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# Some Common Fixed Point Theorems of Rational Contractions Condition in Generalized Banach Space

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## ABSTRACT

In this work, we introduce some common fixed points theorems type rational contractions on generalized Banach space. The provided theorem is a generalization and an extension of many well-known theorems; we present a few examples to illustrate the validity of the results obtained in the paper.

**KEYWORDS**: fixed points, common fixed point, Banach space, Cauchy sequence, rational contractions condition.

الخلاصة

في هذا العمل، نقدم بعض نظريات النقطة الثابتة الشائعة من نوع الانقباضات المنطقية على فضاء باناخ المعمم. النظرية المقدمة هي تعميم وامتداد للعديد من النظريات المعروفة. و نقدم بعض الأمثلة لتوضيح صحة النتائج التي تم الحصول عليها في هذا العمل.

## INTRODUCTION

Banach showed and proved a crucial conclusion in complete metric spaces, namely that each complete metric space has a unique fixed point. Bakhtin and Czerwik, as a generalization of metric spaces, developed the notion of b- metric spaces in 1989 see [1], [2], and [3]. For the study of fixed points in standard metric spaces or other generalized metric spaces, many academics extended and confirmed this theory; we recommend (Boriceanu], Bota *et al.*, Ding et al., Kir and Kiziltunc, Ozturk, and Turkoglu). Many researchers, on the other hand, have used rational type contractive conditions (Mohammad and Pankaj, Sarwar and Rahman], Xie et al., and Seddik, and Taieb see [4-12]. In this work, we show and expand some common fixed-point theorems that are also valid in generalized Banach space; we also provide some specific examples to show the veracity of our results.

A mapping $F : X \to X$ where $(M, \|.\|)$ is a Banach space is said to be a contraction if there exisets $0 \leq k < 1$ such that, for all $x, y \in M$,

$$\|Fx - Fy\| \leq k\|x - y\|. \tag{1.1}$$

**Definition 1.1** [13]: If $M$ nonempty is a linear space having $s \geq 1$, let $\|.\|$ denotes a functon from linear space M into $R$ that satisfies the following axioms:
1. For all $x \in M$ $\|x\| \geq 0$, $\|x\| = 0$ if and only if $x = 0$;
2. For all $x, y \in M$, $\|x + y\| \leq s\,[\|x\| + \|y\|]$;
3. For all $x \in M$, $\alpha \in R$, $\|\alpha x\| = |\alpha|\,\|x\|$;

$(M, \|.\|)$is called generalized normed linear space. If for $s = 1$, it reduces to standard normed linear space.

**Definition1.2** [13]: A Banach space $(M, \|.\|)$ is a normed vector space such that $M$ is complete under the metric induced by the $\|.\|$.

**Definition1.3** [13]: A linear generalized normed space in which every Cauchy sequence is convergent is called generalized Banach space.

**Definition 1.4** [13]: Let $(M, \|.\|)$ be a generalized normed space then the sequence $\{x_n\}$ in $M$ is called
1. Cauchy sequence iff for each $\varepsilon > 0$, there exist $n(\varepsilon) \in N$ such that for all $m, n \geq n(\varepsilon)$ we have $\|x_n - x_m\| < \varepsilon$.
2. Convergent sequence iff there exist $x \in M$ such that for all $\varepsilon > 0$, there exist $n(\varepsilon) \in N$ such that for every $n \geq n(\varepsilon)$ we have $\|x_n -$

$x \| < \varepsilon$.

# MAIN RESULTS

**Lemma 2.1**. Let $(H, \|.\|)$ be a generalized Banach space with a real number $s \geq 1$, and $F$ self-mapping on $H$, assume that $\{u_n\}$ is a sequence in $H$ defined by $u_{n+1} = Fu_n$ if

$\|u_{n-}u_{n+1}\| \leq \alpha \|u_{n-1} - u_n\|$, for all $n \in N$, where $\alpha \in [0, 1)$, $s\alpha < 1$. Then $\{u_n\}$ is a Cauchy sequence and is a converge to some $u^* \in H$ as n$\to +\infty$.

**Proof:** let $p > 0$ we have,

$$\|u_{n-}u_{n+p}\| \leq s[\|u_n - u_{n+1}\| + \|u_{n+1} - u_{n+2}\| + \dots]$$
$$\leq s\|u_n - u_{n+1}\| + s^2 [\|u_{n+1} - u_{n+2}\| + \|u_{n+2} - u_{n+3}\| + \dots]$$
$$\leq s \|u_n - u_{n+1}\| + s^2 \|u_{n+1} - u_{n+2}\| + s^3 \|u_{n+2} - u_{n+3}\| + \cdots\dots$$

by using (1.1), we have

$$\|u_{n-}u_{n+p}\| \leq s\alpha^n \|u_n - u_{n+1}\| + s^2\alpha^{n+1} \|u_{n+1} - u_{n+2}\| + s^3\alpha^{n+2} \|u_{n+2} - u_{n+3}\| + \dots$$
$$\leq (1 + s\alpha + (s\alpha)^2 + \cdots) s\alpha^n \|u_0 - u_1\|$$
$$\leq s\alpha^n \frac{1}{(1-s\alpha)} \|u_0 - u_1\|,$$

by taking the limit as $n \to +\infty$, we obtain. That $\|u_{n-}u_{n+p}\| = 0$, (because $s\alpha < 1$).

Then $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$. Since $(H, \|.\|)$ is a generalized Banach space, $\{u_n\}$ is a converges to some $u^* \in H$ as $n \to +\infty$.

**Theorem (2.1)**: Let $H$ be a generalized Banach space with $\|.\|$ and let $F, Q: H \to H$ be two mappings on $H$ fulfilling the condition:

$$\|Fx - Qy\| \leq a \frac{\|x-y\|}{1+ \|y-Fx\|} + b \frac{\|y-Fx\|\|x-Qy\|}{1+ \|x-Qy\|} + c \frac{\|x-Qy\|(1+\|x-Fx\|)}{1+ \|x-y\|}, \quad (2.1)$$

For all $x, y \in H$, and $a, b$ and $c$ are real number. Then $F$ and $Q$ have a unique common fixed point, where:

$s \geq 1, (a + 2cs < 1)$ and $cs < \frac{1}{2}$.

**Proof:** We define a sequence $\{u_n\}$ in $H$ such that,

$$u_{n+1} = Fu_n,\ u_{n+2} = Qu_{n+1}, \text{ for all } n \in N. \quad (2.2)$$

If, $u_n = u_{n+1}$ for all $n \in N$, and let $n = 2z$, then $u_{2z} = u_{2z+1}$ as well as the (2.1) with $x = u_{2z}$ and $y = u_{2z+1}$, we have,

$$\| u_{2z+1.} - u_{2z+2.}\| = \|Fu_{2z} - Qu_{2z+1.}\|$$
$$\leq a \frac{\| u_{2z.} - u_{2z+1.}\|}{1+ \|u_{2z+1} - Fu_{2z}\|} + b \frac{\|u_{2z+1} - Fu_{2z}\|\|u_{2z} - Qu_{2z+1}\|}{\|u_{2z} - Qu_{2z+1}\| + 1} + c \frac{\|u_{2z} - Qu_{2z+1}\|(1+\|u_{2z} - Fu_{2z}\|)}{1+ \|u_{2z} - u_{2z+1.}\|}$$
$$= a \frac{\|u_{2z} - u_{2z+1}\|}{1+ \|u_{2z+1} - u_{2z+1}\|} + b \frac{\|u_{2z+1} - u_{2z+1}\|\|u_{2z} - u_{2z+2}\|}{\|u_{2z} - u_{2z+2}\| + 1} + c \frac{\|u_{2z} - u_{2z+2}\|(1+\|u_{2z} - u_{2z+1}\|)}{1+ \|u_{2z} - u_{2z+1}\|}$$

$\|u_{2z+1} - u_{2z+2}\| \leq c\|u_{2z} - u_{2z+2}\|$, by condition (2) of the definition (1.1)

$$\|u_{2z+1} - u_{2z+2}\| \leq c[s\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+2}\|]$$
$$= cs \|u_{2z+1} - u_{2z+2}\|]$$

$\|u_{2z+1} - u_{2z+2}\| \leq cs\|u_{2z} - u_{2z+1}\|$, which is a contradiction, (because $cs > \frac{1}{2}$)

$$\|u_{2z+1} - u_{2z+2}\| = 0 \Longrightarrow u_{2z+1} = u_{2z+2},$$

Thus, we have $u_{2z} = u_{2z+1} = u_{2z+2}$. By (2.2), it means $u_{2z} = Fu_{2z} = Qu_{2z}$, that is $u_{2Z}$ is a common fixed point of $F$ and $Q$.

Now, if $u_n \neq u_{n+1}$ for all $n \in N$, and let $n = 2z + 1$, $z \in N$.

$(I)$ We will show that $\{u_n\}$ is a Cauchy sequence, By (2.2) with $x = u_{2z}, y = u_{2z+1}$, we have,

$$\|u_{2z+1} - u_{2z+2}\| = \|Fu_{2z} - Qu_{2z+1}\|$$
$$\leq a \frac{\|u_{2z} - u_{2z+1}\|}{1+ \|u_{2z+1} - Fu_{2z}\|} + b \frac{\|u_{2z+1} - Fu_{2z}\|\|u_{2z} - Qu_{2z+1}\|}{\|u_{2z} - Qu_{2z+1}\| + 1} + c \frac{\|u_{2z} - Qu_{2z+1}\|(1+\|u_{2z} - Fu_{2z}\|)}{1+ \|u_{2z} - u_{2z+1}\|}$$
$$= a \frac{\|u_{2z} - u_{2z+1}\|}{1+ \|u_{2z+1} - u_{2z+1}\|} + b \frac{\|u_{2z+1} - u_{2z+1}\|\|u_{2z} - u_{2z+2}\|}{\|u_{2z} - u_{2z+2}\| + 1} + c \frac{\|u_{2z} - u_{2z+2}\|(1+\|u_{2z} - u_{2z+1}\|)}{1+ \|u_{2z} - u_{2z+1}\|},$$

then

$$\|u_{2z+1} - u_{2z+2}\| \leq a\|u_{2z} - u_{2z+1}\| + c \|u_{2z} - u_{2z+2}\|,$$

by condition (2) of the definition (1.1).

$$\|u_{2z+1} - u_{2z+2}\| \leq a\|u_{2z} - u_{2z+1}\| + c[s\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+2}\|]$$
$$= a\|u_{2z} - u_{2z+1}\| + cs\|u_{2z} - u_{2z+1}\| + cs \|u_{2z+1} - u_{2z+2}\|$$
$$\|u_{2z+1} - u_{2z+2}\| \leq \frac{(a + cs)}{1-cs} \|u_{2z} - u_{2z+1}\|,$$

where $\frac{(a + cs)}{1-cs} = \gamma$.

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Thus, we obtain that, by using Lemma (2.1) and taking limit $n \to \infty$ we get $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$, and is a converges to some $u^* \in H$ as $n \to +\infty$, we get $(II)$ we'll show that $Fu^* = Qu^* = u^*$.

By condition (2) of the definition (1.1), and the condition (2.1), we have

$$\|u^* - Fu^*\| \leq s[\|u^* - u_{2n+2}\| + \|u_{2n+2} - Fu^*\|]$$
$$= s\|u^* - u_{2n+2}\| + s\|Fu^* - Qu_{2n+1}\|$$
$$\leq s\|u^* - u_{2n+2}\| + s\,a\,\frac{\|u^*-u_{2n+1}\|}{1+\|u_{2n+1}-Fu^*\|} + s\,b\,\frac{\|u_{2n+1}-Fu^*\|\|x^*-Qu_{2n+1}\|}{\|u^*-Qu_{2n+1}\|+1} + sc\,\frac{\|u^*-Qu_{2n+1}\|(1+\|u^*-Fu^*\|)}{1+\|u^*-u_{2n+1}\|}.$$

By take the limit as $n \to +\infty$, we get,

$\|u^* - Fu^*\| \leq 0$, thus $Fu^* = u^*$.

Similarly, we get,

$$\|u^* - Qu^*\| \leq s[\|u^* - u_{2n+1}\| + \|u_{2n+1} - Qu^*\|]$$
$$= s\|u^* - u_{2n+1}\| + s\|Fu_{2n} - Qu^*\|$$
$$\leq s\|u^* - u_{2n+2}\| + s\,a\,\frac{\|u^*-u_{2n}\|}{1+\|u^*-Fu_{2n}\|} + s\,b\,\frac{\|u^*-Fu_{2n}\|\|u_{2n}-Qu^*\|}{\|u_{2n}-Qu^*\|+1}$$
$$+ s\,c\,\frac{\|u_{2n}-Qu^*\|(1+\|u_{2n}-Fu_{2n}\|)}{1+\|u^*-u_{2n}\|}$$
$$\|u^* - Qu^*\| \leq sc\,\|u^* - Qu^*\|$$

Since $sc < 1$ (because $+2sc < 1$ ), hence $\|u^* - Qu^*\| = 0$, thus $Qu^* = u^*$.

Thus $u^*$ is a commn fixed point of $F$ and.

$(III)$ Now we will show that $F$ and $Q$ have a unique. Common fixed point.

Assume that $u^*, v^* \in H$ such that $u^* \neq v^*$ are common fixed points of $F$ and $Q$, by (2.1), we have

$$\|u^* - v^*\| = \|Fu^* - Qv^*\|$$
$$\leq a\frac{\|u^*-v^*\|}{1+\|v^*-Fv^*\|} + b\frac{\|u^*-Fu^*\|\|u^*-Qv^*\|}{\|u^*-Qv^*\|+\|v^*-Fu^*\|} + c\,\frac{\|u^*-Qv^*\|(1+\|u^*-Fu^*\|)}{1+\|u^*-v^*\|}$$
$$= a\,\frac{\|u^*-v^*\|}{1+\|v^*-u^*\|} + c\,\frac{\|u^*-v^*\|}{1+\|v^*-u^*\|}$$
$$= (a+c)\,\frac{\|u^*-v^*\|}{1+\|v^*-u^*\|}$$

$\|u^* - v^*\| \leq (a+c)\|u^* - v^*\|$, which is contradiction, since $0 \leq a + c < 1$, we note, $\|u^* - v^*\| = 0, \mathit{thus}\ u^* = v^*$.

**Remark 2.1:** We will take an example that fulfills the condition (2.1) in the theorem (2.1).

**Example 2.1**. Let $H = \{u, i, j\}$, with $\|.\|: H \times H \to [0, +\infty)$ be a mapping that satisfies the following condition (2,1) for all $x, y \in H$, $\|x - y\| = 0$, where $x = y$,

$\|u - i\| = \|i - u\| = \frac{1}{4}$,

$\|u - j\| = \|j - u\| = \frac{1}{10}$,

$\|j - i\| = \|i - j\| = \frac{1}{2}$.

Then $(H, \|.\|)$ be a generalized Banach space with a real number $s = \frac{4}{3} \geq 1$. Consider mapping $F, Q: H \to H$, define by

$$F(u) = u, \quad F(i) = u, F(j) = u,$$
$$Q(u) = u, \quad Q(i) = j, Q(j) = u.$$

Let $= \frac{1}{2}$, $b = 2$ and $c = \frac{1}{8}$, $a + 2sc < 1$, now we'll verify the condition (2.1).

It has the following case to,

$\|Fx^* - Qy^*\| = 0$ the condition (2.1) holds.

$\|Fx^* - Qy^*\| \neq 0$, we showed that following cases:

Case 1. $x = u, y = i$, we can get,

$\|Fx - Qy\| = \frac{1}{10}$, then

$$\frac{1}{10} \leq \frac{1}{2} \times \frac{\frac{1}{4}}{1+\frac{1}{4}} + 2 \times \frac{\frac{1}{4}\left(\frac{1}{10}\right)}{1+\frac{1}{10}} + \frac{1}{8} \times \frac{\left(\frac{1}{10}\right)(1+0)}{1+\frac{1}{4}} = a\frac{\|x-y\|}{1+\|y-Fx\|} + b\frac{\|y-Fx\|\|x-Qy\|}{1+\|x-Qy\|} + c\frac{\|x-Qy\|(1+\|x-Fx\|)}{1+\|x-y\|},$$

Therefore, the condition (2.1) holds .

Case2. $x = i$, $y = i$, we can get $\|Fx - Qy\| = \frac{1}{10}$, then,

$$\frac{1}{10} \leq \frac{1}{2} \times \frac{0}{1+\frac{1}{4}} + 2 \times \frac{\frac{1}{4}(\frac{1}{2})}{1+\frac{1}{2}} + \frac{1}{8} \times \frac{(\frac{1}{2})(1+\frac{1}{4})}{1+0} = a\frac{\|x-y\|}{1+\|y-Fx\|} + b\frac{\|y-Fx\|\|x-Qy\|}{1+\|x-Qy\|} + c\frac{\|x-Qy\|(1+\|x-Fx\|)}{1+\|x-y\|}$$

, the condition (2.1) holds.

Case3. $x = j$, $y = i$, we can get $\|Fx - Qy\| = \frac{1}{10}$, then,

$$\frac{1}{10} \leq \frac{1}{2} \times \frac{\frac{1}{2}}{1+\frac{1}{4}} + 2 \times \frac{\frac{1}{4}(0)}{1+0} + \frac{1}{8} \times \frac{(0)(1+\frac{1}{10})}{1+\frac{1}{2}}$$
$$= a\frac{\|x-y\|}{1+\|y-Fx\|} + b\frac{\|y-Fx\|\|x-Qy\|}{1+\|x-Qy\|} + c\frac{\|x-Qy\|(1+\|x-Fx\|)}{1+\|x-y\|},$$

the condition (2.1) holds.

Then $F$ and $Q$ have a has unique common fixed point,

$(F(u) = u = Qu)$.

**Theorem (2.2):** Let $H$ be a generalized Banach space with $\|.\|$ and let $F, Q: H \to H$ be two mappings on $H$ fulfilling the condition:

$$\|Fx - Qy\| \leq a \frac{\|x-y\| + \|Qy-y\|\|Fx-x\|}{1+ \|Qy-Fx\|}$$
$$+b \frac{\|x-y\| + \|y-Fx\|\|x-Qy\|+\|x-y\| \ \|QFx-Fx\|}{1+ \|Qy-Fx\|}$$
$$+ c \frac{\|FQy-y\|\|y-Fx\|}{s + \|QFx-y\|} + d \|x - Qy\|, \quad (2.3)$$

for all $x, y \in H$, and $a, b$ and $c$ are real number. Then $F$ and $Q$ have a unique common fixed point, where:

$s \geq 1$ , $(0 \leq a + b + 2ds < 1)$ and $(0 \leq a + b + d < 1)$.

**Proof**: We define a sequence. $\{u_n\}$ in $H$ such that $u_{n+1} = Fu_n$, $u_{n+2} = Qu_{n+1}$, for all $n \in N$. (2.4)

If, $u_n = u_{n+1}$, for all $n \in N$, and let $n = 2z, then\ u_{2z} = u_{2z+1}$ as well as the (2.4) with $x = u_{2z}$ and $y = u_{2z+1}$ we have,

$$\|u_{2z+1} - u_{2z+2}\| = \|Fu_{2z} - Qu_{2z+1}\|$$
$$\leq a \frac{\|u_{2z}-u_{2z+1}\| + \|Qu_{2z+1}-u_{2z+1}\|\|Fu_{2z}-u_{2z}\|}{1+ \|Qu_{2z+1}-Fu_{2z}\|}$$

$+b$ ( [‖u_2z − u_(2z + 1) ‖ + ‖u_(2z + 1) − 〖Fu〗 _2z ‖‖u_2z − 〖Qu〗 _(2z + 1) ‖ + ‖u_2z − u_(2z + 1) ‖‖Q 〖Fu〗 _2z − Fu_2z ‖] )/(1 + ‖ 〖Qu〗 _(2z + 1) − Fu_2z ‖)

$$+ c \frac{\|FQu_{2z+1}-u_{2z+1}\|\|u_{2z+1}-Fu_{2z}\|}{s + \|QFu_{2z}-u_{2z+1}\|}$$
$$+ d \|u_{2z} - Qu_{2z+1}\|$$

$$= a \frac{[\|u_{2z} - u_{2z+1}\| + \|u_{2z+2} - u_{2z+1}\|\|u_{2z+1} - u_{2z}\|]}{1 + \|u_{2z+2} - u_{2z+1}\|}$$

$+b$ ([ ‖u_2z − u_(2z + 1) ‖ + ‖u_(2z + 1) − u_(2z + 1) ‖‖u_2z − u_(2z + 2) ‖ + ‖u_2z − u_(2z + 1) ‖ ‖u_(2z + 2) − u_(2z + 1) ‖] )/(1 + ‖u_(2z + 2) − u_(2z + 1) ‖)

$$+ c \frac{\|u_{2z+3}-u_{2z+1}\|\|u_{2z+1}-u_{2z+1}\|}{s + \|u_{2z+2}-u_{2z+1}\|}$$
$$+ d \|u_{2z} - u_{2z+2}\|$$
$$\leq ds [\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+2}\|]$$

$(1 - ds)\|u_{2z+1} - u_{2z+2}\| \leq 0$, which is contradiction we have, limit $n \to \infty \|u_{2z+1} - u_{2z+2}\| = 0$.

Hence $u_{2z+1} = u_{2z+2}$.

Thus, we have $u_{2z} = u_{2z+1} = u_{2z+2}$. By (2.3), it means $u_{2z} = Fu_{2z} = Qu_{2z}$, that is $u_{2z}$ is a common fixed point of $F$ and $Q$.

Now, if $u_n \neq u_{n+1}$ for all $\in N$, and let $n = 2z + 1,\ z \in N$.

$(I)$ We will show that $\{u_n\}$ is a Cauchy sequence, by (2.4) with $x = u_{2z}, y = u_{2z+1}$ we have

$$\|u_{2z+1} - u_{2z+2}\| = \|Fu_{2z} - Qu_{2z+1}\|$$
$$\leq a \frac{[\|u_{2z} - u_{2z+1}\| + \|Qu_{2z+1} - u_{2z+1}\| \|Fu_{2z} - u_{2z}\|]}{1 + \|Qu_{2z+1} - Fu_{2z}\|}$$
$$+b \frac{\begin{bmatrix} \|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - Fu_{2z}\| \\ \|u_{2z} - Qu_{2z+1}\| + \\ \|u_{2z} - u_{2z+1}\|\|QFu_{2z} - Fu_{2z}\| \end{bmatrix}}{1 + \|Qu_{2z+1} - Fu_{2z}\|}$$
$$+ c \frac{\|FQu_{2z+1}-u_{2z+1}\|\|u_{2z+1}-Fu_{2z}\|}{s + \|QFu_{2z}-u_{2z+1}\|}$$
$$+ d \|u_{2z} - Qu_{2z+1}\|$$
$$= a \frac{[\|u_{2z} - u_{2z+1}\| + \|u_{2z+2} - u_{2z+1}\|\|u_{2z+1} - u_{2z}\|]}{1 + \|u_{2z+2} - u_{2z+1}\|}$$
$$+b \frac{[\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+1}\|\|u_{2z} - u_{2z+2}\| + \|u_{2z} - u_{2z+1}\| \ \|u_{2z+2} - u_{2z+1}\|]}{1 + \|u_{2z+2} - u_{2z+1}\|}$$

$$+c \frac{\|u_{2z+3}-u_{2z+1}\| \ \|u_{2z+1}-u_{2z+1}\|}{s + \|u_{2z+2}-u_{2z+1}\|}$$
$$+ d \|u_{2z} - u_{2z+2}\|$$
$$\leq a \ \|u_{2z} - u_{2z+1}\| + b \ \|u_{2z} - u_{2z+1}\|$$
$$+ ds[\|u_{2z} - u_{2z+1}\| + \|u_{2z+1} - u_{2z+2}\|]$$
$$= a\|u_{2z} - u_{2z+1}\| + b \|u_{2z} - u_{2z+1}\|$$
$$+ ds\|u_{2z} - u_{2z+1}\|$$
$$+ ds \|u_{2z+1} - u_{2z+2}\|$$
$$(1 - ds)\|u_{2z+1} - u_{2z+2}\| \leq (a + b + ds) \|u_{2z} - u_{2z+1}\|$$

$\|u_{2z+1} - u_{2z+2}\| \leq \frac{(a + b+ ds)}{(1- ds)} \|u_{2z} - u_{2z+1}\|$, let $\gamma = \frac{(a + b+ ds)}{(1- ds)}$

$\|u_{2z+1} - u_{2z+2}\| \leq \gamma \ \|u_{2z} - u_{2z+1}\|$,

where $0 \leq \gamma < 1$

$(0 \leq a + b + 2ds \leq 1)$.

Using Lemma (2.1) and taking limit n→ ∞ we get $\{u_n\}$ is a Cauchy sequence in $(H, \|.\|)$, and is a converge to some $u^* \in H$ as $n \to +\infty$, we get

$(II)$ We will show that $Fu^* = Qu^* = u^*$.

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By condition (2) of the definition (2.1), and the condition (2.3), we have

$$\|u^* - Fu^*\| \leq s[\|u^* - u_{2n+2}\| + \|u_{2n+2} - Fu^*\|]$$
$$= s\|u^* - u_{2n+2}\| + s\|Fu^* - Qu_{2n+1}\|$$
$$\leq s\|u^* - u_{2n+2}\| + sa\frac{\|u^*-u_{2n+1}\| + \|Qu_{2n+1}-u_{2n+1}\|\|Fu^*-u^*\|}{1+\|Qu_{2n+1}-Fu^*\|}$$
$$+s\,b\frac{[\|u^*-u_{2n+1}\|+\|u_{2n+1}-Fu^*\|\|u^*-Qu_{2n+1}\| + \|u^*-u_{2n+1}\|\|QFu^*-Fu^*\|]}{1+\|Qu_{2n+1}-Fu^*\|}$$
$$+s\,c\frac{\|FQu_{2n+1}-u_{2n+1}\|\|u_{2n+1}-Fu^*\|}{s+\|QFu^*-u_{2n+1}\|}$$
$$+s\,d\|u^* - Qu_{2n+1}\|$$
$$= s\|u^* - u_{2n+2}\|$$
$$+sa\frac{\|u^*-u_{2n+1}\| + \|u_{2n+2}-u_{2n+1}\|\|Fu^*-u^*\|}{1+\|u_{2n+2}-Fu^*\|}$$
$$+sb\frac{[\|u^*-u_{2n+1}\|+\|u_{2n+1}-Fu^*\|\|u^*-u_{2n+2}\| + \|u^*-u_{2n+1}\|\|u^*-Fu^*\|]}{1+\|u_{2n+2}-Fu^*\|}$$
$$+s\,c\frac{\|u_{2n+3}-u_{2n+1}\|\|u_{2n+1}-Fu^*\|}{s+\|QFu^*-u_{2n+1}\|}$$
$$+sd\|u^* - u_{2n+2}\|$$

Take the limit as $n \to +\infty$, we get
$\|u^* - Fu^*\| \leq 0$,
hence $\|u^* - Fu^*\| = 0$, thus $Fu^* = u^*$.
Similarly, we obtain

$$\|u^* - Qu^*\| \leq s[\|u^* - u_{2n+1}\| + \|u_{2n+1} - Qu^*\|]$$
$$= s\|u^* - u_{2n+1}\| + s\|Fu_{2n} - Qu^*\|$$
$$\leq s\|u^* - u_{2n+2}\|$$
$$+s\,a\frac{\|u_{2n}-u^*\| + \|Qu^*-u^*\|\|Fu_{2n}-u_{2n}\|}{1+\|Qu^*-Fu_{2n}\|}$$
$$+sb\frac{[\|u_{2n}-u^*\|+\|u^*-Fu_{2n}\|\|u_{2n}-Qu^*\| + \|u_{2n}-u^*\|\|QFu_{2n}-Fu_{2n}\|]}{1+\|Qu^*-Fu_{2n}\|}$$
$$+sc\frac{\|FQu^*-u^*\|\|u^*-Fu_{2n}\|}{s+\|QFu_{2n}-u^*\|}$$
$$+sd\|u_{2n} - Qu^*\|$$
$$= s\|u^* - u_{2n+2}\|$$
$$+s\,a\frac{\|u_{2n}-u^*\| + \|Qu^*-u^*\|\|u_{2n+1}-u_{2n}\|}{1+\|Qu^*-u_{2n+1}\|}$$
$$+s\,b\frac{\|u_{2n}-u^*\|+\|u^*-u_{2n+1}\|\|u_{2n}-Qu^*\| + \|u_{2n}-u^*\|\|u_{2n+2}-u_{2n+1}\|}{1+\|Qu^*-u_{2n+1}\|}$$
$$+s\,c\frac{\|FQu^*-u^*\|\|u^*-u_{2n+1}\|}{s+\|u_{2n+2}-u^*\|}$$
$$+d\|u_{2n} - Qu^*\|,$$

by take the limit as, $n \to +\infty$, we get
$\|u^* - Qu^*\| \leq d\|u^* - Qu^*\|$
Since $d < 1$, hence $\|u^* - Qu^*\| = 0$, thus $Qu^* = u^*$.
Thus $u^*$ is a common fixed point of $F$ and $Q$.
$(III)$ Now we will show that $F$ and $Q$ have a unique. common fixed point.
Assume that $u^*, v^* \in H$ such that $u^* \neq v^*$ are common fixed points of $F$ and $Q$, by (2.3), we have

$$\|u^* - v^*\| = \|Fu^* - Qv^*\|$$
$$\leq a\frac{\|u^*-v^*\| + \|Qv^*-v^*\|\|Fu^*-u^*\|}{1+\|Qv^*-Fu^*\|} + b\frac{\|u^*-v^*\|+\|v^*-Fu^*\|\|u^*-Fu^*\|+\|u^*-v^*\|\|QFu^*-Fu^*\|}{1+\|Qv^*-Fu^*\|}$$
$$+c\frac{\|FQv^*-v^*\|\|v^*-Fu^*\|}{s+\|QFu^*-v^*\|} + d\|u^* - Qv^*\|$$
$$= a\frac{\|u^*-v^*\| + \|v^*-v^*\|\|u^*-u^*\|}{1+\|v^*-u^*\|}$$
$$+b\frac{\|u^*-v^*\|+\|v^*-u^*\|\|u^*-u^*\|+\|u^*-v^*\|\|u^*-u^*\|}{1+\|v^*-u^*\|}$$
$$+c\frac{\|v^*-v^*\|\|v^*-u^*\|}{s+\|u^*-v^*\|} + d\|u^* - v^*\|$$
$$= a\frac{\|u^*-v^*\|}{1+\|y^*-u^*\|} + b\frac{\|u^*-v^*\|}{1+\|y^*-u^*\|}$$
$$+d\|u^* - v^*\|$$
$$\leq a\|u^* - v^*\| + b\|u^* - v^*\|$$
$$+d\|u^* - v^*\|$$
$$\|u^* - v^*\| \leq (a + b - d)\|u^* - v^*\|$$
$$(a + b - d) < 1,$$

which is contradiction
Since $0 \leq (a + b - d) < 1$,
$\|u^* - v^*\| = 0$. i.e. $u^* = v^*$
we proved that $F$ and $Q$ have a unique common fixed point in $H$ .

**Remark2.2**: The following are the corollaries of Theorem2.2.

**Corollary (2.1)**: Let $H$ be a Generalized Banach spase with $\|.\|$ and let $F, Q: H \to H$ be two mappings on $H$ fulfilling the condition:

$$\|Fx - Qy\| \leq max\{a\frac{\|x-y\| + \|Qy-y\|\|Fx-x\|}{1+\|Qy-Fx\|} + b\frac{\|x-y\| + \|y-Fx\|\|x-Qy\|+\|x-y\|\ \|QFx-Fx\|}{1+\|Qy-Fx\|}, c\frac{\|FQy-y\|\|y-Fx\|}{s+\|QFx-y\|} + d\|x - Qy\|\}, \quad (2.5)$$

for all $x, y \in H$, and $a, b$ and $c$ are real number. Then, $F$ and $Q$ have a unique common fixed point, where:

$$s \geq 1, (0 \leq a + b, 2ds < 1).$$

**Corollary (2.2)**: Let $H$ be a generalized Banach space with $\|.\|$ and let $F, Q: H \to H$ be a two mappings on $H$ fulfilling the condition:

$$\|Fx - Qy\| \leq a \frac{\|x-y\| + \|Qy-y\|\|Fx-x\|}{1+\|Qy-Fx\|} + b \frac{\|x-y\| + \|y-Fx\|\|x-Qy\| + \|x-y\| \; \|QFx-Fx\|}{1+\|Qy-Fx\|} + c \frac{\|FQy-y\|\|y-Fx\|}{s+\|QFx-y\|}, \quad (2.6)$$

for all $x, y \in H$, and $a, b$, and $c$ are real number. Then, $F$ and $Q$ have a unique common fixed point, where:

$$s \geq 1, (0 \leq a + b < 1).$$

**Corollary (2.3)**: Let $H$ be a generalized Banach space with $\|.\|$ and let $F, Q: H \to H$ be a two mappings on $H$ fulfilling the condition:

$$\|Fx - Qy\| \leq c \frac{\|FQy-y\|\|y-Fx\|}{s+\|QFx-y\|} + d\,\|x - Qy\|, \quad (2.7)$$

for all $x, y \in H$, and $a, b$ and $c$ are real number. Then $F$ and $Q$ have a unique common fixed point, where:

$$s \geq 1, (2ds < 1).$$

## CONCLUSIONS

The development of the field of common fixed-point theory depends on the generalization of the Banach Contraction principle on in generalized Banach spaces; we introduced some common fixed-point theorems, including rational condition. We provided examples to explain the theorem (2.1). The common fixed-point contraction mapping principle also has a local version (Corollary (2.1), (2.2), and (2.3) as well as an asymptotic version (Theorem 2.2).

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