On Subclass of Meromorphic Univalent Functions Defined by Multiplier Transformation

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ABSTRACT
In this paper, we will investigate and discuss a new class of meromorphic univalent functions defined by multiplier transformation which is \( R(c, \beta, y, \alpha) \), as well as study the coefficient estimates and growth theorems, and then another line in this work, upon to get the close under the convex linear combination.

KEYWORDS: meromorphic univalent function; multiplier transformation; subordination theorem; close under convex linear combination.

INTRODUCTION
Recently appeared more studied about meromorphic univalent functions defined on some types operators, such as in 2011 Atshan and Joudah [1] studied the class of meromorphic univalent functions with some geometric properties have been got, in 2018 Shabeeb [8] introduced a class \( S(\beta, \alpha) \), of meromorphic univalent functions defined by Ruschweyh derivative and got geometric some properties.

Let \( G \) be the class of functions of the form:

\[
f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k
\]

which are analytic and meromorphic univalent in the perforated unit disc

\[U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.
\]

Suppose \( R \) be a subclass of \( G \) of functions of the form:

\[
f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k
\]

where \( a_k \geq 0, k \in \mathbb{N} \) (2)

Definition/ A function \( f \) be given by (2) then the class \( R(c, \beta, y, \alpha) \) is defined by

\[
\frac{z(L_1(c, \beta) f(z))'}{(L_1(c, \beta) f(z))} + 1 = \frac{4\alpha y(L_1(c, \beta) f(z))'}{(1 + 2y)(L_1(c, \beta) f(z))} < \alpha \quad (3)
\]

For \( 0 < y < 1, \alpha \equiv < 1 \)

Several authors studied geometric properties of this function subclass for other classes, Mille [6]

Where \( L_1(c, \beta) \) is a multiplier transformation studied by Kho and Kim [3], Liue and Srivastave[5]. And then:
\[ L_1(c, B) f(z) = z^{-1} + \sum_{k=1}^{\infty} \left( \frac{k + \beta}{1 + \beta} \right)^c a_k z^k \quad (\beta \geq 0) \]

have different styles to introduce the written text. For example, the introduction of a Functional Specification consists of information that the whole document is yet to explain. If a user guide is written, the introduction is about the product. In a report, the introduction gives a summary about the report contents.

**MAIN RESULTS.**

**Theorem (2.1)** If \( f \in R(c, \odot, y, \rangle \) and if and only if

\[
\sum_{k=1}^{\infty} \left( \frac{k + \beta}{1 + \beta} \right)^c \left( (k + 1) + \alpha(-1 + y(4k - 2)) \right) a_k \leq \alpha(1 + 6y) \quad (5)
\]

Where \( 0 < y < 1, \ 0 < \alpha < 1 \)

The result is sharp for the function.

\[ \sum_{k=1}^{\infty} \left( \frac{k + \beta}{1 + \beta} \right)^c \left( (k + 1) + \alpha(-1 + y(4k - 2)) \right) a_k \leq \alpha(1 + 6y) \quad (6) \]

**Proof** Suppose that the inequality’s (5) holds true and \( |z| = 1 \). Then from (3), we’ve got:

\[ |z(L_1(c, \odot) f(z))' + (L_1(c, \odot) f(z))| \]

\[ -|((1+2y)(L_1(c, \odot) f(z))-4zy(L_1(c, \odot) f(z)))'| \]

\[
\sum_{k=1}^{\infty} (k + 1) \left( \frac{k + \beta}{1 + \beta} \right)^c a_k z^k \leq \alpha(1 + 6y) \quad (7)
\]

\[ \sum_{k=1}^{\infty} (k + 1) \left( \frac{k + \beta}{1 + \beta} \right)^c a_k z^k \leq \alpha(1 + 6y) \quad (8) \]

Eventually, sharpness follows if we take.

\[ f(z) = z^{-1} + \sum_{k=1}^{\infty} \left( \frac{k + \beta}{1 + \beta} \right)^c a_k z^k \]

\[ (1 + 2y)(L_1(c, \odot) f(z))' + (L_1(c, \odot) f(z)) \]

Corollary (2.1) Let \( f \in R(c, \odot, y, \rangle \) then

\[ a_k \leq \left( \frac{k + \beta}{1 + \beta} \right)^c ((k + 1) + \alpha(-1 + y(4k - 2))) \quad (9) \]

Next we get the following distortion and growth theorems for the class \( R(c, \odot, y, \rangle \).

**Theorem (2.2)** Let \( f \in R(c, \odot, y, \rangle \) then
The result is sharp of the next function.

\[ f(z) = z^{-1} + \frac{\alpha(1+6y)}{(1)^{(2+\alpha(-1+2y))}} z \]  

**Theorem (2.3)**: Let \( f(\alpha, \beta, y, \langle \rangle) \) then

\[
\frac{1}{r^2} - \frac{\alpha(1+6y)}{(2+\alpha(-1+2y))} r \leq |L_1(\alpha, \beta) f(z)| \leq \frac{1}{r} + \frac{\alpha(1+6y)}{(2+\alpha(-1+2y))} r
\]  

(\( |z| = r < 1 \))

The result is sharp of the next function:

\[ f(z) = z^{-1} + \frac{\alpha(1+6y)}{(1)^{(2+\alpha(-1+2y))}} z \]  

**Proof**: Let \( f(\alpha, \beta, y, \langle \rangle) \) then by theorem (2.1)

We get.

\[
(1)^{(2+\alpha(-1+2y))} \sum_{k=1}^{n} a_k \leq \frac{(k+\beta)^c}{(1)^{(k+1)+\alpha(-1+\gamma(4k-2))}} \nu_k \]

\[ \leq \alpha(1+6y) \]

\[
\sum_{k=1}^{\infty} a_k \leq \frac{\alpha(1+6y)}{(1)^{(2+\alpha(-1+2y))}}\]

Hence

\[ |L_1(\alpha, \beta) f(z)| \leq \frac{1}{|z|} + \sum_{k=1}^{\infty} \left( \frac{k+\beta}{1+\beta} \right)^c a_k |z|^k \]

\[ \leq \frac{1}{|z|} + (1)^c |z| \sum_{k=1}^{\infty} a_k \]

\[ \leq \frac{1}{|z|} + \frac{\alpha(1+6y)}{(2+\alpha(-1+2y))} r \]

Similarly

\[ |L_1(\alpha, \beta) f(z)| \geq \frac{1}{|z|} - \sum_{k=1}^{\infty} \left( \frac{k+\beta}{1+\beta} \right)^c a_k |z|^k \]

\[ \geq \frac{1}{|z|} - (1)^c |z| \sum_{k=1}^{\infty} a_k \]

\[ \geq \frac{1}{|z|} - \frac{\alpha(1+6y)}{(2+\alpha(-1+2y))} r \]

From (13) and (14), we get (10).
From (18) and (19), we get (15).

**Theorem (2.4)**/ The class $R(c, \circ, y, \odot)$ is close under convex linear combination.

**Proof/** Let $f^1$ and $f^2$ belong to the class $R(c, \circ, y, \odot)$ for $0 \leq \lambda \leq 1$. We must clarify that.

$$[f^1(z) + \lambda f^2(z)] R(c, \circ, y, \odot)$$

And so we have:

$$z^{-1} + \sum_{k=1}^{\infty} \left[ (1 - \lambda) a_{k,1} + \lambda a_{k,2} \right] z^k$$

Then:

$$\sum_{k=1}^{\infty} \left( \frac{k + \beta}{1 + \beta} \right)^{c} ((k + 1) + \alpha(-1 + y(4k - 2))) a_{k,1} + (1 - \lambda) a_{k,2}$$

$$= \lambda \sum_{k=1}^{\infty} \left( \frac{k + \beta}{1 + \beta} \right)^{c} ((k + 1) + \alpha(-1 + y(4k - 2))) a_{k,1}$$

$$+ (1 - \lambda) \sum_{k=1}^{\infty} \left( \frac{k + \beta}{1 + \beta} \right)^{c} ((k + 1) + \alpha(-1 + y(4k - 2))) a_{k,2}$$

$$\leq \lambda \alpha(1 + 6y) + (1 - \lambda) \alpha(1 + 6y)$$

$$= \alpha(1 + 6y)$$

Then by theorem (2.1). We have $h(z) \in R(c, \circ, y, \odot)$.

**Theorem (2.5)**/ Let

$$f^i(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k,i} z^k \in R(c, \beta, y, \alpha), \quad i \in \{1, 2, \ldots, j\}$$

and $0 < \delta < 1$

$$\sum_{i=1}^{j} S_i = 1$$

Such that

The function $X$ defined

$$X = \sum_{i=1}^{j} S_i f^i(z) \in R(c, \beta, y, \alpha)$$

**Proof/** By theorem (2.1) for every $i \in \{1, 2, \ldots, j\}$. We have

$$\sum_{k=1}^{\infty} \left( \frac{k + \beta}{1 + \beta} \right)^{c} ((k + 1) + \alpha(-1 + y(4k - 2))) a_{k,i} \leq 1$$

Since

$$X(z) = \sum_{i=1}^{j} S_i (z^{-1} + \sum_{k=1}^{\infty} a_{k,i} z^k)$$

$$= z^{-1} + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{j} S_i a_{k,i} \right) z^k$$

$$= \sum_{k=1}^{\infty} \left[ \sum_{i=1}^{j} S_i a_{k,i} \right] z^k$$

So:

$$\sum_{k=1}^{\infty} \left[ \sum_{i=1}^{j} S_i a_{k,i} \right] z^k \leq 1$$

Hence $x \in R(c, \circ, y, \odot)$.

**Theorem (2.6)**/ Let $f \in R(c, \beta, y, \alpha)$ then $f$ is univalent meromorphic convex of order $\theta$ ($0 \leq \theta < 1$) in the disc $|z| < R$.

Where:

$$R \inf_{k} \left[ \left( \frac{k + \beta}{1 + \beta} \right)^{c} ((k + 1) + \alpha(-1 + y(4k - 2))) \right]^{1/(k - 1)}$$

**Proof/** It is enough to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| + 2 \leq 1 - \theta \quad (20)$$

for $|z| < R$.

But:

$$\left| \frac{zf''(z)}{f'(z)} \right| + 2 \leq \left| \frac{zf''(z) + 2f'(z)}{f'(z)} \right|$$

$$\leq \frac{\sum_{k=1}^{\infty} (k + 1) a_k |z|^{k-1}}{1 - \sum_{k=1}^{\infty} k a_k |z|^{k-1}}$$

by (20) we have.

$$\sum_{k=1}^{\infty} k (k + 1) a_k |z|^{k-1} \leq 1 - \theta$$

Or

$$\sum_{k=1}^{\infty} \frac{k (k + 1) a_k |z|^{k-1}}{1 - \theta} \leq 1 \quad (21)$$

Since $f \in R(c, \beta, y, \alpha)$, We have.
\[
\sum_{k=1}^{\infty} \frac{\left(\frac{k + \beta}{1 + \beta}\right)^c}{(k + 1) + \alpha(-1 + y(4k - 2))} \alpha(1 + 6y)
\]

Hence (21) will be true if.

\[
\frac{k(k-\theta+2)}{1-\theta} |z|^{k-1} \leq \frac{\left(\frac{k + \beta}{1 + \beta}\right)^c ((k+1)+\alpha(-1+y(4k-2)))}{\alpha(1+6y)}
\]

Or equivalently

\[
|z| \leq \left[ \frac{\left(\frac{k + \beta}{1 + \beta}\right)^c (1-\theta)((k+1)+\alpha(-1+y(4k-2)))}{(k(k-\theta+2))(\alpha(1+6y))} \right]^{\frac{1}{k-1}}
\]

closed under the convex linear combination.

REFERENCES

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