On Intuitionistic Fuzzy Quasi Metric Space

Alaa J. Edan, Sameer Q. Hasan*

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, IRAQ.

*Correspondent contact: dr.sameerqasim@uomustansiriyah.edu.iq

ABSTRACT

In this article the fixed point of intuitionistic fuzzy quasi metric space instabilities on two metrics $d_{1}(f, g)$ and $D_{x}(f, g)$ on complicity space $X$ have been studied and presented in details depended on $(0,1]$-fuzzy contractive map. The illustrative example is proposed to explain the present results using quick sort algorithm.

KEYWORDS: Polycyclic Aromatic Hydrocarbons; leaf Plants.

INTRODUCTION

Firstly, the concept of fuzzy sets is introduced by Zadeh in 1963, [10]. More researchers concept on fuzzy metric space, can be found in Kramos and Michalek [6]. Grabiec in [4] provided conclusive evidence for basic space which is called fuzzy metric spaces and contraction theory. Additionally, George and Veramani in [3] extended the concept of fuzzy space by using the $t$-norm. Some authors have investigated fuzzy fixed-point theorems such as Sojai [9], George and Veramani in [3] also Grabiec [4], Kramosil and Michalek [6] and Sharma [8]. In [5], The Banach fixed point has been studied by applied a context of words on fuzzy metric space. In [7], the fixed point studied on intuitionistic fuzzy metric spaces. In [1] the complexity analysis sport to contractive maps with some important results in fuzzy quasi-metric spaces are provided. In [2,4] the concept of intuitionistic related to fuzzy quasi-metric spaces with application that explain the fixed point theorem have been provided in details

The aim of this paper focused on the fixed point of intuitionistic fuzzy quasi metric space, so we are depended on two metrics for each concept of $M_{x}(f,g,t)$ and $N_{x}(f,g,t)$, for each $f, g \in X$ and $t > 0$.

BASIC DEFINITION

Definition(2.1)[1]:
Consider the following space

$X = \{ f : w \to (0, \infty) : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \}$

Such that $w$ defined as the set of integers with sign nonnegative and $d_{x}$ is the quasi metric on $X$ given by

$d_{x}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left( \frac{1}{g(n)} - \frac{1}{f(n)} \right) V 0$

where $V$ that is proved or zero
In order of defined (2.1) we develop the as following definition.

Definition(2.2):

The complexity intuitionistic fuzzy quasi metric defined as follows:

The auxiliary functions for $\hat{f}, \hat{g} \in X$, $t > 0$, and defined a continuous function $\alpha_{1}, \alpha_{2} : X \to X$ as $\alpha_{1}(f) = \hat{f}$, $\alpha_{2}(\hat{g}) = \hat{g}$

$D_{x}(f, g, t) = \sum_{k=0}^{\infty} 2^{-k} \left( \frac{1}{g(k)} - \frac{1}{f(k)} \right) V 0$ , also
\[ V_x(\hat{g}, \hat{f}, t) = \sum_{k=n}^{\infty} 2^{-k} \left( \left( \frac{i}{f(k)} - \frac{i}{\hat{g}(k)} \right) V_0 \right) \]

where \( t \in (n, n+1] \) such that \( n \in \mathbb{N} \), and \( t \) the time between two algorithm \( \hat{f}, \hat{g} \) provided \( \hat{f}(k) \neq \hat{g}(k) \neq 0 \) for \( k = n \to \infty \).

**Remark (2.3):**

For each \( \hat{f}, \hat{g} \in X \) and \( t > 0 \) we have

\[ D_x(\hat{f}, \hat{g}, t) \leq \sum_{k=0}^{\infty} 2^{-k} \left( \left( \frac{1}{\hat{g}(k)} - \frac{1}{f(k)} \right) V_0 \right) = d_x(\hat{f}, \hat{g}), \]

\[ V_x((\hat{g}, \hat{f}, t) \geq \sum_{k=0}^{\infty} 2^{-k} \left( \left( \frac{1}{\hat{f}(k)} - \frac{1}{\hat{g}(k)} \right) V_0 \right) = d_x(\hat{f}, \hat{g}), \]

In particular, for each \( \hat{f}, \hat{g} \in X \) and \( t \in (0, 1] \), we have

\[ D_x(\hat{f}, \hat{g}, t) = d_x(\hat{f}, \hat{g}) \quad \text{also} \quad V_x((\hat{g}, \hat{f}, t) = d_x(\hat{f}, \hat{g}). \]

**MAIN RESULTS**

**Lemma (3.1):**

For each \( \hat{f}, \hat{g}, \hat{h} \in X \) and \( s > 0 \) it follows

1. \( D_x(\hat{f}, \hat{g}, t+s) \leq D_x(\hat{f}, \hat{h}, t) + D_x(\hat{h}, \hat{g}, s) \)
2. \( V_x((\hat{g}, \hat{f}, t+s) \geq V_x((\hat{h}, \hat{f}, t) + V_x(\hat{h}, \hat{f}, s). \)

**Proof:**

Let \( t \in [n, n+1] \) and \( s \in [m, m+1] \) with \( n, m \in \mathbb{N} \). Then \( t+s \in [n+m, n+m+1] \) or \( t+s \in [n+m+1, n+m+2]. \) Hence

\[ D_x(\hat{f}, \hat{g}, s) \leq \sum_{k=n+m}^{\infty} 2^{-k} \left( \left( \frac{i}{\hat{h}(k)} - \frac{i}{\hat{f}(k)} \right) V_0 \right) \]

\[ \leq \sum_{k=n+m}^{\infty} 2^{-k} \left( \left( \frac{i}{\hat{g}(k)} - \frac{i}{\hat{f}(k)} \right) V_0 \right) \]

\[ + \sum_{k=n+m}^{\infty} 2^{-k} \left( \left( \frac{i}{\hat{f}(k)} - \frac{i}{\hat{g}(k)} \right) V_0 \right) = D_x(\hat{h}, \hat{f}, t) + D_x(\hat{g}, \hat{h}, s). \]

Also

\[ V_x((\hat{g}, \hat{f}, t+s) \geq V_x((\hat{g}, \hat{h}, t) + V_x(\hat{h}, \hat{f}, s). \)

**Definition (3.2):**

1. \( M(x, y, t) + N(x, y, t) \leq 1 \) for all \( t \geq 0; \)
2. \( M(x, y, 0) = 0; \)
3. \( x = y \) if and only if \( M(x, y, t) = M(y, x, t) = 1 \) for all \( t > 0; \)
4. \( M(x, y, t) \land M(y, z, s) \leq M(x, z, t+s) \) for all \( t, s \geq 0; \)
5. \( M(x, y, -) : [0, \infty) \to [0, 1] \) is left continuous; \)
6. \( N(x, y, 0) = 1; \)
7. \( x = y \) if and only if \( N(x, y, t) = N(y, x, t) = 0 \) for all \( t > 0; \)
8. \( N(x, y, t) \lor N(y, z, s) \leq N(x, z, t+s) \) for all \( t, s \geq 0; \)
9. \( N(x, y, -) : [0, \infty) \to [0, 1] \) is left continuous.

**Theorem (3.3):**

For each \( D_x(\hat{f}, \hat{g}, t) \) and \( V_x(\hat{g}, \hat{f}, t) \in X \times X \times [0, \infty) \)

\[ M_x(\hat{f}, \hat{g}, t) = \frac{t}{t+D_x(\hat{f}, \hat{g}, t)}, \]

and \( N_x(\hat{g}, \hat{f}, t) = \frac{1-N_x(\hat{g}, \hat{f}, t)}{t+1-N_x(\hat{g}, \hat{f}, t)} \). Then \( N_x(\hat{g}, \hat{f}, t) = N_{d_x}(\hat{g}, \hat{f}, t) \). Whenever \( t > 1 \) where \( (M_x, N_x, \land, \lor) \) is an intuitionistic fuzzy quasi metric on \( X, \)

\[ d_{\hat{x}}(\hat{f}, \hat{g}, t) = \frac{t}{t+d_{\hat{x}}(\hat{f}, \hat{g})}, \]

and \( N_{\hat{x}}(\hat{g}, \hat{f}, t) = N_{d_{\hat{x}}}(\hat{g}, \hat{f}, t) \). Whenever \( t > 1 \) where \( (M_{\hat{x}}, N_{\hat{x}}, \land, \lor) \) is a intuitionistic fuzzy metric on \( X. \)

1: Since \( M_x(\hat{f}, \hat{g}, t) = 1 \) and \( N_x(\hat{g}, \hat{f}, t) = 0 \)

Then clearly

\[ M_x(\hat{f}, \hat{g}, t) + N_x(\hat{g}, \hat{f}, t) \leq 1 \text{ for all } \hat{f}, \hat{g} \in X \]

and \( t > 0; \)

2: Let \( \hat{f}, \hat{g} \in X \) such that

\[ M_x(\hat{f}, \hat{g}, t) = M_x(\hat{g}, \hat{f}, t) = 1, \]

\[ d_x(\hat{f}, \hat{g}, t) = d_x(\hat{g}, \hat{f}, t) = 0 \]

By remark (2.3) \( d_x(\hat{f}, \hat{g}) = d_x(\hat{g}, \hat{f}) = 0 \)
For all \( \hat{f}, \hat{g} \in X \) and \( t \in (0,1] \), where \( (M_{d_x}, N_{d_x}, \land, \lor) \) is the intuitionistic fuzzy quasi metric induced by \( d_x \).

**Definition (3.5):**

Let \((X, \mathcal{M}, \mathcal{R}, \mathcal{O}, \otimes)\) be intuitionistic fuzzy quasi metric space and \( f : X \rightarrow X \) a self map if it is said that is a fuzzy contractive map if there exists \( k \in (0,1) \) such that
\[
\mathfrak{M}(f(\alpha_1(\omega)), f(\alpha_2(\omega)), t) = k (1 - \mathfrak{M}(\alpha_1(\omega), \alpha_2(\omega), t))
\]
for all \( \alpha_1(\omega), \alpha_2(\omega) \in X \) and \( t > 0 \).

**Definition (3.6):**

Let \((X, \mathcal{M}, \mathcal{R}, \mathcal{O}, \otimes)\) be intuitionistic fuzzy quasi metric space and \( f : X \rightarrow X \) a self map it is said that is \( f \) is an \((0,1] \) fuzzy contractive map if there exists \( k \in (0,1) \) such that
\[
\mathfrak{M}(f(\alpha_1(\omega)), f(\alpha_2(\omega)), t) = k (1 - \mathfrak{M}(\alpha_1(\omega), \alpha_2(\omega), t))
\]
for all \( \alpha_1(\omega), \alpha_2(\omega) \in X \) and \( t > 0 \).

**Theorem (3.7):**

Let \((X, d)\) be a quasi metric space and let \((X, \mathcal{M}, \mathcal{R}, \mathcal{O}, \otimes)\) be intuitionistic fuzzy quasi metric space on \( X \) satisfying
\[
\mathfrak{M}(\alpha_1(\omega), \alpha_2(\omega), t) = \frac{t}{t + d(\alpha_1(\omega), \alpha_2(\omega))}
\]
on the quasi metric space \((X, d^*)\) also \((X, d^t)\) if and only if it is Cauchy sequence, in the intuitionistic fuzzy quasi metric space \((X, \mathcal{M}_{d^t}, \mathcal{R}_{d^t}, \mathcal{O}, \otimes)\)

1. A sequence \( \{ \xi_n \} \) in \( X \) is a Cauchy sequence.

2. A map \( X \rightarrow X \) is \((0,1]\) intuitionistic fuzzy
contractive if and only if it is contractive map in the quasi metric space $(X, d)$.

**Proof (1):**

Let $\{\zeta_n\}$ be a Cauchy sequence in $(X, d^s)$ also $(X, d^s \cdot \mathbb{R})$. Let $\varepsilon \in (0, 1)$, with $t > 1 - \varepsilon$ also $t < \varepsilon$.

There exist $n_0 \in \mathbb{N}$ such that,

$$d^s(\zeta_n, \zeta_m) < \varepsilon \text{ also } d^s(\zeta_n, \zeta_m) > 1 - \varepsilon \text{ for all } n, m \geq n_0.$$

So $\mathcal{M}^i_{d^s}(\zeta_n, \zeta_m), t) > \frac{t}{t+\varepsilon} > 1 - \varepsilon$ also,

$$\mathcal{M}^s_{d^s}(\zeta_n, \zeta_m), t) < \frac{t}{t+\varepsilon} < \varepsilon \text{ for all } n, m \geq n_0.
$$

So $\{\zeta_n\}$ is a Cauchy sequence in $(X, d^s \cdot \mathbb{R})$.

Reciprocally if $\{\zeta_n\}_{n=1}^\infty$ is a Cauchy sequence in the intuitionistic fuzzy quasi metric space $(X, \mathcal{M}^i_{d^s}, \mathcal{M}^s_{d^s}, \mathcal{O}, \mathcal{X})$ given $\varepsilon \in (0, \frac{1}{2})$ there exists $n_0 \in \mathbb{N}$ such that,

$$\mathcal{M}^i_{d^s}(\zeta_n, \zeta_m), t) > \frac{t}{t+\varepsilon} > 1 - \varepsilon \text{ also } \mathcal{M}^s_{d^s}(\zeta_n, \zeta_m), t) < \frac{t}{t+\varepsilon} < \varepsilon \text{ for all } n, m \geq n_0.
$$

Hence $d^s(\zeta_n, \zeta_m) < \frac{t}{t+\varepsilon} < 2\varepsilon$ also,

$$d^s(\zeta_n, \zeta_m) < 1 - \frac{t}{t+\varepsilon} < 1 - 2\varepsilon \text{ for all } n, m \geq n_0,
$$

we conclude that $\{\zeta_n\}$ is a Cauchy sequence in $(X, d^s)$ also $(X, d^s)$. 

**Proof (2):**

Let $f : X \to X$ and $k \in (0, 1)$. Then we have that for all $t \in (0, 1)$ then,

$$\begin{align*}
\mathcal{M}(f(\alpha_1(\omega)), f(\alpha_2(\omega)), t) &\leq k \mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t) + k(1-\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t)) \\
&= k(1-\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t)) + k(1-\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t)) \\
&= k(1-\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t)) + k(1-\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t))
\end{align*}
$$

$k(1-\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t))$.

For all $\alpha_1(\omega), \alpha_2(\omega) \in X$ and $t > 0$.

And $\mathcal{M}(f(\alpha_1(\omega)), f(\alpha_2(\omega)), t) = k \mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t)$

Therefore, $d(f(\alpha_1(\omega)), f(\alpha_2(\omega))) \leq k d(\alpha_1(\omega), \alpha_2(\omega))$.

if and only if

$$\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t) \leq \mathcal{M}(f(\alpha_1(\omega)), f(\alpha_2(\omega)), t) \leq k(1-\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t))$$

$M_d^x(\alpha_1(\omega), \alpha_2(\omega), t)$.

$\mathcal{M}(f(\alpha_1(\omega)), f(\alpha_2(\omega)), t) \leq k(1-\mathcal{M}(\alpha_1(\omega), \alpha_2(\omega), t))$.

for all $\alpha_1(\omega), \alpha_2(\omega) \in X$. This show that $f$ is an (0,1)- intuitionistic fuzzy contractive map on $(X, \mathcal{M}, \mathcal{O}, \mathcal{X})$ if and only if it is a contractive map on the quasi-metric space $(X, d)$.

**Theorem (3.8):**

Let $(X, d^s)$, is bicomplete quasi metric space and $(M_d^x, N_d^x, \Lambda, \nu)$ be intuitionistic fuzzy quasi metric on $X$ such that $M_d^x(\alpha_1(\omega), \alpha_2(\omega), t) = \frac{t}{t+d^x(\alpha_1(\omega), \alpha_2(\omega))}$, also $N_d^x(\alpha_1(\omega), \alpha_2(\omega), t) = \frac{d^x(\alpha_1(\omega), \alpha_2(\omega))}{d^x(\alpha_1(\omega), \alpha_2(\omega))}$, with $t \in (0, 1]$.

If $f : X \to X$ is intuitionistic a fuzzy map which is contractive, then $f$ has a unique fixed point.

**Proof:**

Let $M_d^x(\alpha_1(\omega), \alpha_2(\omega), t) = \frac{d^x(\alpha_1(\omega), \alpha_2(\omega))}{t+d^x(\alpha_1(\omega), \alpha_2(\omega))}$, implies that

$M_d^x(\alpha_1(\omega), \alpha_2(\omega), t) = \frac{t}{t+d^x(\alpha_1(\omega), \alpha_2(\omega))}$ and,

$N_d^x(\alpha_1(\omega), \alpha_2(\omega), t) = \frac{d^x(\alpha_1(\omega), \alpha_2(\omega))}{t+d^x(\alpha_1(\omega), \alpha_2(\omega))}$

since $k \in (0, 1)$ then $\frac{t}{t+k d(\alpha_1(\omega), \alpha_2(\omega))} = \frac{t}{t+d^x(\alpha_1(\omega), \alpha_2(\omega))}$ and,

$M_d^x(\alpha_1(\omega), \alpha_2(\omega), t) + k(1-M_d^x(\alpha_1(\omega), \alpha_2(\omega), t)) = \frac{t}{t+d^x(\alpha_1(\omega), \alpha_2(\omega))}$

$M_d^x(f(\alpha_1(\omega)), f(\alpha_2(\omega), t) = \frac{d^x(\alpha_1(\omega), \alpha_2(\omega))}{t+d^x(\alpha_1(\omega), \alpha_2(\omega))}$

And $N_d^x(f(\alpha_1(\omega), f(\alpha_2(\omega), t) = \frac{d^x(\alpha_1(\omega), \alpha_2(\omega))}{t+d^x(\alpha_\omega(\omega), f(\alpha_2(\omega))}$

If $M_d^x(\alpha_1(\omega), \alpha_2(\omega), t) + k(1-M_d^x(\alpha_1(\omega), \alpha_2(\omega), t) \leq M_d^x(f(\alpha_1(\omega), f(\alpha_2(\omega), t)$.
Implies that \( d(f(\alpha_1(\omega)), f(\alpha_2(\omega))) \leq kd(\alpha_1(\omega), \alpha_2(\omega)) \)

Implies that \( t + d(f(\alpha_1(\omega)), f(\alpha_2(\omega))) \leq t + k d(\alpha_1(\omega), \alpha_2(\omega)) \)

\[
\frac{t}{t + d(f(\alpha_1(\omega)), f(\alpha_2(\omega)))} \geq \frac{M_{dx}(\alpha_1(\omega), \alpha_2(\omega), t)}{M_{dx}(\alpha_1(\omega), \alpha_2(\omega))} + k(1 - M_{dx}(\alpha_1(\omega), \alpha_2(\omega), t))
\]

Implies that \( M_{dx}(f(\alpha_1(\omega), f(\alpha_2(\omega)), t) \geq \frac{M_{dx}(\alpha_1(\omega), \alpha_2(\omega))}{M_{dx}(\alpha_1(\omega), \alpha_2(\omega))} + k \left( 1 - M_{dx}(\alpha_1(\omega), \alpha_2(\omega)) \right) \)

If

\[
\frac{k(N_{dx}(f(\alpha_1(\omega), f(\alpha_2(\omega)), t))}{1 - N_{dx}(\alpha_1(\omega), \alpha_2(\omega))} + k \left( N_{dx}(f(\alpha_1(\omega), f(\alpha_2(\omega)), t) \right) \geq 1 - N_{dx}(f(\alpha_1(\omega), f(\alpha_2(\omega)), t)
\]

Implies that \( d(f(\alpha_1(\omega)), f(\alpha_2(\omega))) \geq kd(\alpha_1(\omega), \alpha_2(\omega)) \)

Implies that \( t + d(f(\alpha_1(\omega)), f(\alpha_2(\omega))) \geq t + k d(\alpha_1(\omega), \alpha_2(\omega)) \)

\[
\frac{k d(\alpha_1(\omega), \alpha_2(\omega))}{t + d(f(\alpha_1(\omega)), f(\alpha_2(\omega)))} \leq \frac{k d(\alpha_1(\omega), \alpha_2(\omega))}{t + k d(\alpha_1(\omega), \alpha_2(\omega))}
\]

\[
\frac{k(N_{dx}(f(\alpha_1(\omega), f(\alpha_2(\omega)), t))}{1 - N_{dx}(\alpha_1(\omega), \alpha_2(\omega))} + k \left( N_{dx}(f(\alpha_1(\omega), f(\alpha_2(\omega)), t) \right) \leq \frac{k(N_{dx}(f(\alpha_1(\omega), f(\alpha_2(\omega)), t))}{1 - N_{dx}(\alpha_1(\omega), \alpha_2(\omega))} + k \left( N_{dx}(f(\alpha_1(\omega), f(\alpha_2(\omega)), t) \right)
\]

Hence, by the Banach fixed point theorem \( f \) has a unique fixed point.

**Example (3.9):**

Let \( T \) be the recurrence equation of a Quicksort algorithm given by:

\[
T(n) = \begin{cases} 0, & \text{if } n = 0 \text{ or } n = 1 \\ \frac{2(n-1)}{n} + \frac{(n+1)}{n} T(n-1), & \text{if } n \geq 2 \end{cases}
\]

We associate to \( T \) the functional \( \Psi : X \rightarrow X \) given by: \( \Psi(f) = T(f) \) and

\[
(\Psi(f))_n = \frac{2(n-1)}{n} + \frac{(n+1)}{n} f_{n-1} \text{ for all } n \geq 2.
\]

Next we show that \( \Psi \) is a contraction on the bicomplete quasi-metric space \( (X, d_x) \), with contraction constant \( 1/2 \). We have:

\[
d_x(\Psi f(n), \Psi g(n)) = \sum_{n=1}^{\infty} 2^{-n} \left( \frac{1}{d_x(f(n))} - \frac{1}{d_x(g(n))} \right) \leq 0
\]

\[
d_x(\Psi f(n), \Psi g(n)) = \sum_{n=2}^{\infty} 2^{-n} \left( \frac{1}{d_x(f(n))} - \frac{1}{d_x(g(n))} \right) \leq 0
\]

**CONCLUSIONS**

The fixed point theorems studied on intuitionistic fuzzy quasi metric space is efficient for studying
the complexity of algorithm technique and solved this problems with two complement fuzzy sets.

REFERENCES


https://doi.org/10.1155/2014/348069

https://doi.org/10.1016/0165-0114(94)90162-7

https://doi.org/10.1016/0165-0114(88)90064-4


https://doi.org/10.1016/j.topol.2006.09.018

https://doi.org/10.1007/s100120200034

https://doi.org/10.22436/jmcs.08.03.01

https://doi.org/10.1016/S0019-9958(65)90241-X

How to Cite