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Best One Sided Multiplier Approximation of Unbounded Functions by Polynomials Operators

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ABSTRACT

The purpose of this paper is present some operators by using polynomials operators of type $G_n(f,x)$, $g_n(f,x)$, $L_n(f)$ and $M_n(f)$, to get the degree of best one- sided multiplier approximation of unbounded functions by algebraic polynomials in $L_{p,\varphi_n}(X), X=[0,1]$, by $\tau(f,\delta)_{p,\psi_n}$.

KEYWORDS: Multiplier convergence; multiplier integral; multiplier modulus.

الخلاصة

الغرض من هذا البحث هو تقديم بعض المؤثرات متعددات الحدود الجبرية من النوع $G_n(f,x)$, $g_n(f,x)$, $L_n(f)$ and $M_n(f)$, للحصول على درجة افضل تقريب مضاعف احادي الجانب للدوال غير المقيدة في الفضاء $L_{p,\varphi_n}(X), X=$ [0,1] بواسطة معدل النعومة $\tau(f,\delta)_{p,\psi_n}$

INTRODUCTION

We will review some researchers who have studied approximation of unbounded functions and who have obtained important results in this field. In (2015), Zaboon [1] studied the Approximation of unbounded functions in locally–global space. In (2014) Jassim and Alaa [2] estimated the degree best one sided approximation unbounded functions by some discrete operater in $L_{p,\omega}-space$. Also Babenko et al. [3] estimated the rate of best one-sided approximation of characterstic polynomials.

Definition(1)[4]

A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergence if there is a sequence

$\{\varphi_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} a_n$, $\varphi_n < \infty$ and we will say that $\{\varphi_n\}$ is a multiplier for the convergence.

Definition(2)[4]

For any real valued function f if there is a sequence $\{\varphi_n\}_{n=0}^{\infty}$ such that:

$\int_X f\,\varphi_n\,(x)\,\mathrm{d}\,x < \infty$, then we say that φ_n is a multiplier for the integral.

Definition(3)

Let $L_{p,\varphi_n}(X)$, $X=[0,1]$, $p \in [0,1)$ be the space of all real valued unbounded functions f, such that $\int_X f\,\varphi_n\,(x)\,\mathrm{d}\,x < \infty$ whit under the norm $\|f\|_{p,\varphi_n} = (\int_X |f\,\varphi_n\,(x)\,|^{p})^{1/p}$, $x \in X$

Definition(4) :

for $f \in L_{p,\varphi_n}(X)$, $X=[0,1]$, $\delta > 0$ we will define the following concepts

1- $\omega(f,\delta)_{p,\varphi_n} = \underset{|h|<\delta}{sep} \|f(x+\mathrm{h}) - f(x)\|_{p,\varphi_n}$

is the multiplier modulus of continuity of function f

2- $\tau_k(f,\delta)_{p,\varphi_n} = \|\omega_k\,(f,0,\delta)\|_{p,\varphi_n}$, $p \in [0,\infty)$, $k \in \mathrm{N}$, $\delta > 0$ is the multiplier averaged modulus of smoothness of f of over k ,where k modulus smoothness for f is defined:

$= \underset{|h|<\delta}{sep} \{ \|\Delta_h^k(f,t)\|_{p,\varphi_n}$:,t ,t+ k h $\omega(f,\delta)_{p,\varphi_n}$ $[x - \frac{ks}{2}, x + \frac{ks}{2}]$ } where $\Delta_h^k(f,x) = \in \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-1} f(x - \frac{kh}{2} + i h), x \pm \frac{kh}{2} \in X$ the k[th] symmetric difference of the function f .

Definition (5):[7]

The degree of best one – sided approximation of f is

$\tilde{E}_n(f) = \inf \{ \| p_n - q_n \|_{L\,p,\psi_n(X)} : q_n(x) \le f(x) \le p_n(x) \}$

Also The degree of best approximation of a function $f \in Lp(X)$ is define by

$E_n(f)p = \inf \{ \| f - p_n \|_{L\,p} : p_n \in P_n \}$

Definition (6)

The degree of best one – sided multiplier approximation of f is

$\tilde{E}_n(f)p,\psi_n = \inf \{ \| p_n - q_n \|_{L\,p,\psi_n(X)} : q_n(x) \le f(x) \le p_n(x) \}$

Also The degree of best multiplier approximation of a function $f \in Lp,\psi_n(X)$ is define by

$E_n(f)p,\psi_n = \inf \{ \| f - p_n \|_{L\,p,\psi_n(X)} : p_n \in P_n \}$

Let us consider the following step function

$$H(x) = \begin{Bmatrix} 0 \ if \ x \in [-1,0] \\ 1 \ if \ \ x \in [0,1] \end{Bmatrix}$$

Fix two sequences of polynomials $\{p_n\}, \{g_n\}, \in \mathbb{P}_n$ such that

$p_n(x) \le H(x) \le g_n(x), x \in$ [a,b] a=0, b=1

and $C_n = \|p_n - g_n\|_{p,\varphi_n} \to 0$ when $n \to \infty$,

we constructed a sequence of polynomial operators as following :

$f \in L_{p,\varphi_n}(X), X = [0,1], n \in$ N , $t \in X$ we defined

$g_n(f,x) = f\varphi_n(0) + \int_X P_n(x-t)(f\varphi_n)'(t)\,dt + \int_X g_n(t-x)(f\varphi_n)'(t)\,dt$

$G_n(f,x) = (f\varphi_n)(0) + \int_X g_n(t-x)(f\varphi_n)'(t)\,dt + \int_X P_n(x-t)(f\varphi_n)'(t)\,dt$

, where $(f\varphi_n)'(x) = \max\{0, (f\varphi_n)'(x)\}, (f\varphi_n)'(x) = \max\{0, -(f\varphi_n)'(x)\}$

Definition(7):[5]

For a function $f \in R[0,1], h \in (0,1)$ and $x \in [0,1]$, set

$L_h(f) = \int_0^1 [f(1-h)x + hs) - \omega(f,(1-h)x + hs, h)]ds$ and

$$M_h(f) = \int_0^1 [f(1-h)x + hs) - \omega(f,(1-h)x + hs, h)]ds$$

Definition(8):

let $f \in L_{p,\varphi_n}(X), X = [0,1], h \in (0,1)$,we define

$L_h(f) = \int_0^1 [f\varphi_n(1-h)x + hs) - \omega(f\varphi_n,(1-h)x + hs, h)]ds$ and

$$M_h(f) = \int_0^1 [f\varphi_n(1-h)x + hs) - \omega(f\varphi_n,(1-h)x + hs, h)]ds$$

Definition(9): [5]

For a function $f \in R[0,1], h \in (0,1)$ and $x \in [0,1]$, set

$A_{n,h}(f,x) = g_n[L_h(f,x)]$ and $B_{n,h}(f,x) = G_n[M_h(f,x)]$

Definition(10):

let $f \in L_{p,\varphi_n}(X), X = [0,1]$, ,we define

$A_{n,h}(f,x) = g_n[L_h(f,x)]$ and $B_{n,h}(f,x) = G_n[M_h(f,x)]$

Definition(11)[6]:

let $X_0 = L_p(X)$ and $X_1 = W_p^r$ and $X_1 \subset X_0$,define the k-function in $L_p(X)$ – space that

$K_r(f,\delta)_p^r = inf_{g \in w_p^r} \{ \|f-g\|_p + \delta^r \|g^r\|_p, \delta > 0 \}$

Now ,will be introduce K -fountained of function $f \in L_{p,\varphi_n}(X)$ such that

$K_r(f,\delta)_{p,\varphi_n}^r = inf_{g \in w_p^r} \{ \|f-g\|_{p,\varphi_n} + \delta^r \|g^r\|_{p,\varphi_n}, \delta > 0 \}$

Definition (12):

let $f \in L_{p,\varphi_n}(X), X = [0,1]$, we define

$I(f,x) = \int_0^x f\,\varphi_n(s)ds$

Auxiliary results:

Lemma(1):[5]

let $f \in L_{p,\varphi_n}(X), X = [0,1]$,then

$g_n(f,x), G_n(f,x) \in \mathbb{P}_n$ such that

$g_n(f) \le f(x) \le G_n(f)$

Lemma(2): $f \in L_{p,\varphi_n}(X), X = [0,1]$,then

$\max \{ \|f - g_n(f)\|_{p,\varphi_n}, \|f - G_n(f)\|_{p,\varphi_n} \} \le C_n \|f'\|_{p,\varphi_n}$

Proof:

$g_n(f,x) = f\varphi_n(0) + \int_X P_n(x-t)(f\varphi_n)'(t)\,dt + \int_X g_n(t-x)(f\varphi_n)'(t)\,dt$

$-g_n(f,x) = -f\varphi_n(0) - \int_X P_n(x-t)(f\varphi_n)'(t)\,dt - \int_X g_n(t-x)(f\varphi_n)'(t)\,dt$

$f\varphi_n(x) - g_n(f,x) = f\varphi_n(x) - (f\varphi_n)(0) - \int_x H_n(t-x)\ (f\varphi_n)'_+(t)dt + \int_x H_n(t-x\ (f\varphi_n)'_-(t)dt$

$= \int_0^x (f\varphi_n)'(t)dt + \int_0^x [p_n(t-x)(f\varphi_n)'(t)dt - g_n(t-x)(f\varphi_n)'(t)]dt$

$= \int_0^1 [H_n(t-x)\ (f\varphi_n)'(t) - p_n(t-x)(f\varphi_n)'(t) + g_n(t-x)(f\varphi_n)'(t)]dt$

$= \int_0^1 [(H_n(t-x) - p_n(t-x))(f\varphi_n)'(t)) + g_n(t-x) - H_n(t-x)(f\varphi_n)'(t)]dt$

$\leq \int_0^1 [g_n(t-x) - p_n(t-x](f\varphi_n)'(t)|dt$

Let $y = t-x$, if $t=1$ then $y = 1-x$

if $t=0$ then $y=-1$

$= \int_0^1 [g_n(y) - P_n(y)](f\varphi_n)'(y+x)|d y$

$\|f - g_n(f,0)\|_{p,\varphi_n} \leq \int_0^1 [\int_{-x}^{1-x} [g_n(y) - P_n(y)](f\varphi_n)'(y+x)|d y]^P d x$

By holder's inequality ,we have

$\leq \int_0^1 ([\int_{-x}^{1-x} [g_n(y) - P_n(y)]\ d y]^{P-1} \int_{-x}^{1-x} |f\varphi_n)'(y+x)|^P g_n(y) - P_n(y)\, d y\, d x$

Let $z = x+y$, if $y = 1-x$ then $z=1$,

if $y=-x$ then $z=0$

$\leq (\int_{-1}^1 [g_n(y) - P_n(y)]\ d y]^{P-1} \int_0^1 |f\varphi_n)'(z)|^P (\int_{z-1}^z [g_n(y) - P_n(y)]\ d y)\ dy$

$\leq (\int_{-1}^1 [g_n(y) - P_n(y)]\ d y]^{P-1} \int_0^1 |f\varphi_n)'(z)|^P d z$

$\leq (\int_{-1}^1 [g_n(y) - P_n(y)]\ d y)^P \|f'\|_{p,\varphi_n}$

$\leq \|g_n - P_n\|_{1,[-1,1]} \|f'\|_{p,\varphi_n}$

Lemma(3): $f \in L_{p,\varphi_n}(X)$, $X = [a,b]$, , a=0,b=1, $0 < h < 1$

, $x \in [0,1]$, then

$\text{Max}\{ \|L_h'(f)\|_{p,\varphi_n}, \|M_h'(f)\|_{p,\varphi_n} \} \leq \frac{3}{h}\tau_k(f,h)_{p,\varphi_n}$

Proof: From definition (7) we have $L_h(f) = \int_0^1 [f\varphi_n(1-h)x + hs) - \omega(f\varphi_n, (1-h)x + hs, h)]ds$ and

$$M_h(f) = \int_0^1 [f\varphi_n(1-h)x + hs) - \omega(f\varphi_n, (1-h)x + hs, h)]ds$$

Then

$L_h'(f,x) = \int_0^1 [f\varphi_n((1-h)x + hs) - \omega(f\varphi_n, (1-h)x + hs, h)]ds$

$L_h'(f,x) = \frac{1-h}{h}[f\varphi_n((1-h)x+h) - f\varphi_n, (1-h)x)] - \frac{1-h}{h}$

$[\omega(f\varphi_n, (1-h)x+h, h)] - \omega(f\varphi_n, (1-h)x, h)]$

$= \frac{1-h}{h}[\omega(f\varphi_n(1-h)x) - \frac{1-h}{h}[\omega(f\varphi_n(1-h)x+h, h)$

$+ \frac{1-h}{h}[\omega(f\varphi_n(1-h)x+h)]$

$= \frac{1-h}{h}[\omega(f\varphi_n(1-h)x, h)]$

Taking nom two the sided , we get

$\|L_h'(f)\|_{p,\varphi_n} \leq \frac{1-h}{h}\|\omega(f(1-h)x, h)\| \leq \frac{1-h}{h}\tau(f,h)_{p,\varphi n}$

Sine

$M_h'(f,x) = \int_0^1 [f\varphi_n((1-h)x + h\, s) - \omega(f\varphi_n, (1-h)x + h\, s, h)]ds$

$M_h'(f) = \frac{1-h}{h}[f\varphi_n[(1-h)x+h) - f\varphi_n((1-h)x)] +$

$\frac{1-h}{h}[\omega\ f\varphi_n(1-h)x+h, h) - \omega\ f\varphi_n(1-h)x, h)]$

$= \frac{1-h}{h}[\omega(f\varphi_n, (1-h)x, h) + \frac{1-h}{h}[\omega(f\varphi_n, (1-h)x, h)]$

$- \frac{1-h}{h}[\omega(f\varphi_n, (1-h)x, h)$

$= \frac{1-h}{h}[\omega(f\varphi_n, (1-h)x, h)]$

Taking norm two sidrd , we get

$\|M_h'(f)\|_{p,\varphi_n} \leq \frac{1-h}{h}\|\omega(f\varphi_n, (1-h)x, h\|_{p,\varphi_n} \leq \frac{1-h}{h}\tau(f,h)_{p,\varphi n}$

$\|M_h'(f)\|_{p,\varphi_n} + \|L_h'(f)\|_{p,\varphi_n} \leq \frac{1-h+1-h}{h}\tau(f,h)_{p,\varphi n} \leq \frac{2}{h}\tau(f,h)_{p,\varphi n}$

$\text{Max}\{ \|L_h'(f)\|_{p,\varphi_n}, \|M_h'(f)\|_{p,\varphi_n} \} \leq \frac{3}{h}\tau_k(f,h)_{p,\varphi_n}$

Lemma(4): *Let* $f \in L_{p,\varphi_n}(X)$, $X = [0,1]$,then we have

$L_h(f,x \leq f(x) \leq M_h(f)$

Proof: since

$$M_h(f) = \int_0^1 [f\varphi_n(1-h)x + hs) - \omega(f\varphi_n, (1-h)x + hs, h)]ds$$

$f(x) - M_h(f,x)$

$= \frac{1}{h}\int_0^h f\varphi_n(x) - f\varphi_n(1-h)x + s) - \omega(f\varphi_n, (1-h)x + s, h)]ds \leq 0$

$\Rightarrow f(x) - M_h(f,x) \leq 0$

$\Rightarrow f(x) \leq M_h(f,x)$ ………….. 1

$L_h(f,x) = \int_0^1 [f\varphi_n\,(1-h)x + hs) - \omega(f\varphi_n\,,(1-h)x + hs, h)]ds$

$L_h(f,x) - f(x) = \frac{1}{h}\int_0^h f\varphi_n\,(1-h)x + hs) - \omega(f\varphi_n\,,(1-h)x + hs, h)]ds$

≤ 0

Since : $L_h(f,x) - f(x) \le 0$

$L_h(f,x) \le f(x)$ ………… 2

from 1 and 2 , we get

$L_h(f,x) \le f(x) \le M_h(f)$

Lemma (5):

let $n \in N$, $h \in (0,1)$ and $f \in L_{p,\varphi_n}(X)$, $X = [0,1]$, then we have

Then $A_{n,h}(f,x) \le f(x) \le B_{n,h}(f,x)$

Proof: $A_{n,h}(f)$ and $B_{n,h}(f) \in \mathbb{P}_n$

From lemma's 1 and 4 , we get

$A_{n,h}(f) = g_n\ (L_h(f)) \le L_h(f) \le f(x)$ ………….. 4

Also

$f(x) \le M_h(f) \le G_n(M_h(f)) = B_{n,h}(f)$………….. 5

from 4 and 5 , we get

$A_{n,h}(f,x) \le f(x) \le B_{n,h}(f,x)$

Theorem 1:

For $f \in L_{p,\varphi_n}(X)$, $X = [0,1]$, $L_h(f)$ and $M_h(f) \in L_{p,\varphi_n}(X)$, $1 \le p < \infty$, then

i- max $\{\|f - L_h(f)\|_{p,\varphi_n}, \|f - M_h(f)\|_{p,\varphi_n}\} \le \frac{2}{(1-h)^{1}/_{p}}\ \tau_k(f,h)_{p,\varphi_n}$

ii- $\|M_h(f) - L_h(f)\|_{p,\varphi_n}\} \le \frac{2}{(1-h)^{1}/_{p}}\ \tau_k(f,h)_{p,\varphi_n}$

proof: since $f(x) \le M_h(f)$

$f(x) - L_h(f) \le M_h(f) - L_h(f)$

$h[f(x) - L_h(f)] \le h[M_h(f) - L_h(f)]$

$(h\|f(x) - L_h(f)\|_{p,\varphi_n})^p \le (h\|M_h(f) - L_h(f)\|_{p,\varphi_n})^p$

$= 2^p \int_0^1 (\int_0^h \omega(f\varphi_n\,,(1-h)x + s, h)ds)^p\,dx$

$\le 2^p h^{p/q} \int_0^1 \int_0^h \omega(f\varphi_n\,,(1-h)x + s, h)ds\,dx$

Let $(1-h)x + s = y \Rightarrow dy = (1-h)dx$

$= \frac{2^p h^{p/q}}{1-h} \int_0^h \int_0^{1-h+s} \omega^p(f,y,h)\,dy\,ds$

$\le \frac{2^p h^{p/q}}{1-h} \int_0^h \int_0^1 \omega^p(f,y,h)\,dy\,ds$

$= \frac{2^p h^{p/q}}{1-h}\ \tau^p\,(f,h)^p_{p,\varphi_n} = \frac{2^p h^p}{1-h}\ \tau^p\,(f,h)_{p,\varphi n}$

$=[\frac{2\ h}{(1-h)^{1/p}}\ \tau\ (f,h)_{p,\varphi_n}]^p$

$h\|f(x) - L_h(f)\|^p_{p,\varphi_n} \le \frac{2\ h}{1-h^{\ 1/p}}\ \tau^p\ (f,h)^p_{p,\varphi n}$ ……….. now ,since 3

$L_h(f) \le -L_h(f)$

$M_h(f) - f \le M_h(f) - L_h(f)$

$(h\|M_h(f) - f(0)\|^p_{p,\varphi_n})^p \le (h\|M_h(f) - L_h(f)\|^p_{p,\varphi_n})^p$

$= 2^p \int_0^1 (\int_0^h \omega\,(f\varphi_n\,,(1-h)\,x + s, h)ds)^p\,dx$

$\le 2^p h^{p/q} \int_0^1 \int_0^h w^p(f\varphi_n\,,(1-h)\,x + s, h)ds\,dx$

$= \frac{2^p h^{p/q}}{1-h} \int_0^h \int_0^{1-h+s} \omega^p(f,y,h)\,dy\,ds$

$= \frac{2^p h^{p/q}}{1-h} \int_0^h \int_0^1 \omega^p(f,y,h)\,d\,y\,ds = \frac{2^p h^{p/q}}{1-h}\ \tau^p\,(f,h)_{p,\varphi_n}$

$= \frac{2^p h^p}{1-h}\ \tau^p\,(f,h)^p_{p,\varphi_n} = [\frac{2\ h}{(1-h)^{1/p}}\ \tau\ (f,h)]^p_{p,\varphi_n}$

$h\|M_h(f) - f(0)\|^p_{p,\varphi_n} \le [\frac{2\ h}{(1-h)^{1/p}}\ \tau\,(f,h)]_{p,\varphi_n}$

$h\|M_h(f) - f(0)\|^p_{p,\varphi_n} \le [\frac{2\ h}{(1-h)^{1/p}}\ \tau\,(f,h)]_{p,\varphi_n}$ …….. 3

from 2 and 3 , we get

$\|M_h(f) - f(0)\|_{p,\varphi_n} + \|L_h(f) - f(0)\|_{p,\varphi_n} \le \frac{4}{(1-h)^{1/p}}\ \tau\ (f,h)_{p,\varphi_n}$

Max $\{\|M_h(f) - f(0)\|_{p,\varphi_n} + \|L_h(f) - f(0)\|_{p,\varphi_n}\} \le C\,(p,h)\,\tau\ (f,h)_{p,\varphi_n}$

Theorem(2):

Let $f \in L_{p,\varphi_n}(X)$, $X = [0,1]$, ,then $\check{E}\,(f)_{p,\varphi_n} \le C\,(p,h)\,\tau\ (f,h)_{p,\varphi_n}$

Proof: $\check{E}\,(f)_{p,\varphi_n} \le \|M_h(f) - L_h(f)\|_{p,\varphi_n} = \|M_h(f) - L_h(f) - f + f\|_{p,\varphi_n}$

$\le \|f - M_h(f)\|_{p,\varphi_n} - \|f - L_h(f)\|_{p,\varphi_n}$

From theorem 1, we get

$\le C\,(p,h)\,\tau\ (f,h)_{p,\varphi_n}$

Lemma(5):

let $f \in L_{p,\varphi_n}(X)$, $X = [0,1]$, $n \in N$, $h \in (0,1)$,

Then $\max\{\|f - A_{n,h}(f)\|_{p,\varphi_n} - \|f - B_{n,h}(f)\|_{p,\varphi_n}\} \le (\frac{2}{1-h} + \frac{3\propto n}{h})\,\tau\ (f,h)_{p,\varphi_n}$

Proof: $\|f - A_{n,h}(f)\|_{p,\varphi_n} = \|f - A_{n,h}(f) - L_h(f) + L_h(f)\|_{p,\varphi_n} \le \|f - L_h(f)\|_{p,\varphi_n} + \|L_h(f) - A_{n,h}(f)\|_{p,\varphi_n}$

From theorem 1 and lemma 2 , we get

$\le \frac{2}{(1-h)^{1/p}}\ \tau\ (f,h)_{p,\varphi_n} + \propto_n \|L_h{}'(f)\|_{p,\varphi_n}$ from lemma 3 we get

$\le \frac{2}{(1-h)^{1/p}}\ \tau\ (f,h)_{p,\varphi_n} + \propto_n \left[\frac{3}{h}\tau\ (f,h)_{p,\varphi_n}\right]$

$\left\|f-A_{n,h}(f)\right\|_{p,\varphi_n} \leq (\frac{2}{1-h}+\frac{3\propto n}{h})\,\tau\ (f,h)_{p,\varphi n}$

Now

$\left\|f-B_{n,h}(f)\right\|_{p,\varphi_n} = \left\|f-B_{n,h}(f)-M_h(f)+M_h(f)\right\|_{p,\varphi_n} \leq \|f-M_h(f)\|_{p,\varphi_n} + \left\|M_h(f)-B_{n,h}(f)\right\|_{p,\varphi_n}$

From theorem 1 and lemma 2 , we get

$\leq \frac{2}{(1-h)^{1/p}}\ \tau\ (f,h)_{p,\varphi_n} + \propto_n \|M_h{}'(f)\|_{p,\varphi_n}$ from lemma 3 we get

$\leq \frac{2}{(1-h)^{1/p}}\ \tau\ (f,h)_{p,\varphi_n} + \propto_n \left[\frac{3}{h}\tau\ (f,h)_{p,\varphi_n}\right]$

$\left\|f-A_{n,h}(f)\right\|_{p,\varphi_n} \leq (\frac{2}{1-h}+\frac{3\propto n}{h})\,\tau\ (f,h)_{p,\varphi n}$

Theorem 4 :

let $f\in L_{p,\varphi_n}(X)$, $X=[0,1]$, Then $\check{E}\,(f)_{\,p,\varphi_n} \leq (\frac{4}{(1-h)^{1/p}}+\frac{6\propto_n}{h})$

$\tau\,(f,h)_{p,\varphi n} \leq C\ (p,h,\propto_n)\,\tau\ (f,h)_{p,\varphi_n}$

Proof: $\check{E}\,(f)_{\,p,\varphi_n} \leq \left\|A_{n,h}(f)-B_{n,h}(f)\right\|_{p,\varphi_n}$

$= \left\|A_{n,h}(f)-B_{n,h}(f)+f-f\right\|_{p,\varphi_n}$

$\leq \left\|f-A_{n,h}(f)\right\|_{p,\varphi_n} + \left\|f-B_{n,h}(f)\right\|_{p,\varphi_n}$

From lemma 5, we get

$= (\frac{4}{(1-h)^{1/p}}+\frac{6\propto_n}{h})\,\tau\ (f,h)_{p,\varphi_n} = C\ (p,h,\propto_n)\,\tau\ (f,h)_{p,\varphi_n}$

Lemma 6:

Let $f \in L_{p,\varphi_n}(X)$, $X=[0,1]$ $V_{n,2}(f) = I(\lambda_n(f')) + \lambda_n[f - I(\lambda_n(f'))]$ and for $r>2$, $V_{n,r}(f) = I\left(V_{n,r-1}(f')\right) + \lambda_n[f - I(V_{n,r-1}(f'))]$ then

$\left\|f-V_{n,2}(f)\right\|_{p,\varphi_n} \leq (\propto_n)^r \|f^r\|_{p,\varphi_n}$, $\propto_n = \|p_n - g_n\|_{p,\varphi_n} \to 0$ when $n\to\infty$

Proof: The proof by induction,

if $r=2$, $f-V_{n,2}(f) = f - I(\lambda_n(f') - \lambda_n[f - I(\lambda_n(f'))]$

$(\int_0^1 |I([f-V_{n,2}(f)]\varphi_n|^p\,dx)^{\frac{1}{p}} = \left(\int_0^1 |I([f - I[\lambda_n(f')] - \lambda_n[f - I[\lambda_n(f')]\varphi_n|^p\,dx\right)^{\frac{1}{p}}$

$\left\|f-V_{n,2}(f)\right\|_{p,\varphi_n} = \|f - I(\lambda_n(f') - \lambda_n[f - I(\lambda_n(f'))]\,\|_{p,\varphi_n}$

By lemma 2, we get

$\propto_n \|(f - I(\lambda_n(f'))\,'\|_{p,\varphi_n} = \propto_n \left\|(f' - (\lambda_n(f')\right\|_{p,\varphi_n}$

By lemma 2 ,we get

$(\propto_n)(\propto_n)\left\|f^{(2)}\right\|_{p,\varphi_n} = (\propto_n)^2 \left\|f^{(2)}\right\|_{p,\varphi_n}$

Suppose that $2\leq k \leq$ r is true

$f-V_{n,r-1}(f) = f - I(V_{n,r}(f') - \lambda_n[f - I\left(V_{n,r}(f')\right)]$

$\left(\int_X |[f-V_{n,r-1}(f)]\varphi_n(x)|^p\,dx\right)^{\frac{1}{p}} = \left(\int_X \left|[f - I(V_{n,r}(f') - \lambda_n\left[f - I\left(V_{n,r}(f')\right)\right]]\varphi_n(x)\right|^p dx\right)^{\frac{1}{p}}$

$\left\|f-V_{n,r-1}(f)\right\|_{p,\varphi_n} = \left\|f - I(V_{n,r}(f') - \lambda_n[f - I\left(V_{n,r}(f')\right)]\right\|_{p,\varphi_n}$

By lemma 2, we get

$\leq (\propto_n)\left\|f - I(V_{n,r}(f')\right\|_{p,\varphi_n} = \left\|f' - V_{n,r}(f')\right\|_{p,\varphi_n}$

By lemma 2, we get

$\propto_n (\propto_n)^r \left\|f^{(r+1)}\right\|_{p,\varphi_n} = (\propto_n)^{r+1}\left\|f^{(r+1)}\right\|_{p,\varphi_n}$

Theorem (5) :

Let $f\in L_{p,\varphi_n}(X)$, $X=[0,1]$,

$f\in W_p^r$, then $\left\|f-V_{n,r}(f)\right\|_{p,\varphi_n} \leq \tau_n(f,\delta)_{p,\varphi_n}$

Proof:

Sinc $\left\|g-V_{n,r}(g)\right\|_{p,\varphi_n} \leq \delta^r \|g^r\|_{p,\varphi_n}$ we have

$\left\|f-V_{n,r}(f)\right\|_{p,\varphi_n} = (\int_x |[f-V_{n,r}(f)]\varphi_n(x)|^{P}\,dx)^{1/p}$

$\leq (\int_X (|f-g|^p\varphi_n(x)dx)^{\frac{1}{p}} + (\int_X (|g - V_{n,r}(g)|^p\varphi_n(x)dx)^{\frac{1}{p}} ++ (\int_x |[V_{n,r}(g) - V_{n,r}(f)]\varphi_n(x)|^{P}\,d\,x)^{1/p}$

$= \|f-g\|_{p,\varphi_n} + \left\|g-V_{n,r}(g)\right\|_{p,\varphi_n} + \left\|V_{n,r}(f)-V_{n,r}(f)\right\|_{p,\varphi_n} \leq \|f-g\|_{p,\varphi_n}$

$+\delta^r \|g^r\|_{p,\varphi_n} + \left\|V_{n,r}(g-f)\right\|_{p,\varphi_n} = \|f-g\|_{p,\varphi_n}$
$+\delta^r \|g^r\|_{p,\varphi_n} + M\|(g-f)\|_{p,\varphi_n}$
$= (1+M)\|(g-f)\|_{p,\varphi_n} + \delta^r \|g^r\|_{p,\varphi_n}$
$\leq K_r(f,\delta^r) \leq \tau_n(f,\delta)_{p,\varphi_n}$

Theorem(6) :
Let $f \in L_{p,\varphi_n}(X)$, $X = [0,1]$, Then
$\check{E}(f)_{p,\varphi_n} \leq$
$\tau(f,h)_{p,\varphi_n}$

Proof: By theorem 5, we have
$\check{E}(f)_{p,\varphi_n} \leq C\, E_n(f)_{p,\varphi_n} \leq \left\|f - V_{n,r}(f)\right\|_{p,\varphi_n}$
$\leq C(P)\,\tau_n(f,\delta)_{p,\varphi_n}$

CONCLUSIONS

From this research, we can get the following:

We obtained some results and found a degree of the best one-sided multiplier approximation of unbounded functions: $f \in L_{p,\psi_n}(X)$-space, $X = [0,1]$, $1 \leq p < \infty, \delta > 0$, using polynomials of type $G_n(f,x)$, $g_n(f,x), L_n(f)$ *and* $M_n(f)$, by means of averaged multiplier modulus of smoothness of f, $\tau_k(f,\delta)_{p,\psi_n}$.

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