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New Arcs in $PG(3,8)$ by Singer Group

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ABSTRACT

In this paper, studied the types of (k, r)-arcs were constructed by action of groups on the three-dimensional projective space over the Galois field of order eight. Also, determined if they form complete arcs or not.

KEYWORDS: Arc; Galois field; Projective space; Singer group.

الخلاصة

في هذا البحث, تم دراسة انواع الاقواس(k, r)-منشئ بفعل عمل الزمر على الفضاء الاسقاطي ذو البعد الثالث على حقل كالواز من الرتبة الثامنة. كذلك تم تحديد فيما اذا كانت تشكل اقواس كاملة ام لا.

INTRODUCTION

Let $PG(n,q)$ be an n-dimensional projective space over the Galois field $GF(q) = F_q$, see [1-3].

The idea of group actions on the finite projective space has been used recently by many authors to find new arcs in particularly projective planes and lines as in [4-8] or to compute new caps in $PG(3,23)$ [9] and in $PG(3,8)$ in [10].

Al-Rikabi et. al. in [11] studied the projective space $PG(3,8)$ were they partitioned the space by subgeometries and subspaces. later on, they used special ten cyclic subgroups S_i of projective general linear $PGL(4,8)$ to do partitioned of the space and then construct caps as in [10].

The first aim of this paper is: formulate arcs, $(k;r)$-arc for $r = 3,4,5,6,7,9,13,21,27$, using the action of the subgroups S_i on $PG(3,8)$ and then classified it to complete arcs and incomplete arcs. The second aim is that: find points that make each incomplete arc complete.

There are many related research, which interested to compute arcs and caps in the projective space of dimension higher that two as in [12-15].

PRINCIPLE DEFINITIONS AND CONCEPTS

Definition 1.1[1][2]: A (k,l)-*set* in $PG(n,q)$ is a set of k l-subspaces. A k-*set* is a $(k,0)$-*set*, that is, a set of k points. The most general type of (k,l)-set that will be considered is a $(k,l;\ r,s;\ n,q)$-*set*; that is, a $(k\ ,l)$-set in $PG(n,q)$ with at most r l-subspaces in any s-subspaces.

For the special cases of $(k,l;\ r,s\ ;n,q)$-set the following are defined:

i- an $(k\ ;r,s;\ n\ ,q)$-*set is* an $(k,0;r,s;n,q)$-set*;*

ii- a k-*arc* is a $(k;n,n-1;n,q)$-set..

Definition 1.2[1][2]: A $(k;r)$-arc is a set of k points in $PG(n,q)$ with $r \geq 3$ such that at most r points of which lie in any plane. A $(k;r)$-arc is complete if it is not contained in a $(k+1;r)$-arc.

Definition 1.3[1][2]: A m-*secant* of an $(k;r)$-arc K in $PG(n,q)$ is a hyperplane $\mathcal{P}$ such that $|K \cap \mathcal{P}| = m$. Let Q be a point of $PG(n,q)$ not on the $(k;r)$-arc K.

Definition 1.4[1]: Let T_i be the total number from i-secants of an $(k;r)$-arc K, hence the type of K with respect to its hyperplanes denoted by $(T_r, T_{r-1}, \dots, T_0)$.

Let $\sigma_i(Q)$ be the number of i-secants through Q. The number $\sigma_r(Q)$ of r-secant is called the *index* of Q with respect to K. Let c_i be the number of points of index i and C_i be the set of points of index i. Therefore $(k;r)$-arc is complete if $c_i = 0$.

Definition 1.5[1][2]: A projectivity τ which permutes the $\theta(n,q)$ points of $PG(n,q)$ in a single cycle is called a cyclic projectivity (Singer cycle) and the group it generates a Singer group

Algorithm

Let τ be primitive element of F_8. In [11,12] the points of the space $PG(3,8)$ have calculated using the non-singular primitive polynomial $f(x) = X^4 - \tau^5 X^3 - X^2 - \tau^3 X - \tau^5$ to construct the companion matrix $T = M(A)$, which is a cyclic projectivity. has been used to construct points, lines and planes. Also, the space partitioned into subgeometries. This matrix is used also to find the planes in (3,8) .

The projective space $PG(3,8)$ has $\theta(3,8) = 585$ points and planes, 4745 lines, 9 points on each line and 73 lines passing through each point.

Let $p_1 = 3, p_2 = 3, p_3 = 5, p_4 = 13$. The ten integers $p_1,\ p_3,\ p_4,\ p_1p_2,\ p_1p_3,\ p_1p_4,\ p_4p_3,\ p_1p_2p_3,\ p_1p_2p_4,\ p_1p_3p_4$ are divided of $\theta(3,8)$. Let $S_i = \langle A^j \rangle$, where j one of these ten integers, are subgroups of $PGL(4,8)$.

The following algorithm is the same as in [10] but with a little modification is used to construct the arcs.

Algorithm 2.1: The procedures that used to prove the main theorem is as follows:

i- Finding the orbits for each non-trivial integer factor of 585 p_i from the action of cyclic group $\langle A^i \rangle$ on $PG(3,8)$.

ii- Finding the intersection between planes and orbits to know the degree of the arc that they formed.

iii- Determined if the arcs are complete or incomplete by finding the points of index zero for each arc.

iv- Adding points to the incomplete arc from the set of points of index zero to make it complete.

Note: The calculations have done using the Gap programming: https://www.gap-system.org/.

Arcs by Subgroups Action on the $PG(3,8)$

Throughout this paper, if $\langle A^i \rangle$ has j orbits, then the symbol A_j^i will denote the orbit j of $\langle A^i \rangle$ and $N^r_{A_j^i}$ =Number of planes which are intersect A_j^i of order r such that $0 \le r \le 73$.

From the action of $\langle A^i \rangle, i = p_1,\ p_3,\ p_4,\ p_1p_2,\ p_1p_3,\ p_1p_4,\ p_4p_3,\ p_1p_2p_3,\ p_1p_2p_4,\ p_1p_3p_4$, on the points of $PG(3,8)$, the following results are deduced:

Main Theorem 3.1:

I. The orbits $A_j^{p_1}; j = 1,2,3$ are complete $(195; 27)$-arcs.

II. The orbits of $A_j^{p_3}; j = 1, \dots, 5$ are complete $(117; 21)$-arcs.

III. The orbits of $A_j^{p_4}; j = 1, \dots, 13$ are complete $(45; 13)$-arcs.

IV. The orbits $A_j^{p_1p_2}; j = 1, \dots, 9.$ are complete $(65; 9)$-arcs **.**

V. The orbits $A_j^{p_1p_3}, j = 1, \dots, 15.$ are complete $(39; 9)$-arcs.

VI. The orbits of $A_j^{p_1p_4};\ j = 1, \dots, 39$ are complete $(15; 7)$-arcs.

VII. The orbits of $A_j^{p_3p_4};\ j = 1, \dots, 65$ are complete $(9; 9)$-arcs.

VIII. 1. The orbits of $A_j^{p_1p_2p_3}, j = 1, \dots, 45$ are incomplete $(13; 4)$-arcs.

2. The maximum complete arcs can be formed from the orbits of $A_j^{p_1p_2p_3}, j = 1, \dots, 45$ are $(14; 4)$-arcs.

3. The maximum complete arcs can be formed from the orbits of $A_j^{p_1p_2p_3}, j = 1, \dots, 45$ are $(19; 5)$-arcs

4. The maximum complete arcs can be formed from the orbits of $A_j^{p_1p_2p_3}, j = 1, \dots, 45$ are $(23; 6)$-arcs.

IX. 1. The orbits of $A_j^{p_1p_2p_4}, j = 1, \dots, 117$ are incomplete 5-arcs.

2. The maximum complete arc can be formed from the orbits of $A_j^{p_1p_2p_4}, j = 1, \dots, 117$ is 7-arcs.

3. The maximum complete arc can be formed from the orbits of $A_j^{p_1p_2p_4}, j = 1, \dots, 117$ is 9-arcs.

X. The orbits of $A_j^{p_1 p_3 p_4}, j = 1, \dots, 195$ are complete 3-arcs.

Proof:

I. The orbits of A^{p_1}: There are three orbits from the action of $\langle A^{p_1} \rangle$ on $PG(3,8)$, of size 195.

$$A_1^3 = \left\{\begin{matrix} 0,3,6,9,12,15,18,21,24,27,30,33,36, \\ 39,42,45,48,51,\dots,579,582 \end{matrix}\right\};$$

$$A_2^3 = \left\{\begin{matrix} 1,4,7,10,13,16,19,22,25,28,31,34,3, \\ 40,43,46,49,\dots,580,583 \end{matrix}\right\};$$

$$A_3^3 = \left\{\begin{matrix} 2,5,8,11,14,17,20,23,26,29,32,35,38, \\ 41,44,47,50,\dots,581,584 \end{matrix}\right\}.$$

The orbits $A_j^{p_1}, j = 1,2,3$ are (195; 27,2; 3,8)-sets of 195 points of degree 27 since $A_j^{p_1}, j = 1,2,3$ intersects each plane in at most 27 points in $PG(3,8)$, as shown in the equation below.

$$N_{A_j^{p_1}}^r = \begin{cases} 195 & if \left|A_j^{p_1} \cap \mathcal{P}_i\right| = 19 \\ 390 & if \left|A_j^{p_1} \cap \mathcal{P}_i\right| = 27 \end{cases};$$

$i = 1, \dots, 585.$

Therefore, the orbits $A_j^{p_1}, j = 1,2,3$ are (195; 27)-arcs additionally, it is complete (195; 27)-arcs since there are no points of index zero for $A_j^{p_1}$; that is, $c_0 = 0$.

II. The orbits of A^{p_3}: There are 5 orbits from the action of $\langle A^{p_3} \rangle$ on $PG(3,8)$ of size 117.

$$A_1^5 = \left\{\begin{matrix} 0,5,10,15,20,25,30,35,40,45, \\ 50,55,60,65,70,\dots,580 \end{matrix}\right\};$$

$$A_2^5 = \left\{\begin{matrix} 1,6,11,16,21,26,31,36,41,46, \\ 51,56,61,66,71,\dots,581 \end{matrix}\right\};$$

$$A_3^5 = \left\{\begin{matrix} 2,7,12,17,22,27,32,37,42, \\ 47,52,57,62,67,72,\dots,582 \end{matrix}\right\};$$

$$A_4^5 = \left\{\begin{matrix} 3,8,13,18,23,28,33,38,43,48, \\ 53,58,63,68,73,\dots,583 \end{matrix}\right\};$$

$$A_5^5 = \left\{\begin{matrix} 4,9,14,19,24,29,34,39,44,49, \\ 54,59,64,69,74,\dots,584 \end{matrix}\right\}.$$

The orbits of $A_j^{p_3}$, $j = 1, \dots, 5$ are (117; 21,2; 3,8)-sets of 117 points of degree 21 since $A_j^{p_3}, j = 1, \dots, 5$ intersect each plane in at most 21 points in $PG(3,8)$ as shown in the equation below.

$$N_{A_j^{p_3}}^r = \begin{cases} 468 & if \left|A_j^{p_3} \cap \mathcal{P}_i\right| = 13 \\ 117 & if \left|A_j^{p_3} \cap \mathcal{P}_i\right| = 21 \end{cases};$$

$i = 1, \dots, 585.$

Therefore, $A_j^{p_3}, j = 1, \dots, 5$ are (117; 21)-arcs. And it is complete (117; 21)-arcs since $c_0 = 0$.

III. The orbits of A^{p_4}: There are 13 orbits from the action of $\langle A^{p_4} \rangle$ on $PG(3,8)$, of size 45.

$$A_1^{13} = \left\{\begin{matrix} 0,13,26,39,52,65,78,91,104,117 \\ 130,143,156,169,182,195,\dots,572 \end{matrix}\right\};$$

$$A_2^{13} = \left\{\begin{matrix} 1,14,27,40,53,66,79,92,105,118, \\ 131,144,157,170,183,196,\dots,573 \end{matrix}\right\};$$

$$A_3^{13} = \left\{\begin{matrix} 2,15,28,41,54,67,80,93,106,119, \\ 132,145,158,171,184,197,\dots,574 \end{matrix}\right\};$$

$$\vdots$$

$$A_{13}^{13} = \left\{\begin{matrix} 12,25,38,51,64,77,90,103,116, \\ 129,142,155,168,181,194,\dots,584 \end{matrix}\right\}.$$

The orbits $A_j^{p_4}, j = 1, \dots, 13$ are intersection of 45 plane in at most 13 points, as shown as follows:

$$N_{A_j^{p_4}}^r = \begin{cases} 540 & if \left|A_j^{p_4} \cap \mathcal{P}_i\right| = 5 \\ 45 & if \left|A_j^{p_4} \cap \mathcal{P}_i\right| = 13 \end{cases},$$

$i = 1, \dots, 585.$

Thus the orbits of $A_j^{p_4}, j = 1, \dots, 13$ are (45; 13,2; 3,8)- sets; that is, (45; 13)-arcs. Since $c_o = 0$, then it is complete arcs.

IV. The orbits of $A^{p_1 p_2}$: There are 9 orbits from the action of $\langle A^{p_1 p_2} \rangle$ on $PG(3,8)$, of size 65.

$$A_1^9 = \left\{\begin{matrix} 0,9,18,27,36,45,54,63,72,81,90, \\ 99,108,117,126,135,144,153,\dots,576 \end{matrix}\right\};$$

$$A_2^9 = \left\{\begin{matrix} 1,10,19,28,37,46,55,64,73,82,91, \\ 100,109,118,127,136,145,\dots,577 \end{matrix}\right\};$$

$$\vdots$$

$$A_9^9 = \left\{\begin{matrix} 8,17,26,35,44,53,62,71,80,89 \\ ,98,107,116,125,134,143,\dots,584 \end{matrix}\right\}.$$

The orbits $A_j^{p_1 p_2}, j = 1, \dots, 9$ are intersection of 520 planes in at most 9 points in $PG(3,8)$, as shown as follows:

$$N_{A_j^{p_1 p_2}}^r = \begin{cases} 65 & if \left|A_j^{p_1 p_2} \cap \mathcal{P}_i\right| = 1 \\ 520 & if \left|A_j^{p_1 p_2} \cap \mathcal{P}_i\right| = 9 \end{cases};$$

$= 1, \dots, 585.$

Therefore, the orbits $A_j^{p_1 p_2}, j = 1, \dots, 9$ are (65; 9; 2; 3,8)- sets; that is, (65; 9)-arcs, and the orbits $A_j^{p_1 p_2}; j = 1, \dots, 9$ are complete (65; 9)-arcs, since there are no points of index zero for $A_j^{p_1 p_2}$; that is, $c_0 = 0$.

V. The orbits of $A^{p_1p_3}$: There are 15 orbits from the action of $\langle A^{p_1p_3}\rangle$ on $PG(3,8)$ of size 39.

$$A_1^{15} = \begin{Bmatrix} 0,15,30,45,60,75,90,105,120, \\ 135,150,165,180,195,210,\dots,570 \end{Bmatrix};$$

$$A_2^{15} = \begin{Bmatrix} 1,16,31,46,61,76,91,106,121, \\ 136,151,166,181,196,211,\dots,571 \end{Bmatrix};$$

$$\vdots$$

$$A_{15}^{15} = \begin{Bmatrix} 14,29,44,59,74,89,104,119,134, \\ 149,164,179,194,209,224,\dots,584 \end{Bmatrix}.$$

The orbits $A_j^{p_1p_3}, j = 1,\dots,15$ are intersection of 78 planes in at most 9 points in $PG(3,8)$ such that

$$N^r_{A_j^{p_1p_3}} = \begin{cases} 195 & if\left|A_j^{p_1p_3} \cap \mathcal{P}_i\right| = 3 \\ 156 & if\left|A_j^{p_1p_2} \cap \mathcal{P}_i\right| = 4 \\ 156 & if\left|A_j^{p_1p_2} \cap \mathcal{P}_i\right| = 6 \\ 78 & if\left|A_j^{p_1p_2} \cap \mathcal{P}_i\right| = 9 \end{cases};$$

$i = 1,\dots,585.$

Thus, the orbits $A_j^{p_1p_3}, j = 1,\dots,15$ are (39; 9,2; 3,8)- sets; that is, (39; 9)-arcs, and it is complete since $c_0 = 0$

VI. The orbits of $A^{p_1p_4}$: There are 39 orbits from the action of $\langle A^{p_1p_4}\rangle$ on $PG(3,8)$, of size 15.

$$A_1^{39} = \begin{Bmatrix} 0,39,78,117,156,195,234,273, \\ 312,351,390,429,468,507,546 \end{Bmatrix};$$

$$A_2^{39} = \begin{Bmatrix} 1,40,79,118,157,196,235,274, \\ 313,352,391,430,469,508,547 \end{Bmatrix};$$

$$A_3^{39} = \begin{Bmatrix} 2,41,80,119,158,197,236,275 \\ 314,353,392,431,470,509,548 \end{Bmatrix};$$

$$A_4^{39} = \begin{Bmatrix} 3,42,81,120,159,198,237,276, \\ 315,354,393,432,471,510,549 \end{Bmatrix};$$

$$\vdots$$

$$A_{39}^{39} = \begin{Bmatrix} 38,77,116,155,194,233,272,311, \\ 350,389,428,467,506,545,584 \end{Bmatrix}.$$

The orbits $A_j^{p_1p_4}, j = 1,\dots,39$ are intersection of 15 planes in at most 7 points, as shown as follows:

$$N^r_{A_j^{p_1p_4}} = \begin{cases} 360 & if\left|A_j^{p_1p_4} \cap \mathcal{P}_i\right| = 1 \\ 210 & if\left|A_j^{p_1p_4} \cap \mathcal{P}_i\right| = 3 \\ 15 & if\left|A_j^{p_1p_4} \cap \mathcal{P}_i\right| = 7 \end{cases};$$

$i = 1,\dots,585.$

Therefore, the orbits $A_j^{p_1p_4}, j = 1,\dots,39$ are (15; 7,2; 3,8)- sets; that is, (15; 7)-arcs, which are complete, since $c_o = 0$.

VII. The orbits of $A^{p_3p_4}$: There are 65 orbits from the action of $\langle A^{p_3p_4}\rangle$ on $PG(3,8)$, of size 9.

$$A_1^{65} = \begin{Bmatrix} 0,65,130,195,260,325,390, \\ 455,520 \end{Bmatrix};$$

$$A_2^{65} = \begin{Bmatrix} 1,66,131,196,261,326,391, \\ 456,521 \end{Bmatrix};$$

$$\vdots$$

$$A_{65}^{65} = \begin{Bmatrix} 64,129,194,259,324,389,454, \\ 519,584 \end{Bmatrix}.$$

The orbits $A_j^{p_3p_4}, j = 1,\dots,65$ are intersection of 9 planes in at most 9 points as shown below.

$$N^r_{A_j^{p_3p_4}} = \begin{cases} 576 & if\left|A_j^{p_3p_4} \cap \mathcal{P}_i\right| = 1 \\ 9 & if\left|A_j^{p_3p_4} \cap \mathcal{P}_i\right| = 9 \end{cases};$$

$i = 1,\dots,585.$

Thus, the orbits $A_j^{p_3p_4}, j = 1,\dots,65$ are (9; 9,2; 3,8)- sets; that is, (9; 9)-arcs and it is complete (9; 9)-caps since $c_0 = 0$.

VIII. The orbits of $A^{p_1p_2p_3}$: There are 45 orbits from the action of $\langle A^{p_1p_2p_3}\rangle$ on $PG(3,8)$, of size 13.

$$A_1^{45} = \begin{Bmatrix} 0,45,90,135,180,225,270, \\ 315,360,405,450,495,540 \end{Bmatrix};$$

$$A_2^{45} = \begin{Bmatrix} 1,46,91,136,181,226,271,316, \\ 361,406,451,496,541 \end{Bmatrix};$$

$$\vdots$$

$$A_{45}^{45} = \begin{Bmatrix} 44,89,134,179,224,269,314, \\ 359,404,449,494,539,584 \end{Bmatrix}.$$

1. The orbits $A_j^{p_1p_2p_3}, j = 1,\dots,45$ are intersection of 52 planes in at most 4 points as shown in an equation:

$$N^r_{A_j^{p_1p_2p_3}} = \begin{cases} 104 & if\left|A_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 0 \\ 195 & if\left|A_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 1 \\ 156 & if\left|A_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 2 \\ 78 & if\left|A_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 3 \\ 52 & if\left|A_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 4 \end{cases};$$

$i = 1,\dots,585.$

For that reason The orbits $A_j^{p_1p_2p_3}, j = 1,\dots,45$ are (13; 4,2; 3,8)- sets; that is, (13; 4)-arcs, and the orbits of $A_j^{p_1p_2p_3}$ are incomplete (13; 4)-arcs since $c_0 = 26$.

2. We can construct a complete arc from the $(k;r)$-arc by including external points such that $C_o = \begin{Bmatrix} 15,30,60,75,105,120,150,165, \\ 195,210,240,255,285,300, \\ 330,345,375,390,420,435, \\ 465,480,510,525,555,570 \end{Bmatrix}$.

The maximum complete $(14;4)$-arc constructed from A_1^{45} is $\xi_1 = A_1^{45} \cup \{15\}; \{15\} \subseteq C_0$. Additionally, we can add any one point of C_0 to the A_1^{45} to get 26 complete $(14;4)$-arcs. And $\xi_j^{p_1p_2p_3}, j = 1, \dots, 45$ are intersection of 70 planes in at most 4 points as shown in an equation:

$$N^r_{\xi_j^{p_1p_2p_3}} = \begin{cases} 88 & if \left|\xi_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 0 \\ 196 & if \left|\xi_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 1 \\ 1477 & if \left|\xi_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 2 \\ 84 & if \left|\xi_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 3 \\ 70 & if \left|\xi_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 4 \end{cases};$$

$i = 1, \dots, 585.$

And $A_j^{p_1p_2p_3}, j = 2, \dots, 45$ are incomplete $(13;4)$-arcs since $c_0 \neq 0$. The maximum complete $(14;4)$-arc constructed from A_j^{45} is $\xi_j^{p_1p_2p_3} = A_j^{45} \cup \{x_j\}$ where $\{x_j\} \subseteq C_0^j$ =set of all external points of A_j^{45}, $j = 1, \dots, 45$ of index zero.

3. Use previous point information in the proof, then a complete $(19;4)$-arc is constructed from the complete $(14;4)$-arc.

Let $\beta_1 = \xi_1 \cup \{30,105,480,84,426\} = A_1^{45} \cup \{15,30,105,480,84,426\}$ such that $\{15,30,105,480,84,426\} \subseteq C_o$, then β_1 is the maximum complete $(19;5)$-arc since $c_0 = 0$. And $\beta_j^{p_1p_2p_3}, j = 1, \dots, 45$ are intersection of 53 planes in at most 5 points as shown in an equation:

$$N^r_{\beta_j^{p_1p_2p_3}} = \begin{cases} 40 & if \left|\beta_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 0 \\ 149 & if \left|\beta_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 1 \\ 148 & if \left|\beta_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 2 \\ 103 & if \left|\beta_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 3 \\ 92 & if \left|\beta_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 4 \\ 53 & if \left|\beta_j^{p_1p_2p_3} \cap \mathcal{P}_i\right| = 5 \end{cases};$$

$i = 1, \dots, 585.$

The maximum complete $(19;5)$-arc constructed form A_j^{45} is $A_j^{45} \cup Z$ where $Z \subseteq C_0^j$ =set of all external points of A_j^{45}, $j = 1, \dots, 45$ of index zero.

4. Let $\mu_1 = A_1^{45} \cup \begin{Bmatrix} 15,30,,60,105,480,165, \\ 240,426,43,86 \end{Bmatrix}$, then μ_1 is the maximum complete $(23;6)$-arc since $c_0 = 0$. And μ_1 intersection of 40 planes in at most 6 points as shown in an equation:

$$N^r_{\mu_1} = \begin{cases} 17 & if\ |\mu_1 \cap \mathcal{P}_i| = 0 \\ 107 & if\ |\mu_1 \cap \mathcal{P}_i| = 1 \\ 150 & if\ |\mu_1 \cap \mathcal{P}_i| = 2 \\ 109 & if\ |\mu_1 \cap \mathcal{P}_i| = 3 \\ 105 & if\ |\mu_1 \cap \mathcal{P}_i| = 4 \\ 57 & if\ |\mu_1 \cap \mathcal{P}_i| = 5 \\ 40 & if\ |\mu_1 \cap \mathcal{P}_i| = 6 \end{cases};$$

$i = 1, \dots, 585.$

The maximum complete $(23;6)$-arc constructed form A_j^{45} is:

$\mu_j^{p_1p_2p_3} = A_j^{45} \cup Z$, where $Z \subseteq C_0^j$ =set of all external points of A_j^{45}, $j = 1, \dots, 45$ of index zero.

IX. The orbits of $A^{p_1p_2p_4}$: There are 117 orbits from the action of $\langle A^{p_1p_2p_4} \rangle$ on $PG(3,8)$, of size 5.

$A_1^{117} = \{0, 117, 234, 351, 468\};$

$A_2^{117} = \{1, 118, 235, 352, 469\};$

$\vdots$

$A_{177}^{117} = \{116,233,350,467,584\}.$

1. The orbits $A^{p_1p_2p_4}, j = 1, \dots, 117$ are intersection of 10 planes in at most 3 points as shown as follows:

$$N^r_{A_j^{p_1p_2p_4}} = \begin{cases} 300 & if\ |A^{p_1p_2p_4} \cap \mathcal{P}_i| = 0 \\ 215 & if |A^{p_1p_2p_4} \cap \mathcal{P}_i| = 1 \\ 60 & if |A^{p_1p_2p_4} \cap \mathcal{P}_i| = 2 \\ 10 & if |A^{p_1p_2p_4} \cap \mathcal{P}_i| = 3 \end{cases};$$

$i = 1, \dots, 585.$

Therefore, the orbits $A^{p_1p_2p_4}, j = 1, \dots, 117$ are $(5;3,2;3,8)$- sets; that is, 5-arcs and it is incomplete 5-caps since $c_0 = 120$, such that

$$C_0 = \left\{ \begin{array}{c} 1,2,4,8,11,16,21,22,32,42,44,51, \\ 57,59,64,69,84,87,88,93,102,105, \\ 111,114,118,119,121,125,128, \\ 133,138,139,149,159,161,168,174, \\ 176,181,186,201,204,205,210,219, \\ 222,228,231,235,236,238,242, \\ 245,250,255,256,266,276,278, \\ 285,291,293,298,303,318,321, \\ 322,327,336,339,345,348,352, \\ 353,355,359,362,367,372,373, \\ 383,393,395,402,408,410,415, \\ 420,435,438,439,444,453,456, \\ 462,465,469,470,472,476,479, \\ 484,489,490,500,510,512,519, \\ 525,527,532,537,552,555,556, \\ 561,570,573,579,582 \end{array} \right\}$$

2. We can construct a complete arc from the k-arc by including external points such that $Z = \{1,22\} \subseteq C_0$. The complete k-arc constructed from A_1^{117} is $\xi_1 = A_1^{117} \cup Z$ is complete 7-arc. And $\xi_2 = A_1^{117} \cup \{1,16\}$, $\xi_3 = A_1^{117} \cup \{1,250\}$, $\xi_4 = A_1^{117} \cup \{1,256\}$, $\xi_5 = A_1^{117} \cup \{1,318\}$, $\xi_6 = A_1^{117} \cup \{1,408\}$,

$\xi_7 = A_1^{117}\{1,582\}$ are complete 7-arcs. And $\xi_j^{p_1p_2p_4}, j = 1, \dots ,117$ are intersection of 35 planes in at most 3 points as shown as follows:

$$N^r_{\xi_j^{p_1p_2p_4}} = \begin{cases} 228 & if \left|\xi_j^{p_1p_2p_4} \cap \mathcal{P}_i\right| = 0 \\ 238 & if \left|\xi_j^{p_1p_2p_4} \cap \mathcal{P}_i\right| = 1 \\ 84 & if \left|\xi_j^{p_1p_2p_4} \cap \mathcal{P}_i\right| = 2 \\ 35 & if \left|\xi_j^{p_1p_2p_4} \cap \mathcal{P}_i\right| = 3 \end{cases};$$

$$i = 1, \dots ,585.$$

$A_j^{p_1p_2p_4}, j = 2, \dots ,117$ are incomplete 5-arcs since $c_0 \neq 0$. The complete 7-arc constructed from A_j^{117}sufficient add for two points of C_0 is $\xi_j^{p_1p_2p_4} = A_j^{117} \cup Z_j$ where $Z_j \subseteq C_0^j$ =set of all external points of $A_j^{117}, j = 1, \dots ,117$ of index zero.

3. We can construct a complete arc from the 9-arc by including external points such that $Z = \{1,64,105,285\} \subseteq C_0$. The maximum complete k-arc constructed from A_1^{117} is $\beta_1 = A_1^{117} \cup Z$ is complete 9-arc. And $\beta_j^{p_1p_2p_4}, j = 1, \dots ,117$ are intersection of 84 planes in at most 3 points as shown as follows:

$$N^r_{\beta_j^{p_1p_2p_4}} = \begin{cases} 168 & if \left|\beta_j^{p_1p_2p_4} \cap \mathcal{P}_i\right| = 0 \\ 261 & if \left|\beta_j^{p_1p_2p_4} \cap \mathcal{P}_i\right| = 1 \\ 72 & if \left|\beta_j^{p_1p_2p_4} \cap \mathcal{P}_i\right| = 2 \\ 84 & if \left|\beta_j^{p_1p_2p_4} \cap \mathcal{P}_i\right| = 3 \end{cases};$$

$$i = 1, \dots ,585.$$

$A_j^{p_1p_2p_4}, j = 2, \dots ,117$ are incomplete 5-arcs since $c_0 \neq 0$. The maximum complete k-arc constructed from A_j^{117}sufficient add four points in C_0 is $\beta_j^{p_1p_2p_4} = A_j^{117} \cup Z_j$ where $Z_j \subseteq C_0^j$ =set of all external points of A_j^{117}, $j = 1, \dots ,117$ of index zero.

X. The orbits of $A^{p_1p_3p_4}$: There are 195 orbits from the action of $A_j^{p_1p_3p_4}$ on $PG(3,8)$, of size 3.

$$A_1^{195} = \{0, 195, 390\};$$
$$A_2^{195} = \{1, 196, 391\};$$
$$\vdots$$
$$A_{195}^{195} = \{194, 389, 584\}.$$

The orbits $A_j^{p_1p_3p_4}, j = 1, \dots ,195$ are intersection of 9 planes in at most 3 points as shown as follows:

$$N^k_{A_j^{p_1p_3p_4}} = \begin{cases} 384 & if \left|A_j^{p_1p_3p_4} \cap \mathcal{P}_i\right| = 0 \\ 192 & if \left|A_j^{p_1p_3p_4} \cap \mathcal{P}_i\right| = 1 \\ 9 & if \left|A_j^{p_1p_3p_4} \cap \mathcal{P}_i\right| = 3 \end{cases};$$

$$i = 1, \dots ,585.$$

Thus the orbits $A_j^{p_1p_3p_4}, j = 1, \dots ,195$ are (3; 3,2; 3,8)-sets; that is, 3 –arcs. Also, it is complete since it is just a line in the plane, and resulting from the intersection of nine planes such that the union of these planes covered the whole space; that is, $c_o = 0$.

CONCLUSIONS

In this paper, we are founded nine types of distinct $(k; r)$- arcs in $PG(3,8)$ with respect to r where $r = 3,4,5, 6,7, 9,13,21,27$ as follows:

Deg.	Orbits	No.	$(\boldsymbol{k};\boldsymbol{r})$-arc
3	A^{195}	195	Complete 3-arc
	A^{117}	117	Incomplete 5-arc
	δ^{117}	14040	Incomplete 6-arc
	ξ^{117}	7020	Complete 7-arc

	β^{117}	3510	Complete 9-arc
4	A^{45} ξ^{45}	45 1170	Incomplete $(13;4)$-arc Complete $(14;4)$-arc
5	β^{45}	45	Complete $(19;5)$-arc
6	μ^{45}	45	Complete $(23;6)$-arc
7	A^{39}	39	Complete $(15;7)$-arc
9	A^{9} A^{15} A^{65}	9 15 65	Complete $(65;9)$-arc Complete $(39;9)$-arc Complete $(9;9)$-arc
13	A^{13}	13	Complete $(45;13)$-arc
21	A^{5}	5	Complete$(117;21)$-arc
27	A^{3}	3	Complete$(195;27)$-arc

Deg.	Type$(T_r, T_{r-1}, \ldots, T_0)$
3	(9,0,192,384) (10,60,215,300) (20,75,228,262) (35,84,238,228) (84,72,261,168)
4	(52,78,156,195,104) (70,84,147,196,88)
5	(53,92,103,148,149,40)
6	(40,57,105,109,150,107,17)
7	(15,0,0,0,210,0,360,0)
9	(520,0,0,0,0,0,0,0,65,0) (78,0,0,156,0,156,195,0,0,0) (9,0,0,0,0,0,0,0,576,0)
13	(45,0,0,0,0,0,0,0,540,0,0,0,0,0)
21	(117,0,0,0,0,0,0,0,468,0,0,0,0,0,0, ... ,0,0)
27	(390,0,0,0,0,0,0,0195,0,0,0,0,0, ... ,0,0,0)

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