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Characterization of P-Semi Homogenous System of Difference Equations

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ABSTRACT

The primary aim of this paper is to define new concepts of systems of difference equations. For A homogenous system $x(n+1) = Ax(n)$ of difference equations we define P-semi homogenous of order m, and adjoint. This definition is generalized to (2×2)-semi-homogeneous system of difference equations of order m, where m is a positive integer number. We study all special cases of this definition and give some examples. Also, we give some characterizations of those definitions, the necessary and sufficient conditions for a homogenous system of difference Equations to be P-semi homogenous of order one or greater than one is given in theorems, as well as some examples and theorems.

KEYWORDS: Difference equation; homogenous system; semi-homogenous; P-semi-homogenous.

الخلاصة

الهدف الأساسي من هذا البحث هو اعطاء مفاهيم جديدة لأنظمة معادلات الفروق للنظام المتجانس x(n+1)=Ax(n) لمعادلات الفروق تم تعريف النظام الشبه المتجانسة من النمط P من الرتبة m ، والنظام المصاحب. تم اعمام التعريف اعلاه على نظام شبه متجانس من معادلات الفروق من الرتبة m ، ذو البعد (2 × 2) حيث m هو عدد صحيح موجب. وتم دراسة جميع الحالات الخاصة لهذا التعريف وتم اعطاء بعض الأمثلة. إضافة لذلك ، تم اعطاء بعض التوصيفات لتلك التعريفات ، الشروط الضرورية والكافية لنظام شبه المتجانسة من النمط P من الرتبة واحد أو أكبر من واحد ، وكذلك بعض الأمثلة والنظريات.

INTRODUCTION

Difference equations have been in existence since ancient times. Two types of functions are basic. First, the functions where the variable x can take every possible value in a given interval. The general form of the difference equation is given by [1-4].

$$a(n+t) = f(a(n)) \qquad (1)$$

where n, t are integer numbers.

In 2016, Al-Asadi [5] studied the properties of difference systems in his paper “properties of difference equations).” in Russian language. In 2021, Abed and Al-Asadi [6] introduced the concept of a self-semi homogenous system of difference equations, which they defined as follows: A homogenous system of the difference equations is called *self-semi homogenous* if there exists a non-zero, nonidentity real matrix M such that the following equation is held

$$f(Ma(n)) = M^m f(a(n)) \qquad (2)$$

where n, m are integer numbers.

This paper introduces new the definitions of a homogeneous system that are generalized semi-homogenous systems which is defined as in follows: A homogenous system of difference equations is called *generalized semi homogenous* if there exists a non-zero, real matrix M such that the following equation is hold

$$F(Mx(n)) = P^k M^m F(x(n)), \qquad (3)$$

where P, k, and m are integer numbers.

There are some special cases discussed in this work as well as the general case, and is given some examples and characteristics for definitions and is proved some theorems, which can be summarized as follows:

1- If $P = 1$, then this system is called semi-homogenous of order m [5].
2- If $P \neq 1$, $k = 1$, then this system is called P-semi-homogenous of order m.

3- If $P \neq 1$, $k \neq 1$, then this system is called P^k-semi-homogenous of order m.

4- If $k = m$, then equation (2) now becomes as follows:

$$F\big(MX(n)\big) = (PM)^k F\big(X(n)\big)$$

and it is called P^k*-semi-homogenous* of order k.

P-semi-homogenous of order m

This section consists of studying P-semi-homogenous of order m, once $m = 1$ and another one $m > 1$.

Consider the system

$$F(X(n)) = BX(n) \qquad (4)$$

Where $n \in \mathbb{Z}\,(integer\ numbers), F = \begin{pmatrix} f \\ g \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $F(X) = \begin{pmatrix} f \\ g \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix} = \begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, that is $F(X)=\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, then we define P-semi homogenous as follows:

Definition 2.1.

System (4) is called P-semi homogenous of order m if there exists a non-zero matrix A and integer number m such that the following equation on stopping sign.

$$F\,(A(c)x(n)) = P(A(c))^m F(x(n)) \qquad (5)$$

where p is an integer number and c is a real number.

Example 2.2. Consider the system $F(x) = Bx$ where $B = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, then there is a matrix $A = \begin{bmatrix} \frac{3}{7} & \frac{-1}{7} \\ \frac{-1}{6} & \frac{1}{3} \end{bmatrix}$ with $P = 3$. Therefore, this system is p-semi-homogenous of order one.

The following theorem shows a matrix A exists in definition 2.1 and gives the general formula for such matrix.

$b_{11}a_{11} = Pa_{11}b_{11} + Pa_{12}b_{21}$ that is $1(\frac{3}{7}) = 3\left(\frac{3}{7}\right)1 + 3\left(\frac{-1}{7}\right)2 = \frac{3}{7}$

$b_{11}a_{12} + b_{12} = Pa_{11}b_{12} + Pa_{12}b_{22}$ that is $1\left(\frac{-1}{7}\right) + 4 = 3\left(\frac{3}{7}\right)4 + 3\left(\frac{-1}{7}\right)3 = \frac{27}{7}$

$b_{21} + b_{22}a_{21} = Pa_{21}b_{11} + Pa_{22}b_{21}$ that is $2 + 3\left(\frac{-1}{6}\right) = 3\left(\frac{-1}{6}\right)1 + 3\left(\frac{1}{3}\right)2 = \frac{3}{2}$

$b_{22}a_{22} = Pa_{21}b_{12} + Pa_{22}b_{22}$ that is $3\left(\frac{1}{3}\right) = 3\left(\frac{-1}{6}\right)4 + 3\left(\frac{1}{3}\right)3 = 1$

Theorem 2.3.

The necessary and sufficient condition for a homogenous system of difference Equations (4) to be P-semi homogenous of order one is that A equal to $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where

$$A_{11} = \frac{Pa_{12}b_{21}}{b_{11} - Pb_{11}}$$

$$A_{12} = \frac{Pb_{12}b_{11} - b_{12}b_{11}}{b_{11}^2 - Pb_{11}^2 - P^2b_{21}b_{12} - Pb_{11}b_{22} + P^2b_{11}b_{22}}$$

$$A_{21} = \frac{Pb_{21}b_{22} - b_{21}b_{22}}{b_{22}^2 - Pb_{22}^2 - P^2b_{21}b_{12} - Pb_{11}b_{22} + P^2b_{11}b_{22}}$$

$$A_{22} = \frac{Pa_{21}b_{12}}{b_{22} - Pb_{22}}$$

Proof:

Necessary condition:

Since F is homogenous of degree one, then there exists a non-zero matrix (say A) such that Equation (5) is hold with $m = 1$. that is,

$$F\,(A(c)x(n)) = PA(c)F(x(n))$$

$$F\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) = P\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \end{pmatrix}$$

$$\begin{pmatrix} f(a_{11}x + a_{12}y) \\ g(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} Pa_{11} & Pa_{12} \\ Pa_{21} & Pa_{22} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \end{pmatrix}$$

$$\begin{pmatrix} b_{11}\,(a_{11}x + a_{12}y\,) + b_{12}y \\ b_{21}x + b_{22}\,(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} Pa_{11}f(x) + Pa_{12}g(y) \\ Pa_{21}f\,(x) + Pa_{22}g\,(y) \end{pmatrix}$$

$$\begin{pmatrix} b_{11}a_{11}x + b_{11}a_{12}y + b_{12}y \\ b_{21}x + b_{22}a_{21}x + b_{22}a_{22}y \end{pmatrix} = \begin{pmatrix} Pa_{11}(b_{11}x + b_{12}y) + Pa_{12}(b_{21}x + b_{22}y) \\ Pa_{21}(b_{11}x + b_{12}y) + Pa_{22}(b_{21}x + b_{22}y) \end{pmatrix}$$

$$b_{11}a_{11} = Pa_{11}b_{11} + Pa_{12}b_{21}$$
$$b_{11}a_{12} + b_{12} = Pa_{11}b_{12} + Pa_{12}b_{22}$$
$$b_{21} + b_{22}a_{21} = Pa_{21}b_{11} + Pa_{22}b_{21}$$
$$b_{22}a_{22} = Pa_{21}b_{12} + Pa_{22}b_{22}.$$

It is easy to show the matrix A equal to $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

Sufficient condition:

Suppose that there is a matrix A equal to $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

To show that the system (4) is P-semi-homogeneous that is

$$F\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}\right) = P\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} f(x) \\ g(y)\end{pmatrix} \quad (6)$$

By substituting the value of the matrix A in (6), we have

$$F\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\begin{pmatrix} x \\ y\end{pmatrix}\right) = P\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\begin{pmatrix} f(x) \\ g(y)\end{pmatrix}$$

Therefore,

$$b_{11}\frac{Pa_{12}b_{21}}{b_{11}(1-P)} = Pb_{11}\frac{Pa_{12}b_{21}}{b_{11}(1-P)} + Pa_{12}b_{21}$$

$$b_{11}\frac{Pa_{12}b_{21}(1-P)}{b_{11}(1-P)} = Pb_{11}\frac{Pa_{12}b_{21}(1-P)}{b_{11}(1-P)} + Pa_{12}b_{21}(1-P)$$

$$Pa_{12}b_{21} = P^2a_{12}b_{21} + Pa_{12}b_{21}(1-P)$$

$$Pa_{12}b_{21} = P^2a_{12}b_{21} + Pa_{12}b_{21} - P^2a_{12}b_{21}$$

$$Pa_{12}b_{21} = Pa_{12}b_{21}.$$

Also,

$$a_{12}b_{11} + b_{12} = \frac{P^2a_{12}b_{12}b_{21}}{b_{11}-Pb_{11}} + Pa_{12}b_{22}.$$

Therefore,

$$b_{11} + \frac{b_{12}}{a_{12}} = \frac{P^2b_{12}b_{21}}{b_{11}-Pb_{11}} + Pb_{22}$$

$$b_{11} + \frac{b_{12}}{\dfrac{-b_{12}(b_{11}-Pb_{11})}{b_{11}^2 - Pb_{11}^2 - P^2b_{21}b_{12} - Pb_{11}b_{22} + P^2b_{11}b_{22}}} = \frac{P^2b_{12}b_{21}}{b_{11}-Pb_{11}} + Pb_{22}.$$

That is,

$$b_{11}^2 - Pb_{11}^2 - b_{11}^2 + Pb_{11}^2 + P^2b_{21}b_{12} + Pb_{11}b_{22} - P^2b_{11}b_{22} = P^2b_{21}b_{12} + Pb_{11}b_{22} - P^2b_{11}b_{22}$$

$$P^2b_{21}b_{12} + Pb_{11}b_{22} - P^2b_{11}b_{22} = P^2b_{21}b_{12} + Pb_{11}b_{22} - P^2b_{11}b_{22}.$$

The third item is

$$a_{21}b_{22} + b_{21} = \frac{P^2a_{21}b_{12}b_{21}}{b_{22}-Pb_{22}} + Pa_{21}b_{11},$$ hence

$$b_{22} + \frac{b_{21}}{a_{21}} = \frac{P^2b_{12}b_{21}}{b_{22}-Pb_{22}} + Pb_{11}$$

$$b_{22} + \frac{b_{21}}{\dfrac{-b_{21}(b_{22}-Pb_{22})}{b_{22}^2 - Pb_{22}^2 - P^2b_{21}b_{12} - Pb_{11}b_{22} + P^2b_{11}b_{22}}} = \frac{P^2b_{12}b_{21}}{b_{22}-Pb_{22}} + Pb_{11}$$

$$b_{22}^2 - Pb_{22}^2 - b_{22}^2 + Pb_{22}^2 + P^2b_{21}b_{12} + Pb_{11}b_{22} - P^2b_{11}b_{22} = P^2b_{21}b_{12} + Pb_{11}b_{22} - P^2b_{11}b_{22}$$

$$P^2b_{21}b_{12} + Pb_{11}b_{22} - P^2b_{11}b_{22} = P^2b_{21}b_{12} + Pb_{11}b_{22} - P^2b_{11}b_{22}$$

The fourth item is

$$b_{22}\frac{Pa_{21}b_{12}}{b_{22}(1-P)} = Pb_{22}\frac{Pa_{21}b_{12}}{b_{22}(1-P)} + Pa_{21}b_{12}$$

$$Pa_{21}b_{12} = P^2a_{21}b_{12} + Pa_{21}b_{12}(1-P)$$

$$Pa_{21}b_{12} = P^2a_{21}b_{12} + Pa_{21}b_{12} - P^2a_{21}b_{12}$$

$$Pa_{21}b_{12} = Pa_{21}b_{12}$$

That is, the left hand is equal to the right hand, and the system (4) is P-semi-homogeneous of order one.

Corollary 2.4. A homogenous system of difference Equations (4) is P-semi homogenous of order one if the following hold:

$$b_{11}a_{11} = Pa_{11}b_{11} + Pa_{12}b_{21}$$
$$b_{11}a_{12} + b_{12} = Pa_{11}b_{12} + Pa_{12}b_{22}$$
$$b_{21} + b_{22}a_{21} = Pa_{21}b_{11} + Pa_{22}b_{21}$$
$$b_{22}a_{22} = Pa_{21}b_{12} + Pa_{22}b_{22}$$

Proof: Direct from Theorem 2.3.

Example 2.5.

Consider the system $F(x) = Bx$ where $B = \begin{bmatrix} 1 & \frac{1}{6} \\ 1 & 1\end{bmatrix}$, then there is a matrix $A = \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1\end{bmatrix}$ with $P = 2$ .We want to show this system is P-semi homogenous of order one by applying Corollary 2.4.

$b_{11}a_{11} = Pa_{11}b_{11} + Pa_{12}b_{21}$, that is $1(-1) = 2(-1)1 + 2\left(\frac{1}{2}\right)1 = -1$

$b_{11}a_{12} + b_{12} = Pa_{11}b_{12} + Pa_{12}b_{22}$ that is $1(\frac{1}{2}) + \frac{1}{6} = 2(-1)(\frac{1}{6}) + 2\left(\frac{1}{2}\right)1 = \frac{2}{3}$

$b_{21} + b_{22}a_{21} = Pa_{21}b_{11} + Pa_{22}b_{21}$ that is $1 + 1(3) = 2(3)(1) + 2(-1)(1) = 4$

$b_{22}a_{22} = Pa_{21}b_{12} + Pa_{22}b_{22}$ that is $1(-1) = 2(3)(\frac{1}{6}) + 2(-1)(1) = -1.$

Corollary 2.6. The special case 3

"$P \neq 1, k \neq 1$, then this system is called P^k-semi-homogenous of order m" is held immediately from case 2 because P and k are integers so P^k is an integer, as an example, we provide the following.

Example 2.7 Considers the system $F(x) = Bx$ where $B = \begin{bmatrix} \frac{1}{3} & \frac{-1}{12} \\ 1 & \frac{-1}{3} \end{bmatrix}$, then by Theorem 2.3, there is a matrix $A = \begin{bmatrix} -1 & \frac{1}{4} \\ 3 & -1 \end{bmatrix}$ with $P^k = 2^2$.

It is clear that this system is 2^2-semi-homogenous or 4-semi-homogenous of order one.

***P*-semi-homogenous of order greater than one.**

This section studied P-semi-homogenous of order greater than one first we need to state the general formula for the power of a matrix and give an example showing this case existed

Theorem 3.1 [5]**.** If a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$A^n = \begin{pmatrix} \alpha_1^n + \frac{(\alpha_1^n-\alpha_2^n)(a_{11}-\alpha_1)}{(\alpha_1-\alpha_2)} & \frac{(\alpha_1^n-\alpha_2^n)(a_{12})}{(\alpha_1-\alpha_2)} \\ \frac{(\alpha_1^n-\alpha_2^n)(a_{21})}{(\alpha_1-\alpha_2)} & \alpha_1^n + \frac{(\alpha_1^n-\alpha_2^n)(a_{22}-\alpha_1)}{(\alpha_1-\alpha_2)} \end{pmatrix},$$

where $\alpha_1$and α_2 are eigenvalues of the matrix A, with $\alpha_1 \neq \alpha_2$.

Example 3.2. Considers the system $F(x) = Bx$ where

$B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, then there exists a matrix

$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$ and $f\big(Ax(n)\big) = PA^m f\big(x(n)\big)$, such that $P = 4,\ L = 3$, and $m = 3$; that is,

$$f\left(\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} x(n)\right) = 4\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}^3 f\big(x(n)\big).$$

Therefore,

$$f\left(\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} x(n)\right) = \begin{bmatrix} 688 & 684 \\ 684 & 688 \end{bmatrix} f\big(x(n)\big),$$

and it is clear to show this system is 4-semi-homogenous of order 3.

Theorem 3.3. The necessary and sufficient conditions for a homogenous system of difference Equations (4) to be P-semi-homogenous of order greater than one is the matrix A equal to

$A = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where:

$$B_{11} = \frac{PA_{12}b_{21}}{a_{11} - PA_{11}}$$

$$B_{12} = \frac{a_{12}a_{22} - PA_{22}a_{12}}{a_{22} - PA_{22} - PA_{11}a_{22} + P^2A_{11}A_{22} - P^2A_{12}A_{21}}$$

$$B_{21} = \frac{-a_{21}a_{11} + PA_{11}a_{21}}{a_{11} - PA_{11} - PA_{22}a_{11} + P^2A_{11}A_{22} - P^2A_{12}A_{21}}$$

$B_{22} = \frac{PA_{21}b_{12}}{a_{22}-PA_{22}}$, and $A^k = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ as in Theorem 3.1

Proof:

The same proof of theorem 2. 3.

Example 3.3.

Consider the system $F(x) = Bx$ where $B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, then by Theorem (3.2), there is a matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ with $P = 2$.

$b_{11}a_{11} = Pa_{11}b_{11} + Pa_{12}b_{21}$ that is

$-(2) = 2(2)(-1) + 2 = -2$

$b_{11}a_{12} + b_{12} = Pa_{11}b_{12} + Pa_{12}b_{22}$ that is

$-1 + (-1) = 2(2)(-1) + 2 = -2$

$b_{21} + b_{22}a_{21} = Pa_{21}b_{11} + Pa_{22}b_{21}$ that is

$1 + 1 = 2(-1) + 2(2) = 2$

$b_{22}a_{22} = Pa_{21}b_{12} + Pa_{22}b_{22}$ that is

$2 = 2(-1) + 2(2) = 2$

Note: From the definition of P^k*-semi-homogenous* of order m, we define a new concept as follows stopping sign.

Definition 3.4. System (4) is called adjoint if there exist two non-zero matrices A and C such that the following equation holds

$F\big(A(c)x(n)\big) = CF(x(n))$.

Theorem 3.5:

The necessary and sufficient condition for a homogenous system of difference Equations (4) to be adjoint is the matrix B equal to $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$,

where:

$$B_{11} = \frac{C_{12}b_{21}}{a_{11} - C_{11}}$$

$$B_{12} = \frac{a_{12}a_{22} - C_{22}a_{12}}{a_{22} - C_{22} - C_{11}a_{22} + C_{11}C_{22} - C_{12}C_{21}}$$

$$B_{21} = \frac{-a_{21}a_{11} + C_{11}a_{21}}{a_{11} - C_{11} - C_{22}a_{11} + C_{11}C_{22} - C_{12}C_{21}}$$

$$B_{22} = \frac{C_{21}b_{12}}{a_{22} - C_{22}}$$

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Proof:

Necessary condition:

Since F is homogenous of degree one, then there exists a non-zero matrix (say B) such that equation (3) is hold. That is,

$$F\big(A(c)X(n)\big) = CF\big(X(n)\big)$$

$$F\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \end{pmatrix}$$

$$\begin{pmatrix} f(a_{11}x + a_{12}y) \\ g(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \end{pmatrix}$$

$$\begin{pmatrix} b_{11}\,(a_{11}x + a_{12}y\,) + b_{12}y \\ b_{21}x + b_{22}\,(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} C_{11}f(x) + C_{12}g(y) \\ C_{21}f\,(x) + C_{22}g\,(y) \end{pmatrix}$$

$$\begin{pmatrix} b_{11}a_{11}x + b_{11}a_{12}y + b_{12}y \\ b_{21}x + b_{22}a_{21}x + b_{22}a_{22}y \end{pmatrix} = \begin{pmatrix} C_{11}(b_{11}x + b_{12}y) + C_{12}(b_{21}x + b_{22}y) \\ C_{21}(b_{11}x + b_{12}y) + C_{22}(b_{21}x + b_{22}y) \end{pmatrix}$$

$$b_{11}a_{11} = C_{11}b_{11} + C_{12}b_{21}$$
$$b_{11}a_{12} + b_{12} = C_{11}b_{12} + C_{12}b_{22}$$
$$b_{21} + b_{22}a_{21} = C_{21}b_{11} + C_{22}b_{21}$$
$$b_{22}a_{22} = C_{21}b_{12} + C_{22}b_{22}$$

It is easy to show that the matrix B equal to $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

Sufficient condition:

Suppose that there is a matrix B equal to $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, to show that the difference system is adjoint that is

$$F\big(A(c)X(n)\big) = CF\big(X(n)\big)$$

That is to show

$$b_{11}a_{11} = C_{11}b_{11} + C_{12}b_{21}$$
$$b_{11}a_{12} + b_{12} = C_{11}b_{12} + C_{12}b_{22}$$
$$b_{21} + b_{22}a_{21} = C_{21}b_{11} + C_{22}b_{21}$$
$$b_{22}a_{22} = C_{21}b_{12} + C_{22}b_{22}$$

By substituting the value of the matrix B in above equations we have

$$a_{11}b_{11} = C_{11}b_{11} + C_{12}b_{21}$$

$$a_{11} = C_{11} + \frac{C_{12}b_{21}}{b_{11}}$$

$$a_{11} = C_{11} + \frac{C_{12}b_{21}}{\frac{C_{12}b_{21}}{a_{11} - C_{11}}}$$

$$a_{11} = C_{11} + a_{11} - C_{11}$$

That is left hand equal to right

Also,

$a_{12}b_{11} + b_{12} = C_{11}b_{12} + C_{12}b_{22}$. Therefore,

$$a_{12}b_{11} + b_{12} = C_{11}b_{12} + \frac{C_{12}C_{21}b_{12}}{a_{22} - C_{22}}$$

$$\frac{a_{12}b_{11}}{b_{12}} + 1 = C_{11} + \frac{C_{12}C_{21}}{a_{22} - C_{22}}$$

$$\frac{a_{12}b_{11}}{\frac{a_{12}a_{22} - C_{22}a_{12}}{a_{22} - C_{22} - C_{11}a_{22} + C_{11}C_{22} - C_{12}C_{21}}} + 1 = C_{11} + \frac{C_{12}C_{21}}{a_{22} - C_{22}}$$

$$b_{11}(a_{22} - C_{22} - C_{11}a_{22} + C_{11}C_{22} - C_{12}C_{21}) + a_{22} - C_{22} = C_{11}a_{22} - C_{11}C_{22} + C_{12}C_{21}$$

$$b_{11} = -1$$

$$C_{11}a_{22} - C_{11}C_{22} + C_{12}C_{21} = C_{11}a_{22} - C_{11}C_{22} + C_{12}C_{21}$$

The third item is

$a_{21}b_{22} + b_{21} = C_{22}b_{21} + C_{21}b_{11}$, hence

$$a_{21}b_{22} + b_{21} = C_{22}b_{21} + \frac{C_{12}C_{21}b_{21}}{a_{11} - C_{11}}$$

$$\frac{a_{21}b_{22}}{b_{21}} + 1 = C_{22} + \frac{C_{12}C_{21}}{a_{11} - C_{11}}$$

$$\frac{a_{21}b_{22}}{\frac{-a_{21}a_{11} + C_{11}a_{21}}{a_{11} - C_{11} - C_{22}a_{11} + C_{11}C_{22} - C_{12}C_{21}}} + 1 = C_{22} + \frac{C_{12}C_{21}}{a_{11} - C_{11}}$$

$$-b_{22}(a_{11} - C_{11} - C_{22}a_{11} + C_{11}C_{22} - C_{12}C_{21}) + a_{11} - C_{11} = C_{22}a_{11} - C_{11}C_{22} + C_{12}C_{21}$$

$$b_{22} = 1$$

$$C_{22}a_{11} - C_{11}C_{22} + C_{12}C_{21} = C_{22}a_{11} - C_{11}C_{22} + C_{12}C_{21}.$$

The fourth item is

$a_{22}b_{22} = C_{22}b_{22} + C_{21}b_{12}$ that is $a_{22} = C_{22} + \frac{C_{21}b_{12}}{b_{22}}$ and $a_{22} = C_{22} + \frac{C_{21}b_{12}}{\frac{C_{21}b_{12}}{a_{22} - C_{22}}}$,therefore

$$a_{22} = C_{22} + a_{22} - C_{22}$$
$$a_{22} = a_{22}$$

Example 3.6. Considers the system $F(x) = Bx$ where $B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, then there are two matrices $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & -4 \\ -4 & -2 \end{bmatrix}$ such that

$$F\,\big(A(c)x(n)\big) = CF(x(n))$$

$b_{11}a_{11} = C_{11}b_{11} + C_{12}b_{21}$ that is $(-1)2 = (-2)(-1) + (-4)(1) = -2$

$b_{11}a_{12} + b_{12} = C_{11}b_{12} + C_{12}b_{22}$ that is $(-1)1 + (-1) = (-2)(-1) + (-4)(1) = -2$

$b_{21} + b_{22}a_{21} = C_{21}b_{11} + C_{22}b_{21}$ that is $1 + 1(1) = (-4)(-1) + (-2)1 = 2$

$b_{22}a_{22} = C_{21}b_{12} + C_{22}b_{22}$ that is $1(2) = (-4)(-1) + (-2)1 = 2$

Remark 3.7.

1- Every P^k-semi homogeneous system is an adjoint system, but in general, the converse is not true, Example 3.5 explains that.

2- In Definition 3.4, if the matrix C can be written as PA, then the converse of 1 will be true.

CONCLUSIONS

In this paper, we present new concepts which are generalized to the (2×2)-semi-homogeneous system of difference equations of order m, where m is positive integer numbers. We research some of their unique cases. For the future work, we intend to generalize our work to the (3×3)-semi-homogeneous system of difference equations of order m.

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