2\|_Q^2+\|\eta_3\|_Q^2+\|u_1\|_\Sigma^2+\|u_2\|_\Sigma^2+\|u_3\|_\Sigma^2+c_4\int_0^t\|\vec{y}_n\|_0^2dt+c_5\int_0^t\|\vec{y}_n\|_0^2dt.$$

Since $\|\vec{y}_n(0)\|_0^2\le b$, $\|\eta_l\|_Q\le m_l$, $\|u_l\|_\Sigma\le c_l$, $\forall l=1,2,3$, putting $c_6=c_4+c_5$, $m^*=b+m_1^2+m_2^2+m_3^2+c_1^2+c_2^2+c_3^2$,

the above inequality yields to

$$\|\vec{y}_n(t)\|_0^2\le m^*+c_6\int_0^t\|\vec{y}_n\|_0^2\,dt\,.$$

By employing the Gronwall-Bellman Lemma (GBL), we obtain

$$\|\vec{y}_n(t)\|_0^2\le m^*e^{c_6T}=c_7\,,\ \forall\, t\in[0,T],$$

which implies to $\|\vec{y}_n(t)\|_{L^\infty(I,L^2(\Omega))}\le c_7$.

But

$$\|\vec{y}_n(t)\|_Q^2=\int_0^T\|\vec{y}_n\|_0^2dt\le T\max_{t\in[0,T]}\|\vec{y}_n(t)\|_0^2\le Tc_6=c_8^2\Rightarrow\ \|\vec{y}_n(t)\|_Q\le c_8\,.$$

***The norm $\|\vec{y}_n(t)\|_{L^2(I,V)}$ is bounded*:** once again, by employing Lemma 1.2 in [16] for the 1^{st} term in the L.H.S. of (21), then by benefitting from the above result which is obtained from its R.H.S., and $\|\vec{y}_n(T)\|_0^2\ge 0$, inequality (21) with $t=T$, turn into

$$\|\vec{y}_n(T)\|_0^2+2\bar{\alpha}\int_0^T\|\vec{y}_n\|_1^2dt\le m^*+c_6\|\vec{y}_n\|_Q^2\Rightarrow$$

$$\int_0^T\|\vec{y}_n\|_1^2\,dt\le\frac{(m^*+c_6c_7)}{2\bar{\alpha}}=c_8^2$$

$$\Rightarrow\|\vec{y}_n\|_{L^2(I,V)}\le p_{11}$$

***The convergence of the solution*:**

Consider $\vec{V}$ has a sequence (seq.) of subspaces $\{\vec{V}_n\}_{n=1}^\infty$, for which $\forall\,\vec{v}=(v_1,v_2,v_3)\in\vec{V}$, there is a seq. $\{\vec{v}_n=(v_{1n},v_{2n},v_{3n})\}\in\vec{V}_n$, $\forall n$, and $\vec{v}_n\longrightarrow\vec{v}$ ST in $\vec{V}\Rightarrow\ \vec{v}_n\longrightarrow\vec{v}$ ST in $(L^2(\Omega))^3$.

Since for each n, with $\vec{V}_n\subset\vec{V}$, problem ((15) – (17)) has a unique TSVS $\vec{y}_n$, hence corresponding to the seq. $\{\vec{V}_n\}_{n=1}^\infty$, the seq. of approximation problems(app. Prs.) like ((15) – (17)) are obtained, and by letting $\vec{v}=\vec{v}_n=(v_{1n},v_{2n},v_{3n})$ for $n=1,2,3,\dots$, in these app. Prs. , they yield to

$$\langle y_{1nt},v_{1n}\rangle+a_1(t,y_{1n},v_{1n})+(b_1(t)y_{1n},v_{1n})_\Omega-((b_4(t)y_{2n},v_{1n})_\Omega-((b_5(t)y_{3n},v_{1n})_\Omega=(f_1(y_{1n}),v_{1n})_\Omega+(u_1,v_{1n})_\Gamma \quad (22a)$$

$$(y_{1n}^0,v_{1n})_\Omega=(y_1^0,v_{1n})_\Omega \quad (22b)$$

$$\langle y_{2nt},v_{2n}\rangle+a_2(t,y_{2n},v_{2n})+(b_2(t)y_{2n},v_{2n})_\Omega+\big((b_6(t)y_{3n},v_{2n}\big)_\Omega-\big((b_4(t)y_{1n},v_{2n}\big)_\Omega=(f_2(y_{2n}),v_{2n})_\Omega+(u_2,v_{2n})_\Gamma \quad (23a)$$

$$(y_{2n}^0,v_{2n})_\Omega=(y_2^0,v_{2n})_\Omega \quad (23b)$$

$$\langle y_{3nt},v_{3n}\rangle+a_3(t,y_{3n},v_{3n})+(b_3(t)y_{3n},v_{3n})_\Omega+\big((b_5(t)y_{1n},v_{3n}\big)_\Omega-\big((b_6(t)y_{2n},v_{3n}\big)_\Omega=(f_3(y_{3n}),v_{3n})_\Omega+(u_3,v_{3n})_\Gamma \quad (24a)$$

$$(y_{3n}^0,v_{3n})_\Omega=(y_3^0,v_{3n})_\Omega \quad (24b)$$

which has a sequence of solutions $\{\vec{y}_n\}_{n=1}^\infty$. From the previous steeps, we got that $\|\vec{y}_n\|_{L^2(Q)}$ and $\|\vec{y}_n\|_{L^2(I,V)}$ are bounded, then by Alaoglu's theorem (Alth.), there is a subsequence of $\{\vec{y}_n\}_{n\in N}$, say again $\{\vec{y}_n\}_{n\in N}$ such that $\vec{y}_n\longrightarrow\vec{y}$ weakly(WK) in $\big(L^2(Q)\big)^3$ and in $\big(L^2(I,V)\big)^3$.

Then through Assumption (A-i), and the bounded norms results from the above steps, once get that $\vec{y}_n\longrightarrow\ \vec{y}$ ST in $\big(L^2(Q)\big)^3$.

Now, consider the WFO ((22) -(24)), take any arbitrary $\vec{v}\in\vec{V}$, then there is a sequence $\{\vec{v}_n\}$, $\vec{v}_n\in\vec{V}_n$, $\forall n$, s.t. $\vec{v}_n\longrightarrow\vec{v}$ ST in $\vec{V}$ (which gives $\vec{v}_n\longrightarrow\vec{v}$ ST in $(L^2(\Omega))^3$). M.B.S of ((22a)- (24a)), by $\varphi_l(t)\in C^1[0,T]$ resp., with $\varphi_l(T)=0$, $\varphi_l(0)\ne 0\ \forall l=1,2,3$, I.B.S. w.r.t. t from 0 to T, and then integration by parts (IBP) the 1^{st} term in the L.H.S. of each equation, to obtain

$$-\int_0^T(y_{1n},v_{1n})\varphi_1'(t)dt+\int_0^T[a_1(t,y_{1n},v_{1n})+(b_1(t)y_{1n}-b_4(t)y_{2n}-b_5(t)y_{3n},v_{1n})_\Omega]\varphi_1dt=\int_0^T(f_1(y_{1n}),v_{1n})_\Omega\varphi_1(t)dt+\int_0^T(u_1,v_1)_\Gamma\varphi_1(t)dt+(y_{1n}^0,v_{1n})_\Omega\varphi_1(0) \quad (25)$$

$$-\int_0^T(y_{2n},v_{2n})\varphi_2'(t)dt+\int_0^T[a_2(t,y_{2n},v_{2n})+(b_2(t)y_{2n}+b_6(t)y_{3n}-b_4(t)y_{1n},v_{2n})_\Omega]\varphi_2dt=\int_0^T(f_2(y_{2n}),v_{2n})_\Omega\varphi_2(t)dt+\int_0^T(u_2,v_2)_\Gamma\varphi_2(t)dt+(y_{2n}^0,v_{2n})_\Omega\varphi_2(0) \quad (26)$$

$$-\int_0^T(y_{3n},v_{3n})\varphi_3'(t)dt+\int_0^T[a_3(t,y_{3n},v_{3n})+(b_3(t)y_{3n}+b_5(t)y_{1n}-b_6(t)y_{2n},v_{3n})_\Omega]\varphi_3dt=$$

ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 34, Issue 1, 2023 DOI: http://doi.org/10.23851/mjs.v34i1.1241

$\int_0^T (f_3\,(y_{3n}), v_{3n})_\Omega \varphi_3(t)dt +$
$\int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt + (y^0_{3n}, v_{3n})_\Omega \varphi_3(0)$ (27)

since for $l = 1,2,3$ $y_{ln} \longrightarrow y_l$ WK in $L^2(Q)$, $y^0_{ln} \longrightarrow y^0_l$ ST in $L^2(\Omega)$, and since

$$\left.\begin{matrix} v_{ln} \longrightarrow v_l \text{ ST in } L^2(\Omega) \\ v_{ln} \longrightarrow v_l \text{ ST in } V \end{matrix}\right\} \Rightarrow$$

$$\begin{cases} v_{ln}\varphi_l' \longrightarrow v_l\varphi_l' \text{ ST in } L^2(Q) \\ v_{ln}\varphi_l \longrightarrow v_l\varphi_l \text{ ST in } L^2(I,V) \end{cases}$$

Then the following converges are concluded

$\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)dt + \int_0^T [a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n} - b_4(t)y_{2n} - b_5(t)y_{3n}, v_{1n})_\Omega]\varphi_1(t)dt \longrightarrow$
$\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1 - b_4(t)y_2 - b_5(t)y_3, v_1)_\Omega]\varphi_1(t)dt$ (28a)

$(y^0_{1n}, v_{1n})_\Omega \varphi_1(0) \longrightarrow (y^0_1, v_1)_\Omega \varphi_1(0)$ (28b)

$\int_0^T (y_{2n}, v_{2n})\varphi_2'(t)dt + \int_0^T [a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n} + b_6(t)y_{3n} + b_4(t)y_{1n}, v_{2n})_\Omega]\varphi_2(t)dt \longrightarrow$
$\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [a_2(t, y_2, v_2) + (b_2(t)y_2 + b_6(t)y_3 + b_4(t)y_1, v_2)_\Omega]\varphi_2(t)dt$ (29a)

$(y^0_{2n}, v_{2n})_\Omega \varphi_2(0) \longrightarrow (y^0_2, v_2)_\Omega \varphi_2(0)$ (29b)

$\int_0^T (y_{3n}, v_{3n})\varphi_3'(t)dt + \int_0^T [a_3(t, y_{3n}, v_{3n}) + (b_3(t)y_{3n} + b_5(t)y_{1n} + b_6(t)y_{2n}, v_{3n})_\Omega]\varphi_3(t)dt \longrightarrow$
$\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [a_3(t, y_3, v_3) + (b_3(t)y_3 + b_5(t)y_1 + b_6(t)y_2, v_3)_\Omega]\varphi_3(t)dt$ (30a)

$(y^0_{3n}, v_n)_\Omega \varphi_3(0) \longrightarrow (y^0_3, v_3)_\Omega \varphi_3(0)$ (30b)

On the other hand, let ($\forall l = 1,2,3$,) $w_{ln} = v_{ln}\varphi_l$ and $w_l = v_l\varphi_l$ then $w_{ln} \longrightarrow w_l$ ST in $L^2(Q)$, from employing the Assumption (A-i), and proposition 3.1 in [17], the int. $\int_0^T (f_l(y_{ln}), w_{ln})_\Omega dt$ is continuous w.r.t. (y_{ln}, w_{ln}), since $\vec{y}_n \longrightarrow \vec{y}$ ST in $(L^2(Q))^3$ a, then

$\int_0^T (f_l(y_{ln}), v_{ln})_\Omega \varphi_l(t)dt \longrightarrow$
$\int_0^T (f_l(y_l), v_l)_\Omega \varphi_l(t)dt,\ \forall l = 1,2,3$

From this result and from ((28)-(30)), ((25) - (27)) become

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1 - b_4(t)y_2 - b_5(t)y_3, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(y_1), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y^0_1, v_1)_\Omega \varphi_1(0)$ (31)

$-\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [a_2(t, y_2, v_2)_\Omega + (b_2(t)y_2 + b_6(t)y_3 + b_4(t)y_1, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(y_2), v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt + (y^0_2, v_2)_\Omega \varphi_2(0)$ (32)

$-\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [a_3(t, y_3, v_3)_\Omega + (b_3(t)y_3 + b_5(t)y_1 - b_6(t)y_2, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3(y_3), v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt + (y^0_3, v_3)_\Omega \varphi_3(0)$ (33)

Now, the following two cases will be considered:

Case1: choose $\varphi_l \in D[0,T]$, i.e., $\varphi_l(0) = \varphi_l(T) = 0$, $\forall l = 1,2,3$, utilizing in ((30) - (32)), then employing I.B.P for the 1^{st} terms in the L.H.S. of the obtained equations, yield

$\int_0^T (y_{1t}, v_1)\varphi_1(t)dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_4(t)y_2, v_1)_\Omega - (b_5(t)y_3, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(y_1), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt$ (34)

$\int_0^T (y_{2t}, v_2)\varphi_2(t)dt + \int_0^T [a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_6(t)y_3, v_2)_\Omega + (b_4(t)y_1, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(y_2), v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt$ (35)

$\int_0^T (y_{3t}, v_3)\varphi_3(t)dt + \int_0^T [a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_5(t)y_1, v_3)_\Omega - (b_6(t)y_2, v_3)_\Omega]\varphi_2(t)dt = \int_0^T (f_3(y_3), v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt$ (36)

i.e., $\vec{y}$ is the TSVS of the W.F. ((11a) – (13a)).

Case 2: choose $\varphi_l \in C^1[0,T]$, $\forall l = 1,2,3$ such that $\varphi_l(T) = 0$ & $\varphi_l(0) \neq 0$, Using I.B.P for 1^{st} term in the L.H.S. of ((34) - (36)), one gets

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_4(t)y_2, v_1)_\Omega - (b_5(t)y_3, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(y_1), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1(0), v_1)_\Omega \varphi_1(0)$ (37)

$-\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [a_2\,(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_6(t)y_3, v_2)_\Omega + (b_4(t)y_1, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(y_2), v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt + (y_2(0), v_2)_\Omega \varphi_2(0)$ (38)

$-\int_0^T (y_3, v_3)\varphi_3' + \int_0^T [a_3\,(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_5(t)y_1, v_3)_\Omega - (b_6(t)y_2, v_3)_\Omega]\varphi_2(t)dt =$
$\int_0^T (f_3(y_3), v_3)_\Omega \varphi_3(t)dt +$

$\int_0^T (u_3, v_3)_\Gamma\, \varphi_3(t)\, dt + (y_3(0), v_3)_\Omega \varphi_3(0)$ (39)

The following results are obtained from subtracting ((31)-(33)) from ((37)- (29))

$(y_l^0, v_l)_\Omega \varphi_l(0) = (y_l(0), v_l)_\Omega \varphi_l(0) \Rightarrow$
$(y_l^0, v_l)_\Omega = (y_l(0), v_l)_\Omega$

i.e. the initial condition (11b)-(13b) are conclude.

The strong convergence for $\vec{y}_n$ in $L^2(I,V)$:

by substituting $v_l = y_l$ and $v_l = y_{ln}$ $\forall l = 1,2,3$. in ((11a) and(15a))-((13a) and (17a)) resp, integrating the resulting equations from $t = 0$ to $t = T$, finally collecting together the equations which are obtained from ((11a)-(13a)), and those obtained from ((16a) -(18a)), with utilizing Assumption (A-iii), we get

$$\int_0^T \langle \vec{y}_t, \vec{y}\rangle\, dt + \int_0^T c(t, \vec{y}, \vec{y}) dt = \int_0^T [(f_1(y_1), y_1)_\Omega + (f_2(y_2), y_2)_\Omega + (f_3(y_3), y_3)_\Omega] dt + \int_0^T (u_1, y_1)_\Gamma dt + \int_0^T (u_2, y_2)_\Gamma dt + \int_0^T (u_3, y_3)_\Gamma dt \quad (40a)$$

$$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n\rangle\, dt + \int_0^T c(t, \vec{y}_n, \vec{y}_n) dt = \int_0^T [(f_1(y_{1n}), y_{1n}) + (f_2(y_{2n}), y_{2n}) + (f_3(y_{3n}), y_{3n})] dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (u_2, y_{2n})_\Gamma dt + \int_0^T (u_3, y_{3n})_\Gamma dt \quad (40b)$$

Employing Lemma 1.2 in [16] for the 1st term in the L.H.S. of (40a), to conclude that

$$\tfrac{1}{2}\|\vec{y}(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}(0)\|_0^2 + \int_0^T c(t, \vec{y}, \vec{y}) dt = \int_0^T [(f_1(y_1), y_1)_\Omega + (f_2(y_2), y_2)_\Omega + (f_3(y_3), y_3)_\Omega] dt + \int_0^T (u_1, y_1)_\Gamma dt + \int_0^T (u_2, y_2)_\Gamma dt + \int_0^T (u_3, y_3)_\Gamma dt \quad (41a)$$

$$\tfrac{1}{2}\|\vec{y}_n(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T c(t, \vec{y}_n, \vec{y}_n) dt = \int_0^T [(f_1(y_{1n}), y_{1n})_\Omega + (f_2(y_{2n}), y_{2n})_\Omega + (f_3(y_{3n}), y_{3n})_\Omega] dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (u_2, y_{2n})_\Gamma dt \int_0^T (u_3, y_{3n})_\Gamma dt \quad (41b)$$

Since:

$$\tfrac{1}{2}\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 + \int_0^T c(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})\, dt = A - B - C \quad (42)$$

where

$A = \tfrac{1}{2}\|\vec{y}_n(T)\|_0^2 - \tfrac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T c\big(t, \vec{y}_n(T), \vec{y}_n(T)\big) dt,$

$B = \tfrac{1}{2}(\vec{y}_n(T), \vec{y}(T))_\Omega - \tfrac{1}{2}(\vec{y}_n(0), \vec{y}(0))_\Omega + \int_0^T c\big(t, \vec{y}_n(T), \vec{y}(T)\big)\, dt,$

$C = \tfrac{1}{2}(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T))_\Omega - \tfrac{1}{2}(\vec{y}(0), \vec{y}_n(0) - y(0))_\Omega + \int_0^T c\big(t, \vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) dt$.

Since

$\vec{y}_n^0 = \vec{y}_n(0) \longrightarrow \vec{y}^0$ ST in $\big(L^2(\Omega)\big)^2$ (43a)

$\vec{y}_n(T) \longrightarrow \vec{y}(T)$ ST in $\big(L^2(\Omega)\big)^2$ (43b)

then

$(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0))_\Omega \longrightarrow 0$ & $(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T))_\Omega \longrightarrow 0$ (43c)

$\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 \longrightarrow 0$ & $\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 \longrightarrow 0$ (43d)

and since $\vec{y}_n \longrightarrow \vec{y}$ WK in $\big(L^2(I,V)\big)^2$, then

$\int_0^T c\big(t, \vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) dt \longrightarrow 0$ (43e)

From proposition (3.1) in [17], the int. $\int_0^T (f_l(y_{ln}), y_{ln})_\Omega\, dt$ is con. w.r.t. y_l , then

$$\int_0^T [(f_1(y_{1n}), y_{1n})_\Omega + (f_2(y_{2n}), y_{2n})_\Omega + (f_3(y_{3n}), y_{3n})_\Omega] dt \longrightarrow \int_0^T [(f_1(y_1), y_1)_\Omega + (f_2(y_2), y_2)_\Omega + (f_3(y_3), y_3)_\Omega] dt . \quad (43f)$$

From $y_{ln} \longrightarrow y_l$ ST in $L^2(Q)$, $\forall l = 1,2,3$.

Now, when $n \to \infty$ in both sides of (42), one has the following results:

1. The first two terms in the L.H.S. of (42) are vanished (from (43d))
2. From Eq.(43f)
   Eq. (A) = $\sum_{l=1}^3 \int_0^T [(f_l(y_{ln}), y_{ln})_\Omega + (u_l, y_{ln})_\Gamma]\, dt \longrightarrow \sum_{l=1}^3 \int_0^T [(f_l(y_l), y_l)_\Omega + (u_l, y_l)_\Gamma]\, dt$
3. Eq.(B) →L.H.S.of (41a) $= \sum_{l=1}^3 \int_0^T [(f_l(y_l), y_l)_\Omega + (u_l, y_l)_\Gamma]\, dt$
4. From (43c) and (43e) the three terms in (C) are vanished.

From the above steps, (42) gives that $\int_0^T c(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y}) dt \longrightarrow 0$,

which means that,

$\bar{\alpha} \int_0^T \|\vec{y}_n - \vec{y}\|_1^2\, dt \longrightarrow 0 \Rightarrow \vec{y}_n \longrightarrow \vec{y}$ St in $\big(L^2(I,V)\big)^3$

***Uniqueness of the solution*:**

Let $\vec{y}$, $\vec{\hat{y}}$ be two TSVSs of the WFO((11a)–(13a)), substituting each equation from the other and then setting $\vec{y} = \vec{y} - \vec{\hat{y}}$, one obtains,

$$\langle (y_1 - \hat{y}_1)_t, y_1 - \hat{y}_1\rangle + a_1(t, y_1 - \hat{y}_1, y_1 - \hat{y}_1) + (b_1(t)(y_1 - \hat{y}_1) - b_4(t)(y_2 - \hat{y}_2), y_1 - \hat{y}_1)_\Omega - (b_5(t)(y_3 - \hat{y}_3), y_1 - \hat{y}_1)_\Omega = (f_1(y_1) - f_1(\hat{y}_1), y_1 - \hat{y}_1)_\Omega \quad (44)$$

$$\langle (y_2 - \hat{y}_2)_t, y_2 - \hat{y}_2\rangle + a_2(t, y_2 - \hat{y}_2, y_2 - \hat{y}_2) + (b_2(t)(y_2 - \hat{y}_2) + b_6(t)(y_3 - \hat{y}_3), y_2 - \hat{y}_2)_\Omega + (b_4(t)(y_1 - \hat{y}_1), y_2 - \hat{y}_2)_\Omega = (f_2(y_2) - f_2(\hat{y}_2), y_2 - \hat{y}_2)_\Omega \quad (45)$$

ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 34, Issue 1, 2023 DOI: http://doi.org/10.23851/mjs.v34i1.1241

$\langle(y_3-\hat{y}_3)_t, y_3-\hat{y}_3\rangle + a_3(t, y_3-\hat{y}_3, y_3-\hat{y}_3) + (b_3(t)(y_3-\hat{y}_3) + b_5(y_1-\hat{y}_1), y_3-\hat{y}_3)_\Omega - (b_6(t)(y_2-\hat{y}_2), y_3-\hat{y}_3)_\Omega = (f_3(y_3) - f_3(\hat{y}_3), y_3-\hat{y}_3)_\Omega$ (46)

Collecting ((44)-(46)) together, utilizing Lemma1.2 in [16],

$\frac{1}{2}\frac{d}{dt}\|\vec{y}-\vec{\hat{y}}\|_0^2 + \bar{\alpha}\|\vec{y}-\vec{\hat{y}}\|_1^2 \le \sum_{l=1}^3 (f_l(y_l) - f_l(\hat{y}_l), y_l-\hat{y}_l)_\Omega$ (47)

Since the 2^{nd} term of the L.H.S. is positive, I.B.S w.r.t. t from 0 to t, and then utilizing Assumption (A-ii) on the R.H.S., it becomes

$\|(\vec{y}-\vec{\hat{y}})(t)\|_0^2 \le 2\ \sum_{l=1}^3 \int_0^t \int_\Omega L_l |y_l - \hat{y}_l|^2 dxdt \le \int_0^t 3L \|\vec{y}-\vec{\hat{y}}\|_0^2 dt,$

$L = \max\{L_1, L_2, L_3\}$

Utilizing the GBL $\forall t \in I$, it yields

$\|(\vec{y}-\vec{\hat{y}})(t)\|_0^2 \le 0\exp(\int_0^T 2Ldt) = 0$,

Again, I.B.S. of (46) w.r.t. t from 0 to T, utilizing using the given initial condition and the above result for the R.H.S., one has

$\int_0^T \frac{d}{dt}\|\vec{y}-\vec{\hat{y}}\|_0^2 dt + 2\bar{\alpha}\int_0^T \|\vec{y}-\vec{\hat{y}}\|_1^2 dt$

$\le L\int_0^T \|\vec{y}-\vec{\hat{y}}\|_0^2 dt$

$\Rightarrow 2\bar{\alpha}\int_0^T \|\vec{y}-\vec{\hat{y}}\|_1^2 dt \le L\int_0^T \|\vec{y}-\vec{\hat{y}}\|_0^2 dt$

$\le 0 \Rightarrow \|\vec{y}-\vec{\hat{y}}\|_{L^2(I,V)} = 0$

Existence of A CCBOCV:

To study the existence of a CCBOCV, the following theorem and lemma are important:

***Theoem 2*:**

(a) In addition to Assumptions (A), if $\vec{y}$, $\vec{y}+\overrightarrow{\Delta y}$ are the TSVS corresponding to the CCBCVs $\vec{u}$, $\vec{u}+\overrightarrow{\Delta u}$ in $(L^2(\Sigma))^3$, then

$\|\overrightarrow{\Delta y}\|_{L^\infty(I,L^2(\Omega))} \le K\|\overrightarrow{\Delta u}\|_\Sigma$, $\|\overrightarrow{\Delta y}\|_{L^2(Q)} \le K\|\overrightarrow{\Delta u}\|_\Sigma$, and $\|\overrightarrow{\Delta y}\|_{L^2(I,V)} \le K\|\overrightarrow{\Delta u}\|_\Sigma$.

(b) With Assumptions (A), the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ from$(L^2(\Sigma))^3$ into $(L^\infty(I, L^2(\Omega)))^3$, or in to $(L^2(I,V))^3$, or in to $(L^2(Q))^3$ is cont.

<u>Proof:</u>

(a) Let $\vec{u}, \vec{\hat{u}} \in (L^2(\Sigma))^3$, and let $\overrightarrow{\Delta u} = \vec{\hat{u}} - \vec{u}$, hence by Theorem (1), there are STVSs $\vec{y} = (y_1 = y_{u_1}, y_2 = y_{u_2}, y_3 = y_{u_3})$ and $\vec{\hat{y}} = (\hat{y}_1 = \hat{y}_{\hat{u}_1}, \hat{y}_2 = \hat{y}_{\hat{u}_2}, \hat{y}_3 = \hat{y}_{\hat{u}_3})$ of ((11)-(13)), i.e.

$\langle\hat{y}_{1t}, v_1\rangle + a_1(t, \hat{y}_1, v_1) + (b_1(t)\hat{y}_1, v_1)_\Omega - (b_4(t)\hat{y}_2, v_1)_\Omega - (b_6(t)\hat{y}_3, v_1)_\Omega = (f_1(x,t,\hat{y}_1), v_1)_\Omega + (\hat{u}_1, v_1)_\Gamma$ (48a)

$(\hat{y}_1(0), v_1)_\Omega = (y_1^0, v_1)_\Omega$ (48b)

$\langle\hat{y}_{2t}, v_2\rangle + a_2(t, \hat{y}_2, v_2) + (b_2(t)\hat{y}_2, v_2)_\Omega + (b_6(t)\hat{y}_3, v_2)_\Omega + (b_4(t)\hat{y}_1, v_1)_\Omega = (f_2(x,t,\hat{y}_2), v_2)_\Omega + (\hat{u}_2, v_2)_\Gamma$ (49a)

$(\hat{y}_2(0), v_2)_\Omega = (y_2^0, v_2)_\Omega$ (49b)

$\langle\hat{y}_{3t}, v_3\rangle + a_3(t, \hat{y}_3, v_3) + (b_3(t)\hat{y}_3, v_3)_\Omega + (b_5(t)\hat{y}_1, v_3)_\Omega - (b_6(t)\hat{y}_2, v_3)_\Omega = (f_3(x,t,\hat{y}_3), v_3)_\Omega + (\hat{u}_3, v_3)_\Gamma$ (50 a)

$(\hat{y}_3(0), v_3)_\Omega = (y_3^0, v_3)_\Omega$ (50b)

Subtracting ((11)-(12)) from ((48)-(50)) resp., letting $\Delta y_l = \hat{y}_l - y_l$, $\Delta u_l = \hat{u}_l - u_l$, for $l = 1,2,3.$ to get

$\langle\Delta y_{1t}, v_1\rangle + a_1(t, \Delta y_1, v_1) + (b_1(t)\Delta y_1, v_1)_\Omega - (b_4(t)\Delta y_2, v_1)_\Omega - (b_5(t)\Delta y_3, v_1)_\Omega = (f_1(y_1+\Delta y_1), v_1)_\Omega - (f_1(y_1), v_1)_\Omega + (\Delta u_1, v_1)_\Gamma$ (51a)

$(\Delta y_1(0), v_1)_\Omega = 0$ (51b)

$\langle\Delta y_{2t}, v_2\rangle + a_2(t, \Delta y_2, v_2) + (b_2(t)\Delta y_2, v_2)_\Omega + (b_6(t)\Delta y_3, v_2)_\Omega + (b_4(t)\Delta y_1, v_2)_\Omega = (f_2(y_2+\Delta y_2), v_2)_\Omega - (f_2(y_2), v_2)_\Omega + (\Delta u_2, v_2)_\Gamma$ (52a)

$(\Delta y_2(0), v_2)_\Omega = 0$ (52b)

$\langle\Delta y_{3t}, v_3\rangle + a_3(t, \Delta y_3, v_3) + (b_3(t)\Delta y_3, v_3)_\Omega + (b_5(t)\Delta y_1, v_3)_\Omega - (b_6(t)\Delta y_2, v_3)_\Omega = (f_3(y_3+\Delta y_3), v_3)_\Omega - (f_3(y_3), v_3)_\Omega + (\Delta u_3, v_3)_\Gamma$ (53a)

$(\Delta y_3(0), v_3)_\Omega = 0$ (53b)

By using $v_l = \Delta y_l$, $\forall l = 1,2,3$, in ((51a)-(53a)) resp., collecting the resulting equations, utilizing Lemma 2.1 in [16] for the 1^{st} term and Assumption (A-iii) for the second term in the L.H.S. of the equation, it yields to

$\frac{1}{2}\frac{d}{dt}\|\overrightarrow{\Delta y}\|_0^2 + \bar{\alpha}\|\overrightarrow{\Delta y}\|_1^2 \le |(f_1(y_1+\Delta y_1) - f_1(y_1), \Delta y_1)| + |(f_2(y_2+\Delta y_2) - (f_2(y_2), \Delta y_2)| + |(f_3(y_3+\Delta y_3) - f_3(y_3), \Delta y_3)| + |(\Delta u_1, \Delta y_1)_\Gamma| + |(\Delta u_2, \Delta y_2)_\Gamma| + |(\Delta u_3, \Delta y_3)_\Gamma|$ (54)

Since the 2^{nd} term in L.H.S. of (54) is positive, I.B.S w.r.t. t from 0 to t, using Assumption (A-ii) for the first three terms in its R.H.S. and then the CSIN for the last three terms in the same side, to get $\forall t \in [0,T]$:

$\|\overrightarrow{\Delta y}(t)\|_0^2 \le 2L_1\int_0^t \|\Delta y_1\|_0^2 dt + \int_0^T \|\Delta u_1\|_\Gamma^2 dt + \int_0^t \|\Delta y_1\|_\Gamma^2 dt + 2L_2\int_0^t \|\Delta y_2\|_0^2 dt + \int_0^T \|\Delta u_2\|_\Gamma^2 dt + \int_0^t \|\Delta y_2\|_\Gamma^2 dt + 2L_3\int_0^t \|\Delta y_3\|_0^2 dt + \int_0^T \|\Delta u_3\|_\Gamma^2 dt + \int_0^t \|\Delta y_3\|_\Gamma^2 dt$

Using the trace theorem for the last term in R.H.S., setting $c_2 = \max(2L_1, 2L_2, 2L_3)$, to get

$\|\overrightarrow{\Delta y}(t)\|_0^2 \le \|\overrightarrow{\Delta u}\|_\Sigma^2 + c_2 \int_0^t \|\overrightarrow{\Delta y}\|_0^2 dt +$
$c_3 \int_0^t \|\overrightarrow{\Delta y}\|_0^2 dt \le \|\overrightarrow{\Delta u}\|_\Sigma^2 + L_3 \int_0^t \|\overrightarrow{\Delta y}\|_0^2 dt,$

where $L_3 = c_2 + c_3$

By using GBL, to obtain $\forall t \in [0,T]$

$\|\overrightarrow{\Delta y}(t)\|_0^2 \le \|\overrightarrow{\Delta u}\|_\Sigma^2 e^{\int_0^T L_3 dt} = e^{L_3 T}\|\overrightarrow{\Delta u}\|_\Sigma^2 =$
$K^2\|\overrightarrow{\Delta u}\|_\Sigma^2 \Rightarrow \|\overrightarrow{\Delta y}(t)\|_0 \le K\|\overrightarrow{\Delta u}\|_\Sigma$,

Thus $\|\overrightarrow{\Delta y}\|_{L^\infty(I,L^2(\Omega))} \le K\|\overrightarrow{\Delta u}\|_\Sigma$

And, $\|\overrightarrow{\Delta y}\|_{L^2(Q)}^2 = \int_0^T \|\overrightarrow{\Delta y}(t)\|_0^2 dt \le$
$\max_{t\in[0,T]} \|\overrightarrow{\Delta y}(t)\|_0^2 \int_0^T dt \le TK^2\|\overrightarrow{\Delta u}\|_\Sigma^2$

Then $\|\overrightarrow{\Delta y}\|_{L^2(Q)} \le K\|\overrightarrow{\Delta u}\|_\Sigma$, where $K^2 = TK^2$, and K denotes to various constants.

Repeating the same manner that is utilized in the above steps for the R.H.S. of (54), with $t = T$, to get

$\int_0^T \frac{d}{dt}\|\overrightarrow{\Delta y}\|_0^2 + 2\bar{\alpha}\int_0^T \|\overrightarrow{\Delta y}\|_1^2 dt \le \|\overrightarrow{\Delta u}\|_\Sigma^2 +$
$L_3 \int_0^T \|\overrightarrow{\Delta y}\|_0^2 dt \Rightarrow \|\overrightarrow{\Delta y}(T)\|_0^2 + 2\bar{\alpha}\int_0^T \|\overrightarrow{\Delta y}\|_1^2 dt$
$\le \|\overrightarrow{\Delta u}\|_\Sigma^2 + L_3 \|\overrightarrow{\Delta y}\|_Q^2 \Rightarrow \|\overrightarrow{\Delta y}\|_{L^2(I,V)}^2 \le K^2\|\overrightarrow{\Delta u}\|_\Sigma^2$,

where $K^2 = \frac{(1+L_3K^2)}{2\bar{\alpha}} \Rightarrow$

$\|\overrightarrow{\Delta y}\|_{L^2(I,V)} \le K\|\overrightarrow{\Delta u}\|_\Sigma$, where K denote to various constants.

(**b**) Let $\overrightarrow{\Delta u} = \vec{\bar{u}} - \vec{u}$ and $\overrightarrow{\Delta y} = \vec{\bar{y}} - \vec{y}$ where $\vec{\bar{y}}$,$\vec{y}$ are the correspond STVSs to the CCBCVs $\vec{\bar{u}}$,$\vec{u}$ utilizing the first result in part (a) of this theorem, to conclude

$\|\vec{\bar{y}} - \vec{y}\|_{L^\infty(I,L^2(\Omega))} \le K\|\vec{\bar{u}} - \vec{u}\|_\Sigma$

Then $\vec{\bar{y}} \to \vec{y}$ in $\left(L^\infty(I, L^2(\Omega))\right)^3$ when $\vec{\bar{u}} \to \vec{u}$ in $\left(L^2(\Sigma)\right)^3$, hence $\vec{u} \longmapsto \vec{y}$ is Lip. Con. from $\left(L^2(\Sigma)\right)^3$ in to $\left(L^\infty(I, L^2(\Omega))\right)^3$,

The other two results in this part are obtained using the same manner.

Lemma 1:

With Assumptions (B), the CF. $G_l(\vec{u})$ is cont. on $(L^2(\Sigma))^3$, for each $k = 0,1,2,3$.

Proof:

From Assumptions (B)

$\|g_{li}(x,t,y_i)\| \le \gamma_{li}(x,t) + c_{li}\|y_i\|^2$,
$\|h_{li}(x,t,u_i)\| \le \delta_{li}(x,t) + d_{li}\|u_i\|^2$

Using proposition 3.1in [17], the ints $\int_Q g_{li}(x,t,y_i)\,dxdt$ and $\int_\Sigma h_{li}(x,t,u_i)\,d\sigma$ are conts on $L^2(Q)$ and on $L^2(\Sigma)$ resp. $\forall l = 1,2,3$, $\forall k = 0,1,2,3$, hence $G_l(\vec{u})$ is cont. on $(L^2(\Sigma))^3$.

Theorem 3:

Beside the Assumptions (A), and (B). If $\vec{U}$ is compact, $\overrightarrow{W}_A \ne \emptyset$, $G_0(\vec{u})$ is convex w.r.t. $\vec{u}$ for fixed $(x,t,\vec{y})$. Then there exists a CCBOCV.

Proof: From the assumptions on U_l $(l = 1,2,3)$, then $W_1 \times W_2 \times W_3$ is WK compact (WKC). Since $\overrightarrow{W}_A \ne \emptyset$, then there is $\vec{\bar{u}} \in \overrightarrow{W}_A$ and a minimizing seq. $\{\vec{u}_k \in \overrightarrow{W}\}$, $\forall k$, s.t.

$\lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\bar{u}}\in\overrightarrow{W}_A} G_0(\vec{\bar{u}})$

Since $\vec{u}_k \in \overrightarrow{W}$, $\forall k$ and $\overrightarrow{W}$ is WKC, then the seq. $\{\vec{u}_k\}$ has a seubseq. say again $\{\vec{u}_k\}$, s.t.

$\vec{u}_k \longrightarrow \vec{u} \in \overrightarrow{W}$ WK in $(L^2(\Sigma))^3$, and $\|\vec{u}_k\|_\Sigma \le c$, $\forall k$.

From Theorem (1), for each CCBCV $\vec{u}_k$, the WK of the TSEs has a unique TSVS $\vec{y}_k = \vec{y}_{\vec{u}_k}$ and $\|\vec{y}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ and $\|\vec{y}_k\|_{L^2(I,V)}$ are bounded, then by Alth. the seq. $\{\vec{y}_k\}$ has a subseq. say again $\{\vec{y}_k\}$ for which $\vec{y}_k \longrightarrow \vec{y}$ WK in $(L^\infty(I, L^2(\Omega)))^3$, $(L^2(Q))^3$ and $(L^2(I,V))^3$.

Also, from the same above indicated theorem we got that $\|\vec{y}_{tk}\|_{L^2(I,V^*)}$ is bounded and since $(L^2(I,V))^3 \subset (L^2(Q))^3 \cong ((L^2(Q))^*)^3 \subset (L^2(I,V^*))^3$. Hence by the by Alth. the seq. $\{\vec{y}_k\}$ has a subsequence say again $\{\vec{y}_k\}$ for which $\vec{y}_k \longrightarrow \vec{y}$ ST in $(L^2(Q))^3$.

Now, since for any k, y_{lk} is the TSVS corresponding to the CCBCV u_{lk}, $\forall l = 1,2,3$, therefore

$$\langle y_{1kt}, v_1\rangle + a_1(t, y_{1k}, v_1) + (b_1(t)y_{1k}, v_1)_\Omega - (b_4(t)y_{2k}, v_1)_\Omega - (b_5(t)y_{3k}, v_1)_\Omega = (f_1(x,t,y_{1k}), v_1)_\Omega + (u_{1k}, v_1)_\Gamma, \quad (55)$$

$$\langle y_{2kt}, v_2\rangle + a_2(t, y_{2k}, v_2) + (b_2(t)y_{2k}, v_2)_\Omega + (b_6(t)y_{3k}, v_2)_\Omega + (b_4(t)y_{1k}, v_2)_\Omega = (f_2(x,t,y_{2k}), v_2)_\Omega + (u_{2k}, v_2)_\Gamma \quad (56)$$

$$\langle y_{3kt}, v_3\rangle + a_3(t, y_{3k}, v_3) + (b_3(t)y_{3k}, v_3)_\Omega + (b_5(t)y_{1k}, v_3) - (b_6(t)y_{2k}, v_3)_{\Omega_\Omega} = (f_3(x,t,y_{3k}), v_3)_\Omega + (u_{3k}, v_3)_\Gamma \quad (57)$$

M.B.S of ((55)- (57)) by $\varphi_l(t) \in C^1[I]$ resp, with $\varphi_l(T) = 0$, $\forall l = 1,2,3$ and then I.B.S w.r.t. t from 0 to T, and using I.B.P for the 1^{st} terms in the L.H.S. of each equation, i.e.

$-\int_0^T (y_{1k}, v_1)\acute{\varphi}_1(t)dt + \int_0^T [a_1(t, y_{1k}v_1) + (b_1(t)y_{1k} - b_4(t)y_{2k} - b_5(t)y_{3k}, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1(x,t,y_{1k}), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_{1k}, v_1)_\Gamma \varphi_1(t)\, dt + (y_{1k}(0), v_1)_\Omega \varphi_1(0)$ (58)

$-\int_0^T (y_{2k}, v_2)\acute{\varphi}_2(t)dt + \int_0^T [a_2(t, y_{2k}v_2) + (b_2(t)y_{2k} + b_6(t)y_{3k} + b_4(t)y_{1k}, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2(x,t,y_{2k}), v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_{2k}, v_2)_\Gamma \varphi_2(t)\, dt + (y_{2k}(0), v_2)_\Omega \varphi_2(0)$ (59)

$-\int_0^T (y_{3k}, v_3)\acute{\varphi}_3(t)dt + \int_0^T [a_3(t, y_{3k}v_3) + (b_3(t)y_{3k} + b_5(t)y_{1k} - b_6(t)y_{2k}, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3(x,t,y_{3k}), v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_{3k}, v_3)_\Gamma \varphi_3(t)\, dt + (y_{3k}(0), v_3)_\Omega \varphi_3(0)$ (60)

Now, since $u_{lk} \longrightarrow u_l$ WK in $L^2(\Sigma)$, then

$\int_0^T (u_{lk}, v_l)_\Gamma \varphi_l(t)\, dt \longrightarrow \int_0^T (u_{lk}, v_l)_\Gamma \varphi_l(t)\, dt, \ \forall l = 1,2,3$ (61)

Now, to proof the above WFO((58)-(60)) convergence to the WFO((11)-(13)), we note that except the terms that include the controls(L.H.S. of (61)) in ((58)-(60)), all other terms are similar to their corresponding term in the ((25)-(27)) in the proof of Theorem (1), therefore to avoid the reputations for the steps, the same manner which were used to proof the convergence ((25)-(27)), can be used also here to get that $\vec{y}$ is a SVS of the WFO((11)-(13)).

From Lemma (1), the int. $\int_Q g_{0l}(x,t,y_{lk})\, dxdt$ $\int_\Sigma h_{0l}(x,t,u_{lk})d\sigma$ are cont. w.r.t. y_l and u_l resp., since $\vec{y}_k \longrightarrow \vec{y}$ ST in $(L^2(Q))^3$, hence

$\int_Q g_{0l}(x,t,y_{lk})\, dxdt \longrightarrow \int_Q g_{0l}(x,t,y_l)\, dxdt$, for each $l = 1,2,3.$ (62)

From the hypotheses on h_{0l} , $h_{0l}(x,t,u_l)$ is WK lower semi cont. w.r.t. u_l, for each $l = 1,2,3$ then with using (62), to obtain

$\int_Q g_{0l}(x,t,y_l)\, dxdt + \int_\Sigma h_{0l}(x,t,u_l)\, d\sigma \leq \lim_{k\to\infty} \inf \int_\Sigma h_{0l}(x,t,u_{lk})d\sigma + \int_Q g_{0l}(x,t,y_l)\, dxdt$

$= \lim_{k\to\infty} \inf(\int_\Sigma (h_{0l}(x,t,u_{lk})d\sigma + \int_Q g_{0l}(x,t,y_{lk})\, dxdt)$

$+ \lim_{k\to\infty} \inf \int_Q (g_{0l}(x,t,y_l) - g_{0l}(x,t,y_{lk})\, dxdt$

$= \lim_{k\to\infty} \inf (\int_\Sigma h_{0l}(x,t,u_{lk})\, d\sigma + \int_Q g_{0l}(x,t,y_{lk})\, dxdt$

$\Rightarrow G_o(\vec{u}) \leq \lim_{k\to\infty} \inf G_0(\vec{u}_k)$, hence

$G_0(\vec{u}) \leq \lim_{k\to\infty} \inf G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\vec{u}}\in \overrightarrow{W}_A} G_0(\vec{\vec{u}})$

$\Rightarrow G_0(\vec{u}) = \min_{\vec{\vec{u}}\in \overrightarrow{W}_A} G_0(\vec{\vec{u}})$

$\Rightarrow \vec{u}$ is a CCBOCV.

CONCLUSIONS

In this paper, the existence theorem for a unique STVS of the WFO for the NLTPBVP is stated and demonstrated via the MGa , and the first compactness theorem under suitable assumptions and with given CCBTCV. Furthermore, the continuity operator between the STVS of the WFO for the NLTPBVP and the corresponding CCBTCV is stated and demonstrated. The continuity of the LIP. Operator between the STVS of the WFO for the QNLPBVP and the corresponding CCBTCV is proved. The existence of a CCBOTCV is stated and demonstrated under suitable conditions.

Disclosure and conflict of interest: The authors declare that they have no conflicts of interest.

REFERENCES

[1] M.G. Cojocaru, A.S. Jaber, " Optimal Control of a Vaccinating Game Toward Increasing Overall Coverage,," J. appl. math. phys., Vol.6, pp. 754-769, 2018. https://doi.org/10.4236/jamp.2018.64067

[2] E. Staffetti E, X. Li,Y. Matsuno, M. Soler, "Optimal Control Techniques in Aircraft Guidance and Control,," Int. J. Aerosp. Eng. 2 pages, 2019. https://doi.org/10.1155/2019/3026083

[3] I. Syahrini, R. Masabar, A. Aliasuddin, S. Munzir, Y. Hazim, "The Application of Optimal Control Through Fiscal Policy on Indonesian Economy,," J. Asian Finance Econ. Bus. Vol.8, No. 3, pp. 0741-0750, 2021.

[4] G. Rigatos, M. Abbaszadeh, "Nonlinear Optimal Control for Multi-DOF Robotic Manipulators with Flexible Joints,," Optim. Control Appl. Methods, Vol.42, no. 6,pp. 1708-1733, 2021. https://doi.org/10.1002/oca.2756

[5] D. Derome D, H. Razali, A. Fazlizan, A. Jedi, K.P. Roberts. Determination of Optimal Time -Average Wind Speed Data in the Southern Part of Malaysia. Baghdad Sci. J. Vol.19, no.5, pp.1111-1122, 2022.

[6] P. Lin, W.Wang, "Optimal Control Problems for Some Ordinary Differential Equations with Behavior of Blowup or Quenching,," Math. Control Relat.

Fields.Vol.8, no. 4, pp. 809-828, 2018. https://doi.org/10.3934/mcrf.2018036

[7] A. Manzoni, A. Quarteroni, S. Salsa, Optimal Control of Partial Differential Equations: Analysis, Approximation, and Applications (Applied Mathematical Sciences, 207). 1st edition. New York: Spriger; 2021, p. 515. https://doi.org/10.1007/978-3-030-77226-0_1

[8] Y. Wang, X. Luo, S. Li, "Optimal Control Method of Parabolic Partial Differential Equations and Its Application to Heat Transfer Model in Continuous Cast Secondary Cooling Zone," Adv.Math. Phys, Vol. 2015, Article ID 585967, 10 pages, 2015. https://doi.org/10.1155/2015/585967

[9] Sh. Du · Z. Cai, Adaptive Finite Element Method for Dirichlet Boundary Control of Elliptic Partial Differential Equations," J. Sci. Comput., vol.89, no36, pp.1-25, 2021. https://doi.org/10.1007/s10915-021-01644-3

[10] I. Aksikas, J. J. Winkin, and D. Dochain, "Optimal LQ-feedback control for a Class of First-Order Hyperbolic Distributed Parameter Systems," ESAIM-Control, Optim. and Calc. Var., Vol. 14, no. 04, pp. 897-908, 2008. https://doi.org/10.1051/cocv:2008015

[11] J. A. Ali Al-Hawasy, A. A. H. Naeif," The Continuous Classical Boundary Optimal Control of a Couple Nonlinear Parabolic Partial Differential Equations," Special Issus: 1st Scientific International Conference, College of Science, Al-Nahrain University, Part I, pp.123-136, 21-22/11/2017. https://doi.org/10.22401/ANJS.00.1.17

[12] J. A. Ali Al-Hawasy, S. J. M. Al-Qaisi," The Continuous Classical Boundary Optimal Control of a Couple Nonlinear Elliptic Partial Differential Equations with State Constraints," *Al-Mustansiriyah Journal of Science*. Vol. 30, Issue 1, pp.143-151, 2019. https://doi.org/10.23851/mjs.v30i1.464

[13] J. A. Ali Al-Hawasy, "The Continuous Classical Boundary Optimal Control of Couple Nonlinear Hyperbolic Boundary Value Problem with Equality and Inequality Constraints," Baghdad Sci. J. Vol.16, no.4, Pp.1064-1074, Supplement 2019. https://doi.org/10.21123/bsj.2019.16.4(Suppl.).1064

[14] J. A. Ali Al-Hawasy, N. A. Th. Al-Ajeeli,"The Continuous Classical Boundary Optimal Control of Triple Nonlinear Elliptic Partial Differential Equations with State Constraints," Iraqi Journal of Science, Vol. 62, No. 9, pp. 3020-3030, 2021. https://doi.org/10.24996/ijs.2021.62.9.17

[15] L. H Ali J. A. Al-Hawasy," Boundary Optimal Control for Triple Nonlinear Hyperbolic Boundary Value Problem with State Constraints," Iraqi Journal of Science, Vol 62, No 6, pp. 2009-2021, 2021. https://doi.org/10.24996/ijs.2021.62.6.27

[16] R. Temam, Navier-Stokes equations. Amsterdam-New York: North-Holand Publishing Company; 1977. p. 470.

[17] J. Al-Hawasy, "The Continuous Classical Optimal Control of a couple Nonlinear Hyperbolic Partial Differential Equations with Equality and Inequality Constraints. Iraqi J. Sci.,Vol. 57, No. 2c, pp. 1528-1538, 2016.

How to Cite

J. A. A. Al-hawasy and Y. H. Rashid, “Classical Continuous Boundary Optimal Control Vector Problem for Triple Nonlinear Parabolic System”, *Al-Mustansiriyah Journal of Science*, vol. 34, no. 1, pp. 77–86, Mar. 2023.