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Bernoulli's polynomials Approach for Special Type of Integro Differential Equation

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ABSTRACT

In this paper, Bernoulli's polynomials approach is employed for solving approximately linear Fractional Fredholm integro-differential equations via Petrove-Glerkain method. The Fractional derivatives are described in the sense Caputo. The approximate solution is compared with the exact solution to confirm the validity and efficiency of the method to a same as before. Some illustrative examples are presented.

KEYWORDS: Callculus fractional; Bernoulli polynomial approach; Caputo fractional Fredholm; Petrov-Galerkian.

الخلاصة

في هذا البحث، وضفنا متعدد الحدود برنولي لحل معادلات فريدهولم التكاملية التفاضلية الخطية تقريبًا عبر طريقة بيتروف-كالركن يتم وصف المشتقات الكسرية بالمعنى Caputo. تتم مقارنة الحل التقريبي مع الحل الدقيق للتأكد من صحة وكفاءة الطريقة كما كانت من قبل. يتم تقديم بعض الأمثلة التوضيحية.

INTRODUCTION

Fractional calculus is an important branch of applied mathematics, the fractional differential equations involving the Caputo and other fractional derivatives, which are a generalization of classical differential equations, have attracted widespread attention. This type of differentiation and integration may be considered as a generalization to the useful definition of differentiation and integration analytically, and hence finding accurate numerical solutions [1]. Fractional differential equations involving the Caputo and other fractional derivatives, which are a generalization of classical differential equations, have pulled in far reaching consideration [2]. Many numerous problems can be modeled by fractional Integro-differential equations from varied sciences and engineering applications. Besides most issues which cannot be unraveled, utilizing numerical strategies, will be exceptionally accommodating.

As of late, a few numerical strategies to unravel linear fractional Integro- differential conditions (LFFIDEs) have been given and cite to this reference. The existence and uniqueness of solutions to fractional differential equations have been explored [3,4].

Furthermore, a few strategies have as of late been proposed for looking for solutions counting numerical solution and approximate solution such as the Homotopy perturbation method HPM and variational iteration method VIM [5], by least squares method and shifted Chebyshev polynomial[6], Taylor expansion method[7]و and a time-fractional equation via the quintic non-polynomial spline[8].

The numerical solution of the following linear fractional Integro-differential problem is the subject of this paper:

$$D_\alpha^* u(x) = f(x) + \int_0^1 k(x,t)u(t)dt \qquad (1)$$

with the following additional requirements:

$$u^i(0) = \beta_i ,\, n-1 < \alpha \le n,\, n \in \mathrm{N}, \qquad (2)$$

where $D_\alpha^* u(x)$ indicates the α the Caputo fractional derivative of u(x); $f(x)$, $K(x,t)$ are given functions, x and t are real variables varying in the interval [a, b], and u(x) is the unknown function to be determined..

In this paper, we show how the approximately methods which are based on the petrove-Galerkain method ($\mathcal{PGM}$) can be used to solve (LFVIDE's) to obtain approximate solutions via the Bernoulli bases.

1. Basic Definitions of Fractional Derivatives

Some essential concepts and properties of fractional calculus theory are provided in this section, which are required for the formulation of the issue.

Definition (1.1) [1]**:**

A real function f(t), t > 0, is said to be in the space C_{μ} , μ∈ R , if there exists a real number P > μ, such that f(t)n$t^p h_1(t)$, where $f_1(t) \in (0,\infty)$, and it is said to be in space C_{μ}^{n} if and only if $f^n C_{\mu}, n \in N$.

Definition (1.2) [1]:

Let $f \in C_{-1}^{m}1, m \in N \cup \{0\}$. Then the caputo fractional derivatives of *f*(x) is defined as:

$$D^{\alpha}f(x) = \begin{cases} J^{M-\alpha}f^{m}(x), m-1<\alpha\le m, m\in N \\ \frac{D^{m}f(x)}{Dx^{m}}, \quad \alpha = m \end{cases}$$

Hence , we have following properties

1) $J^{\alpha}J^{v}f = J^{\alpha+v}f$, $\alpha, v > 0$, $f \epsilon C_{\mu}, \mu > 0$
2) $J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}, \alpha>0, \gamma>-1, x>0$
3) $J^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0^{+})\frac{x^{k}}{k!}$, $x > 0, m-1<\alpha\le m$
4) $J^{\alpha}D^{\alpha}f(x) = f(x)$, $x>0, m-1<\alpha\le m$
5) D^{α}C = 0, C is constant

$$D^{\alpha}x^{\beta} = \begin{cases} 0 \quad \beta \in N_0, < [\alpha] \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}\beta \in N_0, \beta \ge [\alpha] \end{cases}$$

where $[\alpha]$ denoted the smallest integer greater than or egual to α and $N_0\{0,1,2,\cdots\}$

2. The Derivative for Bernoulli Polynomials

The unknown function appears as a linear combination of defined basic functions in the collocation Bernoulli technique.

Polynomials, orthogonal functions, wavelets functions, and spline approximation are frequently used as basic functions.

Unknown coefficients are items that must be computed to obtain a rough solution.

Bernoulli polynomials are polynomials that have Bernoulli coefficients.

$$\frac{te^{zt}}{e^{t}-1} = \sum_{0}^{\infty} B_n(\tau)\frac{\tau^n}{n!} \tag{3}$$

Bernoulli polynomials have an explicit formula that is written as" and cite this sentence.

$$B_n(\tau) = \sum_{n=0}^{m}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+k)^m \quad \text{or}$$

$$B_n(\tau) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k \tag{4}$$

The few Bernoulli polynomials can be expressed as:

$B_0(\tau) = 1$, $B_1(\tau) = \tau - \frac{1}{2}$,$B_2(\tau) = \tau^2 - \tau + \frac{1}{6}$

$B_3(x) = \tau^3 - \frac{3}{2}\tau^2 + \frac{1}{2}\tau$, ,

$B_4(\tau) = \tau^4 - 2\tau^3 + \tau^2 - \frac{1}{30}$,

$B_5(\tau) = \tau^5 - \frac{5}{2}\tau^4 + \frac{5}{3}\tau^3 - \frac{1}{6}\tau$,

$B_6(\tau) = \tau^6 - 3\tau^5 + \frac{5}{2}\tau^4 - \frac{1}{2}\tau^2 + \frac{1}{42}$.

3. Algorithm of (Pgm) for solving (Lffides) via Bernoulli basis

In this section, the Petrov-Galerkian method (PGM) is used to investigate the approximation solution of the linear Fredholm fractional integro-differential eq.(1) using Bernoulli polynomials of six degrees in the interval [0,1].

$$D_{*}^{\alpha}u(x) = f(x) + \int_a^b k(x,t)u(t)dt \ , \quad u(0) = \beta , x \in [a,b] \tag{5}$$

Our approach being by taking the fractional integration to both side of eq.(5) we get

$$u(x) = u(0) + I^{\alpha}f(x) + I^{\alpha}(\int_a^b k(x,t)u(t)dt \tag{6}$$

To approximate solution of eq.(5), we use the Bernoulli polynomials basis on [a,b] as:

$$u(x) = \sum_{i=0}^{n} a_i b_{i,n}(x) \tag{7}$$

Where ($a_i, i = 0,1,2,\dots,n$) are unknown constant to be determined substituting eq.(7) in to eq.(6),we get:

$$\sum_{i=0}^{n} a_i b_{i,n}(x) = u(0) + I^{\alpha}f(x) + I^{\alpha}(\int_a^b k(x,t)\sum_{i=0}^{n} a_i b_{i,n}(t)d \tag{8}$$

Hence:

$$\sum_{i=0}^{n} a_i b_{i,n}(x) - I^{\alpha}(\int_a^b k(x,t)\sum_{i=0}^{n} a_i b_{i,n}(t)dt = u(0) + I^{\alpha}f(x) \tag{9}$$

In the next step, apply (PGM) for eq.(5) is a numerical method for finding $u(x) = \sum_{i=0}^{n} a_i b_{i,n}(x) \in x_n$. Such that a_i is unknown and must be determined from eq. (9) which can be written as:

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$$< \sum_{i=0}^{n} a_i b_{i,n}(x) - I^{\alpha}\left(\int_a^b k(x,t) \sum_{i=0}^{n} a_i b_{i,n}(t) dt\,, b^*_{j,n} > \right.$$

$$< \ u(0) + I^{\alpha} f(x), b^*_{j,n} > \qquad (10)$$

Thus

$$\int_0^1 \{ \sum_{i=0}^{n} a_i b_{i,n}(x) - I^{\alpha}\left(\int_a^b k(x,t) \sum_{i=0}^{n} a_i b_{i,n}(t) dt\right\} b^*_{j,n} = \int_0^1 \{ u(0) + I^{\alpha} f(x) dx \} b^*_{j,n} \qquad (11)$$

Then, eq.(11) is equivalent to linear system can be formed as follows:

$$\mathrm{L}(\mathrm{x}, a_i) = \int_0^1 \{ \sum_{i=0}^{n} a_i b_{i,n}(x) - I^{\alpha}\left(\int_a^b k(x,t) \sum_{i=0}^{n} a_i b_{i,n}(t) dt\right\} b^*_{j,n} \qquad (12)$$

$$m_j = \int_0^1 [\, u(0) + I^{\alpha} f(x) dx] b^*_{j,n}$$

We can represent the system eq.(12) as a matrix form:

$$\mathrm{LA} = \mathrm{M} \qquad (13)$$

Where

$$\mathrm{L} = \begin{bmatrix} \int_0^1 L(x,a_0) b^*_{0,n}\, dt & \cdots & \int_0^1 L(x,a_n) b^*_{0,n}\, dt \\ \vdots & \ddots & \vdots \\ \int_0^1 L(x,a_0) b^*_{n,n}\, dt & \cdots & \int_0^1 L(x,a_n) b^*_{n,n}\, dt \end{bmatrix},$$

$$\mathrm{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \end{bmatrix} \qquad (14)$$

Then solving the eq.(14) to calculate the value a_i

4. Application Examples on LFFIDE by ($\mathcal{P}\mathbb{G}\mathcal{M}$)

Consider we have LFFIDE

$$D^{\alpha}\mathrm{u}(\mathrm{x}) = \mathrm{e}^{x} + 0.2021509 - \frac{2}{17}\int_0^1 u(\mathrm{t})\, \mathrm{dt}$$

Initial value u(0)= 1

The exact solution is given as: $\mathrm{u}(\mathrm{x}) = \mathrm{e}^{x}$

$$L(x,a_0) = \int_0^1 \{ (a_0 + a_1(x - \tfrac{1}{2}) + a_2(x^2 - x + \tfrac{1}{6}) + a_3(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x) + \tfrac{2}{17}\, I^{\alpha}(\int_0^1 (a_0 + a_1(t - \tfrac{1}{2}) + a_2(t^2 - t + \tfrac{1}{6}) + a_3(t^3 - \tfrac{3}{2}t^2 + \tfrac{1}{2}t)) dt \}\ dx * 1$$

$$L(x,a_0) = \int_0^1 \{ (a_0 + a_1(x - \tfrac{1}{2}) + a_2(x^2 - x + \tfrac{1}{6}) + a_3(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x) + \tfrac{2}{17}\, I^{\alpha}(t a_0 + \left(\tfrac{t^2}{2} - \tfrac{1}{2}t\right) a_1 + \left(\tfrac{t^3}{3} - \tfrac{t^2}{2} + \tfrac{1}{6}t\right) a_2 + \left(\tfrac{t^4}{4} - \tfrac{t^3}{2} + \tfrac{t^2}{4}\right) a_3]_0^1 \}\, dx * 1$$

$$= \int_0^1 \{ (a_0 + a_1(x - \tfrac{1}{2}) + a_2(x^2 - x + \tfrac{1}{6}) + a_3(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x) + \tfrac{x^{\alpha}}{\Gamma(\alpha+1)} a_0 \} dx * 1$$

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \bigoplus z)$$

<u>Case (1)</u> where $\alpha \in [0,1]$ choose it randomly

When $\alpha = 1$ then

$$L(x,a_0) = \int_0^1 \{ (a_0 + a_1(x - \tfrac{1}{2}) + a_2(x^2 - x + \tfrac{1}{6}) + a_3\left(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x\right) + \tfrac{2}{17} x a_0 \} dx * 1$$

$$= \left(x + \tfrac{x^2}{17}\right) a_0 + \left(\tfrac{x^2}{2} - \tfrac{1}{2}x\right) a_1 + \left(\tfrac{x^3}{3} - \tfrac{x^2}{2} + \tfrac{1}{6}x\right) a_2 + \left(\tfrac{t^4}{4} - \tfrac{t^3}{2} + \tfrac{t^2}{4}\right) a_3]_0^1$$

$$= 1.0589\, a_0$$

$$L(x,a_1) = \int_0^1 \{ (a_0 + a_1(x - \tfrac{1}{2}) + a_2(x^2 - x + \tfrac{1}{6}) + a_3\left(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x\right) + \tfrac{2}{17} x a_0 \} dx * (x - \tfrac{1}{2})$$

$$= \left(\tfrac{2x^3}{51} + \tfrac{8x^2}{17} - \tfrac{x}{2}\right) a_0 + \left(\tfrac{x^3}{3} - \tfrac{x^2}{2} + \tfrac{1}{4}x\right) a_1 + \left(\tfrac{x^4}{4} - \tfrac{x^3}{2} + \tfrac{x^2}{3} - \tfrac{1}{12}x\right) a_2 + \left(\tfrac{x^5}{5} - \tfrac{x^4}{2} + \tfrac{5x^3}{12} - \tfrac{x^2}{8}\right) a_3]_0^1$$

$$= 0.0098 a_0 + \ 0.0833 a_1 - 0.0083 a_3$$

$$L(x,a_2) = \int_0^1 \{ (a_0 + a_1(x - \tfrac{1}{2}) + a_2(x^2 - x + \tfrac{1}{6}) + a_3\left(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x\right) + \tfrac{2}{17} x a_0 \} dx * (x^2 - x + \tfrac{1}{6})$$

$$= \left(\tfrac{x^4}{34} + \tfrac{5x^3}{17} - \tfrac{25x^2}{51} + \tfrac{1}{6}x\right) a_0 + \left(\tfrac{x^4}{4} - \tfrac{x^3}{2} + \tfrac{x^2}{3} - \tfrac{1}{12}x\right) a_1 + \left(\tfrac{x^5}{5} - \tfrac{x^4}{2} + \tfrac{4x^3}{9} - \tfrac{x^2}{6} + \tfrac{1}{36}x\right) a_2 + \left(\tfrac{x^6}{6} - \tfrac{x^5}{2} + \tfrac{13x^4}{24} - \tfrac{x^3}{4} + \tfrac{x^2}{24}\right) a_3]_0^1$$

$$= 0.0055 a_2$$

$$L(x,a_3) = \int_0^1 \{ (a_0 + a_1(x - \tfrac{1}{2}) + a_2(x^2 - x + \tfrac{1}{6}) + a_3\left(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x\right) + \tfrac{2}{17} x a_0 \} dx * (x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x)$$

$$= \left(\tfrac{2x^5}{85} + \tfrac{7x^4}{34} - \tfrac{49x^3}{102} + \tfrac{1}{4}x^2\right) a_0 + \left(\tfrac{x^5}{5} - \tfrac{x^4}{2} + \tfrac{5x^3}{12} - \tfrac{x^2}{8}\right) a_1 + \left(\tfrac{x^6}{6} - \tfrac{x^5}{2} + \tfrac{13x^4}{24} - \tfrac{x^3}{4} + \tfrac{x^2}{24}\right) a_2 + \left(\tfrac{x^7}{7} - \tfrac{x^6}{2} + \tfrac{13x^5}{20} - \tfrac{3x^4}{8} + \tfrac{x^3}{12}\right) a_3]_0^1$$

$$= -0.0009 a_0 - 0.0083 a_1 + 0.0011 a_3$$

$$m_j = \int_0^1 [u(0) + \ I^{\alpha}(f(x))]\, Q^*_{i,n}$$

$$m_j = \int_0^1 [1 + \ I^{\alpha}(\mathrm{e}^{x} + 0.2022)] * 1\, dx$$

$$m_0 = \int_0^1 [1 + \ I^{\alpha}\,(1.2022 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3)] * 1\, dx$$

$$m_0 = \int_0^1 (1 + \tfrac{1.2022 x^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{\Gamma(2) x^{\alpha+1}}{\Gamma(\alpha+2)} + \tfrac{\Gamma(3) x^{\alpha+2}}{2\Gamma(\alpha+3)} + \tfrac{\Gamma(4) x^{\alpha+3}}{6\Gamma(\alpha+4)}) dx$$

$\alpha = 1$

$m_0 = \int_0^1 (1 + 1.2022x + 0.5x^2 + 0.1666x^3 + 0.0416x^4)dx$

$m_0 = x + 0.6011x^2 + 0.1666x^3 + 0.0416x^4 + 0.0083x^5]_0^1$

$m_0 = 1.8176$

$m_1 = \int_0^1 (1 + 1.2022x + 0.5x^2 + 0.1666x^3 + 0.0416x^4) * \left(x - \frac{1}{2}\right) dx$

$m_1 = -0.5x + 0.1994x^2 + 0.3174x^3 + 0.1041x^4 + 0.0291x^5 + 0.0069x^6]_0^1$

$m_1 = 0.1569$

$m_2 = \int_0^1 (1 + 1.2022x + 0.5x^2 + 0.1666x^3 + 0.0416x^4) * \left(x^2 - x + \frac{1}{6}\right) dx$

$m_2 = 0.1666x - 0.3998x^2 - 0.0396x^3 + 0.1825x^4 + 0.0681x^5 + 0.0208x^6 + 0.0059x^7]_0^1$

$m_2 = 0.0045$

$m_2 = \int_0^1 (1 + 1.2022x + 0.5x^2 + 0.1666x^3 + 0.0416x^4) * (x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)dx$

$m_2 = 0.25x^2 - 0.2996x^3 - 0.1383x^4 + 0.1071x^5 + 0.0451x^6 + 0.0148x^7 + 0.0052x^8]_0^1$

$m_2 = -0.0157,$

$$\begin{bmatrix} 1.0589 & 0 & 0 & 0 \\ 0.0098 & 0.0833 & 0 & -0.0083 \\ 0 & 0 & 0.0055 & 0 \\ -0.0009 & -0.0083 & 0 & 0.0011 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1.8176 \\ 0.1569 \\ 0.0045 \\ -0.0157 \end{bmatrix}$$

$a_0 = 1.7165$, $a_1 = 1.6094$, $a_2 = 0.8182$, $a_3 = -0.7245$

Thus, the approximate solution of equation, when $\alpha = 1$ becomes:

y(x)=$1.7165 + 1.6094\left(x - \frac{1}{2}\right) + 0.8182\left(x^2 - x + \frac{1}{6}\right) - 0.7245(x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)$.

In the same way, we find α=0.5 and α= 0.8 As shown in the following table 1.

Table 1. Numerical results of example 1.

	Exact	Approximate solution					
X	$u(x) = e^x$	α=1	Absolute Error	α=0.5	Absolute Error	α= 0.8	Absolute Error
0.1	1.105171	1.109332	0.004161	1.342628	0.237457	1.181397	0.076226
0.2	1.221402	1.204304	0.017098	1.557014	0.335612	1.298135	0.076733
0.3	1.349858	1.328681	0.021177	1.747256	0.397398	1.446381	0.096523
0.4	1.491824	1.478116	0.013708	1.923521	0.431697	1.619965	0.128141
0.5	1.648721	1.648262	0.000459	2.095978	0.447257	1.812715	0.163994
0.6	1.822118	1.834772	0.012654	2.274796	0.452678	2.018458	0.196340
0.7	2.013752	2.031125	0.017373	2.470141	0.456389	2.231024	0.217272
0.8	2.225540	2.239496	0.013956	2.692183	0.466643	2.444240	0.218700
0.9	2.459903	2.449016	0.010887	2.951089	0.491186	2.651936	0.192033
1.0	2.718281	2.657512	0.060769	3.257028	0.538747	2.847940	0.129659

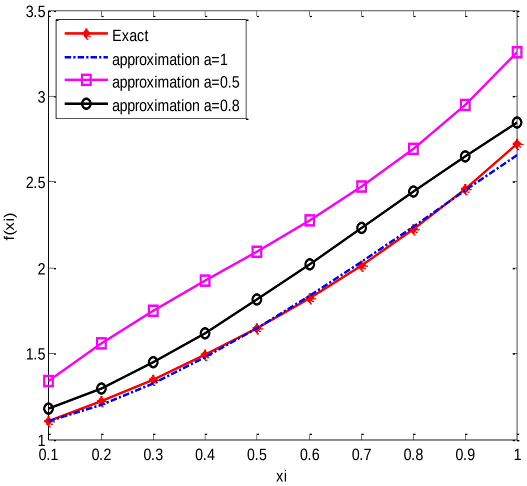

Figure 1. Numerical results of example 1.

5. Application example on L-FFIDE by Petrov-Galerkin method

Consider L-FFIDE

$D^\alpha u(x) = cosx - 0.000051 + \frac{1}{3}\int_0^1 u(t)\, dt$

with initial condition u(0)= 0

And the exact solution: u(x) = $sinx$

By using PGM method, we can solve the above example as follows:

Table 2. Numerical results of example 2.

	Exact	Approximate solution					
X	$u(x) = sinx$	α=1	Error	α=0.5	Error	α= 0.8	Error
0.1	0.099834	0.103037	0.003203	0.438074	0.338240	0.106441	0.006607
0.2	0.198669	0.239245	0.040576	0.637547	0.438878	0.255734	0.057065
0.3	0.295521	0.364623	0.069102	0.794284	0.498763	0.389522	0.094001
0.4	0.389419	0.479993	0.090574	0.908283	0.518864	0.508942	0.119523
0.5	0.479428	0.586177	0.106749	0.993671	0.514243	0.615131	0.135703
0.6	0.564648	0.683997	0.119349	1.055157	0.490509	0.709226	0.144578
0.7	0.644234	0.774275	0.130041	1.099804	0.455570	0.792364	0.148130
0.8	0.717398	0.857833	0.140435	1.134676	0.417278	0.865682	0.148284
0.9	0.783421	0.935493	0.152069	1.166836	0.383415	0.930317	0.146896
1.0	0.841667	1.008077	0.166410	1.203346	0.361679	0.987406	0.145739

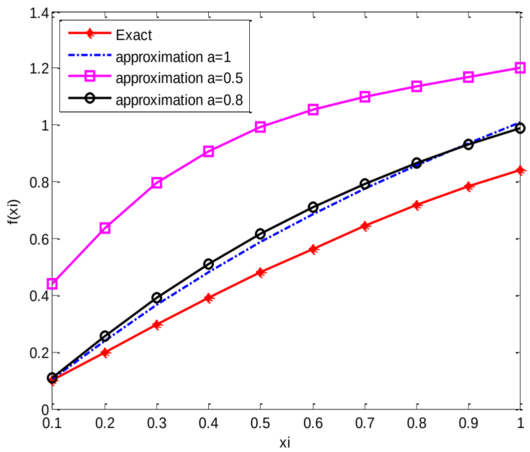

Figure 2. Numerical results of example 2.

CONCLUSIONS

We studied the linear Fractional Fredholm integro-differential equations (LFFIDE) by using Bernoulli's polynomials technique. For implementing the approximate solution to (LFFIDE), we used the Petrove-Glerkain method to test the approximate solution against the exact solution. The following conclusions are remarked:

1. The proposed numerical methods are efficient and accurate to estimate the solution of these equation.
2. In most cases, the (LFFIDEs) are usually difficult to solve analytically, so we can solved by approximation method.
3. This method can be applied to nonlinear (LFFIDEs).

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