New Types of Separation Axioms via Wp-Open Sets

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ABSTRACT
In this work, new kinds of separation axioms using $w_p$ open sets, some results, properties, examples, and the relationship between these concepts have been given to support our work.

KEYWORDS: $w$-open, pre-open, regular, and normal spaces

INTRODUCTION
In our work the definitions of $w_p$-$T_i$, $i=0, 1, 2, 3, 4$ are defined by using $w_p$-open set. We also offer some notions by utilizing $w_p$-open sets and studying some of their facts. Throughout this paper $(X, \tau)$ or $X$ is always a topological space. The intersection of $w_p$ closed sets containing $A$ we call it $w_p$ closer of $A$. the largest $w_p$ open set contained in $A$ we call it $w_p$ interior of $A$. There are more researches about $w$-open set [1-5].

$w_p$-OPEN SETS AND SOME RESULTS

Definition 1 [6, 7]: A set $A$ taken from space $X$ called preopen set if $A \subseteq \text{int} (\text{cl}(A))$.

The researchers in our introduction define $w$-open set as follows:

Definition 2: The set $A$ taken from space $X$ we call it $w$-open if we find open set $U$ containing any point $x$ in $A$ such that $U$-$A=\text{countable}$.

Definition 3 [8]: A set $A$ taken from space $X$, we call it $w_p$ open if for every point $x$ belongs to $A$ we have a preopen set $U$ containing $x$, in which $U$-$A=\text{countable}$.

So a subset $F$ of space $X$ is $w_p$ closed if $X$-$F$ is $w_p$ open.

Remark 1: Every open set is $w_p$-open set but the converse maybe not true.

Example 1: In the indiscrete space $(R, \tau_{ind})$ the set of whole rational numbers is $w_p$-open but not open

Lemma 1: A subset $U$ of space $X$ is $\omega_p$-open if and only if every point in $U$ is an $\omega_p$-interior point.

Proof: Suppose $U$ is $\omega_p$-open, then it is $\omega_p$-neighborhood to each of its points. So every point is an $\omega_p$-interior point. Conversely, since $U = \bigcup_{x \in U} \{x\}$ and every point has $\omega_p$-open set $V_x$ such that $x \in V_x \subseteq U$ then $U = \bigcup_{x \in U} \{x\}$ and the union of $\omega_p$-open sets is $\omega_p$-open so $U$ is $\omega_p$-open set.

Definition 4: The space $X$ will be named:
1- $\omega_p T_0$ –space if for different elements $x, y$ at $X$, we find $\omega_p$ open set $W$ in $X$ containing $x$ but not $y$ or vice versa.
2- $\omega_p T_1$ –space if for different elements $x, y$ in $X$, there are $\omega_p$ open sets $W_1$, $W_2$ in $X$ such that $x \in W_1, y \notin W_1$ and $y \in W_2, x \notin W_2$.
3- $\omega_p T_2$ –space if for all different elements $x, y$ in $X$, there are $\omega_p$ open sets have no mutual point $W_1, W_2$ in $X$ such that $x \in W_1$ and $y \in W_2$. 
Example 2: Let $X=\{1, 2, 3\}$ and $\tau=\{\emptyset, X, \{1\}, \{3\}, \{1, 3\}\}$, so $(X, \tau)$ is $\omega_pT_0$-space and, $\omega_pT_1$-$\omega_pT_2$-space, but neither $T_1$ nor $T_2$-space.

Remark 2:
1-If $(X, \tau)$ is $T_1$-space, that it is $\omega_pT_1$-space where $i=0, 1, 2$.
2- If $(X, \tau)$ is $\omega_pT_2$-space, then it is $\omega_pT_1$ and $\omega_pT_0$-spaces.

Example 3:
1- The indiscrete space $(Z, \tau_{ind})$ is $\omega_pT_0$-space, $\omega_pT_1$ and $\omega_pT_2$-space but not $T_0, T_1$ and $T_2$-space.
2- The co-finite topological space $(R, \tau_{cof})$ is $\omega_pT_0$-space and $\omega_pT_1$-space, but not $\omega_pT_2$-space.

Proposition 1: Suppose $Y$ be an open subspace of space $X$, if $W$ is pre-open in $X$, then $W \cap Y$ is pre-open in $Y$.

Proof: Since $W$ is pre-open in $X$, then $W \subseteq \overline{W}$ so $W \cap Y \subseteq \overline{W} \cap Y = (\overline{W} \cap Y)^\gamma \subseteq (W \cap Y)^\gamma \subseteq (W \cap Y)^\gamma$ (since $Y$ is open)

$\subseteq (W \cap Y)^\gamma \cap Y^\gamma = (W \cap Y \cap Y)^\gamma$

therefore $W \cap Y \subseteq (W \cap Y)^\gamma$.

Proposition 2: If $W$ is $\omega_p$-open in a topological space $(X, \tau_X)$ and $(Y, \tau_Y)$ is a partial open set of $X$, then $W \cap Y$ is $\omega_p$-open set in $Y$.

Proof: Set $x \in W \cap Y$, so $x \in W$ with $x \in Y$, hence there is pre-open $G$ in $X$, with $x \in G$ and $G-W$ be countable, but $(G-W) \cap Y \subseteq (G-W) \cap Y$ is countable, and $(G-W) \cap Y = (G \cap Y) - (W \cap Y)$ will become countable, where $G \cap Y$ is pre-open set in $Y$ (by proposition 1), we get $W \cap Y$ is $\omega_p$-open set in $Y$.

Corollary 1: If $M$ is $\omega_p$-closed in the space $(X, \mu)$ with $(Y, \mu_Y)$ is subspace from $X$, then $M \cap Y$ is $\omega_p$-closed set in $Y$.

Definition 6: A function $f$ from space $X$ into space $Y$ we call it pre-open function when image of all preopen set at $X$ is preopen set at $Y$.

Lemma 1: If $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is bijective pre-open function, then the image of all $\omega_p$-open set at $X$ is $\omega_p$-open set at $Y$.

Proof: If $H$ is an $\omega_p$-open set at $X$, and $y \in f(H)$, so there is $x \in X$ so that $f(x)=y$ (because $f$ is bijective), since it is $y \in f(H)$, then $x=f^{-1}(y) \in f^{-1}(f(H))=f(H) \in f(H)$ (because one to one), hence $x \in H$ which is $\omega_p$-open, thus we have preopen set $W$ at $X$ where $x \in W$ and $W-H$ will be countable, so $f(W-H)$ is countable subset in $Y$, however $f(W-H)=f(W)-f(H)$, since $f(W)$ is preopen at $Y$ (because of $f$ is pre-open function), with $x \in W$, so $f(x)=y \in f(W)$, therefore $f(H)$ is $\omega_p$-open set in $Y$.

Proposition 3: A property of spaces which is $\omega_pT_i$-space, $i=0, 1, 2$ is open hereditary property.

Proof: Take $Y$ is a subspace of $\omega_pT_0$-space $X$ and $x, y$ as distinct points in $Y$, hence $x, y$ are distinct points in $X$ which is $\omega_pT_0$-space, so there is $\omega_p$-open subset $W$ in $X$ such that $x \in W, y \notin W$. We have $W \cap Y$ is $\omega_p$-open subset of $Y$(since $f$ is bijective), (by lemma 1) with $x \in W \cap Y, y \notin W \cap Y$ (because $x \in W$ and $x \in Y$ but $y \notin W$), therefore $Y$ is $\omega_pT_0$-space, which means $\omega_pT_0$-space is hereditary property.

Proposition 4: Set $f$ is function of $X$ into $Y$ be homomorphism then whether $X$ $\omega_pT_i$-space then $Y$ is $\omega_pT_i$, where $i=0, 1, 2$.

Proof: To prove $Y$ is $\omega_pT_0$-space, whenever $X$ is $\omega_pT_0$-space and $y_1, y_2$ are distinct points in $Y$, then there are distinct points $x_1, x_2$ in $X$ with $f(x_1)=y_1$, $f(x_2)=y_2$ (since $f$ is a bijective function), so there exists $\omega_p$-open subset $W$ of $X$ with $x_1 \in W$ and $x_2 \notin W$ (because $X$ is $\omega_pT_0$-space) there is open set $G$ containing $x$ with $G-W$=countable. So $f(G-W)=f(G) - f(W)$ is countable. $f(G) \subseteq f(G), f(G) = f(G) \subseteq f(G)$ then $f(G)$ is preopen set so $f(W)$ is $\omega_p$-open set $f(x_1)=y_1 \in f(W)$ and $f(x_2)=y_2 \notin f(W)$, where $f(W)$ will be $\omega_p$-open partial set of $Y$, therefore $Y$ is $\omega_pT_0$-space. By same method, we can prove the other part.

Proposition 5: A topological space $(X, \tau_X)$ is $\omega_pT_1$-space if and only if any singleton set $\{x\}$ of $X$ is $\omega_p$-closed subset of $X$. 

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Proof: Suppose any singleton subset \{x\} of \( X \) is \( \omega_p \)-closed subset of \( X \) for any \( x \in X \), and let \( x, y \) be distinct points in \( X \), so \( \{x\}^c, \{y\}^c \) are \( \omega_p \)-open sets containing \( y, x \) respectively, so \( X \) is \( \omega_p \)-T_1-space.

Conversely, let \( X \) be an \( \omega_p \)-T_1-space, to prove \( \{x\} \) is \( \omega_p \)-closed subset of \( X \), that means to prove \( \{x\}^c \) is \( \omega_p \)-open partial set from \( X \). Let \( y \in \{x\}^c \), so \( y \neq x \) and there is \( \omega_p \)-open set \( W \) at \( X \) such that \( y \in W, x \notin W \), so \( y \in W \subseteq \{x\} \), thus \( \{x\}^c \) is \( \omega_p \)-open set, therefore \( \{x\} \) is \( \omega_p \)-closed, but \( x \) is arbitrary point in \( X \), that means every singleton subset of \( X \) is \( \omega_p \)-closed.

**Proposition 6:** A topological space \((X, \tau_X)\) is \( \omega T_0 \)-Space iff \( cl_{\omega_p} (\{x\}) \neq cl_{\omega_p} (\{y\}) \) for each distinct points \( x \) and \( y \) in \( X \).

**Proof:** Suppose \( cl_{\omega_p} (\{x\}) \neq cl_{\omega_p} (\{y\}) \) with different elements \( x \) and \( y \) in \( X \), so there is at least one element exists in one of them and not in the other, say \( a \in cl_{\omega_p} (\{x\}), a \notin cl_{\omega_p} (\{y\}) \), and suppose \( x \notin cl_{\omega_p} (\{y\}) \), because if \( x \in cl_{\omega_p} (\{y\}) \) then \( cl_{\omega_p} (\{x\}) \subseteq cl_{\omega_p} (\{y\}) \).

Conversely, if \( X \) is \( \omega_p T_0 \)-space and \( x, y \) are distinct points in \( X \), so there is \( \omega_p \)-open set \( U \) of \( X \) with \( x \in U \) and \( y \notin U \), then \( X-U \) is \( \omega_p \)-closed set contains \( y \) but not \( x \), from definition of \( cl_{\omega_p} (\{y\}) \) we get, \( cl_{\omega_p} (\{y\}) \subseteq X-U \), which means \( x \notin cl_{\omega_p} (\{y\}) \) but \( x \in cl_{\omega_p} (\{x\}) \), so that \( cl_{\omega_p} (\{x\}) \neq cl_{\omega_p} (\{y\}) \).

**Definition 7:** Any space \( X \) called :
1- \( \omega_p \)-regular space for any point \( x \in X \) and with \( M \) closed at \( X \) and \( x \notin M \), there is no mutual points of \( \omega_p \)-open sets \( W_1, W_2 \) in \( X \) at which \( x \in W_1 \) and \( M \subseteq W_2 \).
2- \( \omega_p \)-regular space if for any point \( x \in X \) with all \( \omega_p \)-closed subset \( M \) of \( X \) with \( x \notin M \), so we have sets \( W_1, W_2 \in \tau \), with \( W_1 \cap W_2 = \emptyset \) in which \( x \in W_1 \) and \( M \subseteq W_2 \).
3- \( \omega_p \)-regular space if for any point \( x \in X \) with all \( \omega_p \)-closed subset \( M \) of \( X \) with \( x \notin M \), so we have \( \omega_p \)-open sets \( W_1, W_2 \) in \( X \), with \( W_1 \cap W_2 = \emptyset \) in which \( x \in W_1 \) and \( M \subseteq W_2 \).

**Remark 3:**
1) Regular space is \( \omega_p \)-regular but the converse is not true.
2) Every \( \omega_p \)- regular space is \( \omega_p \)-regular.
3) Every \( \omega_p \)- regular space is \( \omega_p \)-regular.

**Example 4:**
1- \( X=\{1,2,3\}, \tau=\{\emptyset, X,\{1\}\} \), we have \( \tau^c=\{X,\emptyset,\{2,3\}\} \).So \( X \) is not regular but it is \( \omega_p \)-regular.
2- Let \( X=\{1,2,3\}, \tau=\text{indirec} \) is regular, \( \omega_p \)-regular, and \( \omega_p \)-regular space, but not \( \omega_p \)-regular.

**Example 5:** The co-finite supra topological space \((Z, \tau_{cof})\) is \( \omega_p \)-regular and \( \omega_p \)-regular space while it is neither \( \omega_p \)-regular, nor a regular.

**Theorem 1:** A space \((X, \tau)\) be \( \omega_p \)-regular space iff for all point \( x \) in \( X \) with all \( \omega_p \)-neighborhood \( K \) to \( x \), having neighborhood \( W \) in \( X \) of \( x \) with \( cl(W) \subseteq K \).

**Proof:** If \( X \) be \( \omega_p \)-regular space, let \( x \in X \) and \( K \) be a \( \omega_p \)-neighborhood to \( x \), so there exists \( \omega_p \)-open set \( E \) at \( X \) and \( x \in E \subseteq K \), hence \( E^c \) is \( \omega_p \)-closed set at \( X \) and \( x \notin E^c \), but \( X \) is \( \omega_p \)-regular, thus we have two different sets open \( W, B \) at \( x \), \( x \in W, E^c \subseteq B \), then \( W \) is neighborhood to \( x \) and \( W \subseteq B^c \), where \( B^c \) is closed set in \( X \), therefore \( cl(W) \subseteq cl(B^c) = B^c \), which means \( cl(W) \subseteq B^c \). From (1) & (2) we have, \( cl(W) \subseteq B^c \subseteq E \subseteq K \Rightarrow cl(W) \subseteq K \).

Conversely, Set \( x \in X \) and \( M \) is \( \omega_p \)-closed set at \( X \) with \( x \notin M \), so \( x \in M^c \) which is \( \omega_p \)-open set in \( X \), then \( M^c \) is \( \omega_p \)-neighborhood to \( x \), so we have neighborhood \( W \) at \( X \) to \( x \) such that \( cl(W) \subseteq M^c \), since \( W \) is the neighborhood
to $x$, then there will be open set $W_1$ in $X$ with $x \in W_1 \subseteq W$, from $cl(W) \subseteq M^c$ we get $M \subseteq (cl(W))^c$, thus $(cl(W))^c$ is open subset of $X$ and since $W \cap W^c = \emptyset$, then $W_1 \cap (cl(W))^c = \emptyset$ (because $W_1 \subseteq W$ and because $W \subseteq cl(W)$, so $(cl(W))^c \subseteq W^c$). so for all point $x$ in $X$ with all $\omega_p$ closed set $M$ in $X$ where $x \notin M$ there is disjoint open sets $W_1, (cl(W))^c$ such that $x \in W_1$ and $M \subseteq (cl(W))^c$, that implies $X$ is $\omega_p^*$regular.

**Proposition 7:**
Every open subspace of $w_p$ regular space is $w_p$ regular.

**Proof:**
If $Y$ be subspace of an $\omega_p$ regular space $X$, take $M$ as a closed set at $Y$ and $q$ as any point at $Y$ such that $q \notin M$, so there is closed set $M'$ in $X$ such that $M = M' \cap Y$, it obvis $q \notin M$, because if opposite, we get $q \in M \cap Y$ that equal $M$ which is a contradiction, so $q \notin M$, but $X$ is $\omega_p$ regular, so that there are two $\omega_p$ open sets $W, B$ in $X$ with $Q \in W$, $M' \subseteq B$, with $W \cap B = \emptyset$, thus $W \cap B \cap Y$ are $\omega_p$ open in $Y$, in which $q \in W \cap Y$ and $M' \cap Y = M \subseteq B \cap Y$ and $(W \cap Y) \cap (B \cap Y) = (W \cap B) \cap Y = \emptyset \cap Y$ that equal $\emptyset$, so $Y$ is $\omega_p$ regular space.

**Lemma 2:**
Let $f: X \rightarrow Y$ be a homeomorphism function if $A$ is preopen in $X$, then $f(A)$ is preopen in $Y$.

**Proof:**
Set $A$ is preopen in $X$ then $A \subseteq int(cl(A))$ so by take $f$ for two sides then $f(A) \subseteq f(int(cl(A))$ ....(1) but is continuous , then $f(cl(A)) \subseteq clf(A)$ by take interior for two sides we get $int(f(cl(A)) \subseteq int(cl(A))$ ....(2) so by (1) and (2) we have $f(A) \subseteq int(cl(A))$ that is $f(A)$ is preopen in $Y$.

**Lemma 3:**
Let $f$ from $X$ into $Y$ be homeomorphism then if $G$ is preopen and $G \subseteq Y$, then $f^{-1}(G)$ is also preopen in $X$.

**Proof:**
Since $G$ is pre-open at $Y$ so $G \subseteq G^c$ then by take $f^{-1}$ for both sides we have $f^{-1}(G) \subseteq f^{-1}(G^c)=f^{-1}(G)^c$ so $f^{-1}(G) \subseteq f^{-1}(G)^c$ so $f^{-1}(G)$ is preopen in $X$.

**Lemma 4:**
Let $f$ from $X$ into $Y$ be homeomorphism function if $A$ be $\omega_p$ open and $A \subseteq X$, then $f(A)$ $\omega_p$ open and $f(A) \subseteq Y$.

**Proof:**
Set $A$ be $\omega_p$ open at $X$ to prove $f(A)$is $\omega_p$open at $Y$ , let $y \in f(A)$ then , $x = f^{-1}(y) \in f^{-1}(f(A)) = A$ but $A$ is $\omega_p$ open set then there exists preopen set $G$ in $X$ containing $x$ such that $G \cap A = \emptyset$ is countable so $f(G\cap A) = \text{countable}$ since $f$ is bijective but $f(G-A) = f(G) - f(A)$ but $f(G)$ is also preopen set by above lemma , then $f(G) - f(A) = \text{countable}$ , then $f(A)$ is also $\omega_p$ open set .

**Lemma 5:**
Let $f$ be a homeomorphism function then the inverse image of $\omega_p$ open set is also $\omega_p$ open.

**Proposition 8:**
1- The property of space $X$ being $\omega_p$regular is topological property.
2- The property of space $X$ we call it $\omega^*_p$ regular is topological property.
3- A property of space $X$ we call it $\omega^{**}_p$ regular is topological property.

**Proof:**
1- Suppose $(X, \tau_X)$ is $\omega_p$regular, with $f$ is homeomorphism function from a topological space $X$into a topological space $Y$, let $M$ be closed in $Y$ and $q$ is arbitrary point in $Y$ in which $q \notin M$, so there is point $p \in X$ with $f(p)=q$, we get $f^{-1}(M)$ is closed at $X$ (because $f$ is continuous function) and $f^{-1}(q) = p \notin f^{-1}(M)$, but $X$ is an $\omega_p$ regular space, so there are two $\omega_p$ open sets $W, B$ in $X$ where $p \in W$, $f^{-1}(M) \subseteq B$, with $W \cap B = \emptyset$, so $f(p) = q \in f(W)$ with $f(f^{-1}(M)) = M \subseteq f(B)$ where $f(W), f(B)$ are $\omega_p$ open sets in $Y$ (by lemma **), likewise $f(W) \cap f(B) = f(W \cap B) = f(\emptyset) = \emptyset$, hence $Y$ is $\omega_p$ regular, therefore the property space $X$ being $\omega_p$ regular is topological property.
2- If \( f : X \to Y \) is a homeomorphism function and X be \( \omega_p^* \) regular space to prove Y is also \( \omega_p^* \) regular, let \( q \in M \) be \( \omega_p \) closed set in Y and \( q \not\in M \) then \( p=f^{-1}(M) \) is \( \omega_p \) closed in X by (lemma 4) and \( X \omega_p^* \) regular space, then we have two open sets H, K with \( p \in H \) and \( f^{-1}(M) \subseteq K \) and \( H \cap K = \emptyset \), \( q \neq f(p) \in f(H) \) and \( M = f f^{-1}(M) \subseteq f(K) \) also \( f(H) \) and \( f(K) \) will be also open sets sincef is homeomorphism and \( f(H) \cap f(K) = f(H \cap K) = f(\emptyset) = \emptyset \) then Y is \( \omega_p^* \) regular space.

3- Prove by the same context.

**Theorem 2:**
A topological space \((X, \tau_X)\) is \( \omega_p \)-regular iff for every \( x \in X \) and for all open set \( U \) in \( X \) contained \( x \), there is \( \omega_p \)-open \( V \) in \( X \) with \( x \in V \subseteq \text{cl}_{\omega_p}(V) \subseteq U \).

**Proof:**
Suppose \( X \) is \( \omega_p \)-regular space, set \( x \in X \) and \( U \) is an open in \( X \) such that \( x \in U \), so \( U^c \) is closed in \( X \) that not containing \( x \), but \( X \) is \( \omega_p \)-regular space, so there are two \( \omega_p \)-open sets \( V, W \) with \( x \in V, U^c \subseteq W \) and \( V \cap W = \emptyset \), so \( V \subseteq W^c \), thus \( \text{cl}_{\omega_p}(V) \subseteq \text{cl}_{\omega_p}(W^c) = W^c \ldots (1) \), and since \( U^c \subseteq W \), then \( W^c \subseteq U \ldots (2) \), from (1) and (2) we get, \( x \in V \subseteq \text{cl}_{\omega_p}(V) \subseteq W^c \subseteq U \), which means \( x \in V \subseteq \text{cl}_{\omega_p}(V) \subseteq U \).

Conversely, let \( M \) be closed in \( X \) and \( x \) is arbitrary point in \( X \) with \( x \not\in M \), so \( M^c \) is open in \( X \) containing \( x \), then from a hypothesis there is \( \omega_p \)-open set \( V \) in \( X \) in which \( x \in V \subseteq \text{cl}_{\omega_p}(V) \subseteq M^c \), since \( \text{cl}_{\omega_p}(V) \subseteq M^c \), hence \( M^c = M \subseteq (\text{cl}_{\omega_p}(V))^c \) (since \( \text{cl}_{\omega_p}(V) \) is \( \omega_p \) closed set, so \( (\text{cl}_{\omega_p}(V))^c \) is \( \omega_p \) open set), and since \( V \subseteq \text{cl}_{\omega_p}(V) \), hence \( V \cap (\text{cl}_{\omega_p}(V))^c = \emptyset \). Therefore, there are two \( \omega_p \)-open sets \( V, (\text{cl}_{\omega_p}(V))^c \) in \( X \) such that \( x \in V, M \subseteq \text{cl}_{\omega_p}(V)^c \), and \( V \cap (\text{cl}_{\omega_p}(V))^c = \emptyset \), then \( X \) is \( \omega_p \)-regular space.

**Definition 8:**
A topological space \((X, \tau)\) we call it:

1- \( \omega_p T_3 \)-space if it be \( \omega_p \)-regular \( T^1 \)-space.

2- \( \omega_p^* T_3 \)-space if it is \( \omega_p^* \)-regular \( T^1 \)-space.

3- \( \omega_p^{**} T_3 \)-space if it is \( \omega_p^{**} \)-regular space and supra \( T^1 \)-space.

**Example 6:**
The discrete topological space \((R, \tau_D)\) is \( \omega_p, \omega_p^* \) and \( \omega_p^{**} T^3 \)-space.

**Remark 4:**
1- Every \( \omega_p T_3 \)-space is \( \omega_p \)-regular.

2- \( \omega_p \)-regular space need not be \( T^2 \)-space \((R,\tau_{ind})\).

3- \( \omega_p T^1 \)-space need not be \( \omega_p \)-regular \((R,\tau_{cof})\).

**Example 7:**
\((R,\tau_{ind})\) is \( \omega_p \)-regular and \( \omega_p T^2 \)-space but neither \( \omega_p^* \)-regular nor \( \omega_p T^3 \)-space.

**Definition 9** [4]:
If \( X \) is space we called \( X \) to be Excluded space, if \( \tau_{Ex} = \{U: U \subseteq X, \{x\} \not\subseteq U, \text{for some } x \in X \} \cup \{X\} \). Excluded space is neither \( T^1 \) nor regular space.

**Definition 10** [4]:
Set \( X \) be space we called \( X \) to be Included space if \( \tau_{In} = \{U: U \subseteq X, \{x\} \subseteq U, \text{for some } x \in X \} \cup \{\emptyset\} \). Included space is \( T^1 \) but not regular so it is not \( T^3 \).

**Example 8:**
To show that \((X, \tau_{In})\) is \( \omega_p T^3 \) since it is \( T^1 \) space now let \( \{2\} \subseteq X \) (since \( 1 \not\in \{2\} \)) i.e. \( R \subseteq \{2\} \subseteq \emptyset \) R so \( \{2\} = (R \setminus \{2\})^c \subseteq \text{closed } R \) and \( 3 \in R \) with \( 3 \not\in \{2\} \), since \( \{1,3\} \subseteq \text{open } R \) so it is \( \omega_p \)-open, T.p. \( \{2\} \) is \( \omega_p \)-open, \( 2 \subseteq \{2\} \subseteq \emptyset \), \( \{2\} \subseteq R \) containing \( \{2\} \), \( \{2\} \) is \( \omega_p \)-open, \( \{2\}^c = \emptyset \Rightarrow \{2\} \not\subseteq \{2\}^c \) i.e. \( \omega_p \)-open.

The following scheme is helpful.

**Proposition 9:**
For all \( \omega_p T_3 \) space be \( \omega_p T_2 \)-space.

**Proof:**
Suppose \((X, \tau_X)\) is \( \omega_p T_3 \) space, let \( x, y \) any different points in \( X \), we have \( X \) is \( T^1 \) space (from definition of \( \omega_p T_3 \) space), so \( \{x\} \) is closed and \( y \not\in \{x\} \), since \( X \) is \( \omega_p \)-regular so there are \( \omega_p \)-open sets \( W, B \) with \( \{x\} \subseteq W, y \in B \) also \( W \cap B \) equal \( \emptyset \), since \( x \in \{x\} \subseteq W \), therefore \( X \) is \( \omega_p T^2 \)-space.
Proposition 10:
1- Every $\omega_pT_3$-space is $\omega_pT_1$-space.
2- Every $\omega_p^*T_3$-space is $\omega_pT_2$-space.
3- Every $\omega_p^{**}T_3$-space is $\omega_pT_2$-space.

Proof:
1- As in proposition (9).
2- Set $x, y \in X$ and $x, y$ are not equal, since $X$ is $\omega_p^*T_3$-space $\Rightarrow$ $x \in T_3 \Rightarrow \{x\}$ is closed in $X$. $x \in [x], [x]$ is closed , $y \in [x]$ so $[x]$ is $\omega_p$ closed but $X$ is $\omega_p^*$-regular so then exists two open sets $W, B$ such that $[x] \subseteq W, y \in B$ but every open is $\omega_p$ open so we have $W, B$ are two $\omega_p$-open sets and $x \in [x] \subseteq W$, therefore $X$ is $\omega_pT_2$-space.
3- Similarly to the above proposition.

Definition 11:
A space $X$ we call it $\omega_p$-normal if for every closed two sets $F_1, F_2$ in $X$ then there exist two different $\omega_p$-open sets $H, K$ containing $F_1, F_2$ respectively.

Definition 12:
A space $X$ we call it $\omega_p^*$-normal if for any two $\omega_p$-closed sets $F_1, F_2$ in $X$, we have two different $\omega_p$-open sets $H, K$ containing $F_1, F_2$ respectively.

Definition 13:
A space $X$ we call it $\omega_p^{**}$-normal if for any two $\omega_p$-closed sets $F_1, F_2$ in $X$, we have two different $\omega_p$-open set $H, K$ containing $F_1$ and $F_2$ respectively.

Remarks 5:
1- Every normal space is $\omega_p$-normal but the converse is not true.
2- Every $\omega_p^*$-normal is $\omega_p$-normal.
3- Every $\omega_p^{**}$-normal is $\omega_p^*$-normal.
4- Every $\omega_p^{**}$-normal space is $\omega_p$-normal but the converse is not true.

Proposition 11:
A space $X$ is $\omega_p$-normal if to each closed set $F$ with an open set $U$ containing $F$ there exist an $\omega_p$-open set $V$, containing $F$ so that $F \subseteq V \subseteq \omega_p cl( V ) \subseteq U$. 

Theorem 3:
A space $X$ is $\omega_p$-normal iff for all closed set $F$ with any set $G$ containing $F$, there is an $\omega_p$-open set $V$ with $F \subseteq V \subseteq G$.

Proof:
Set $F$ is any closed set and $G$ is an open set with $F \subseteq G$, so $G^c$ is closed set and $F \cap G^c = \emptyset$. But $X$ is an $\omega_p$-normal, then we have $\omega_p$-open sets $U, V$ with $G^c \subseteq U, \emptyset \subseteq V$ and $U \cap V = \emptyset$ so that $V \subseteq U^c$ so $\omega_p cl(V) \subseteq \omega_p cl(U^c) \subseteq U^c \subseteq (1)$, since $U^c$ is $\omega_p$-closed set, but $G^c \subseteq U$, then $U^c \subseteq G \subseteq (2)$ by (1) and (2) we get $\omega_p cl (V) \subseteq G$ therefore there exist $\omega_p$-open set $V$ such that $F \subseteq V$ and $\omega_p cl( V ) \subseteq G$.

Conversely. Let $L$ and $M$ be closed subsets of $X$ with $L \cap M = \emptyset$ so that $L \subseteq M^c$, so by hypothesis there is an $\omega_p$-open set $V$ with $L \subseteq V$ and $\omega_p cl (V) \subseteq M^c$ so $M \subseteq (\omega_p cl(V))^c$ also $V \cap (\omega_p cl(V))^c = \emptyset$ then $V$ and $(\omega_p cl(V))^c$ are two different $\omega_p$-open sets, with $L \subseteq V, M \subseteq (\omega_p cl(V))^c$, therefore, $X$ is $\omega_p$-normal space.

CONCLUSIONS
In this paper, we introduce new types of separation axioms via $W_p$-open sets. In addition to this, we get many results and the most important of which are:

1. If $W$ is $\omega_p$ - open in a topological space $(X, \tau_X)$ and $(Y, \tau_Y)$ is a partial set of $X$, then $W \cap Y$ is $\omega_p$ - open set in $Y$.

2. A property of spaces which is $\omega_p T_i$-space, $i = 0, 1, 2$ is hereditary property.

3. A space $(X, \tau_X)$ be $\omega_p^*$ regular space iff for all point $x \in X$ with all $\omega_p$-neighborhood $K$ to $x$, having neighborhood $W$ in $X$ of $x$ with $cl(W) \subseteq K$.

4. A topological space $(X, \tau_X)$ is $\omega_p$ regular iff for every $x \in X and for all open set $U$ in $X$ contained $x$, there is $\omega_p$-open $V$ in $X$ with $x \in V \subseteq cl_{\omega_p}(V) \subseteq U$.

5. A space $X$ is $\omega_p$-normal iff for all closed set $F$ with any set $G$ containing $F$, there is an $\omega_p$-open set $V$ with $F \subseteq V \subseteq G$.

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