Certain Subclasses of Meromorphic Univalent Function Involving Differential Operator

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ABSTRACT

The main object of the present paper is to introduce the class of meromorphic univalent function $K^*(\sigma,\tau,S)$ defined by differential operator with study some geometric properties like coefficient inequality, growth theorem and distortion theorem, radii of starlikeness and convexity of $f(z)$ in the class $K^*(\sigma,\tau,S)$. Also the concept of convolution (Hadamard product) investigate and Neighborhoods of the elements of class $K^*(\sigma,\tau,S)$ are obtained.

KEYWORDS: Meromorphic univalent function, Differential operator, Hadamard product, Starlike function, Convex function, Neighborhood.

INTRODUCTION

Denote by $\Sigma$ the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are analytic and univalent in the punctured open unit disk

$$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$$

A function $f \in \Sigma$ is said to be meromorphic starlike if

$$R \left( \frac{zf''(z)}{f'(z)} \right) < 0, z \in U^*$$

We denote by $\Sigma^*$ the class of all meromorphic starlike function.

A function $f \in \Sigma$ is said to be meromorphic convex if

$$R \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 0, z \in U^*$$

The class of all meromorphic convex functions will be denoted by $\Sigma^c$.

Let $f \in \Sigma$ be of the form (1) and $\gamma, \delta$ be real numbers with $\gamma \geq \delta \geq 0$, then the analogue of the differential operator given in [6] is defined as follows:

$$D^\gamma_{\gamma,\delta} f(z) = f(z)$$

$$D^\gamma_{\gamma,\delta} f(z) = D^\gamma_{\gamma,\delta} f(z)$$

$$= \gamma \delta (z^2 f(z))'' + (\gamma - \delta) \frac{(z^2 f(z))'}{z} + (1 - \gamma + \delta) f(z)$$

$$D^S_{\gamma,\delta} f(z) = D_{\gamma,\delta} (D^S_{\gamma,\delta} - 1 f(z)),$$

$z \in U^* , \sigma \in N \{1,2,3 \ldots \}$.

If $f \in \Sigma$ is given by (1), then we have

$$D^S_{\gamma,\delta} f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} B(\gamma, \delta, l) a_n z^l,$$

$z \in U^*$

(2)

where

$$B(\gamma, \delta, l) = [(l + 2)\gamma \delta + \gamma - \delta ](l + 1) + 1.$$

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Note that for $\delta = 0$ and $\gamma = 1$, we obtain the differential operator introduced by [1].
In this paper, we shall try to obtain the coefficient estimates of the class $K^*(\sigma, \tau, S)$, growth and distortion theorems, radii of starlikeness and convexity, convolution and neighborhoods of the elements for the class $K^*(\sigma, \tau, S)$.

**A SET OF MAIN RESULTS**

In this section, we define the following subclasses of meromorphic function utilizing the operator $D_{\gamma, \delta}^S$.

**Definition 1:**
A function $f(z) \in \Sigma$ of the form (1) is in the class $K^*(\sigma, \tau, S)$ if it satisfies the following inequality
\[
\frac{z(D_{\gamma, \delta}^S f(z))'' + 2(D_{\gamma, \delta}^S f(z))'}{z(D_{\gamma, \delta}^S f(z))'' + 2\tau(D_{\gamma, \delta}^S f(z))'} < \sigma \quad (3)
\]
(0 $\leq \tau < 1$ , $0 < \sigma \leq 1$ , $S = 1, 2, 3, ...$)

This class was studied by many researchers by (for example Juma [3] and Mahmoud [5]).

1. **Coefficient Inequality:**

   We derive the coefficient inequality for the class $K^*(\sigma, \tau, S)$ in the next theorem.

**Theorem 1:**

The function $f(z)$ given by (1) is in the class $K^*(\sigma, \tau, S)$, if and only if
\[
\sum_{l=1}^{\infty} B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))] a_l \\
\leq 2\sigma(1 - \tau) \quad (4)
\]

The result is sharp for the function $f(z)$ is given by:

\[
f(z) = \frac{1}{z} + \frac{2\sigma(1 - \tau)}{Bl[l(1 + \sigma) + (1 + \sigma(2\tau - 1))] z^l}, l \geq 1 \quad (5)
\]

**Proof:** Suppose (4) holds, and if $|z| = 1$. Then by (3), we have
\[
\left|\frac{z(D_{\gamma, \delta}^S f(z))'' + 2(D_{\gamma, \delta}^S f(z))'}{z(D_{\gamma, \delta}^S f(z))'' + 2\tau(D_{\gamma, \delta}^S f(z))'}\right| < \sigma
\]
\[
\left|\frac{z(D_{\gamma, \delta}^S f(z))'' + 2(D_{\gamma, \delta}^S f(z))'}{z(D_{\gamma, \delta}^S f(z))'' + 2\tau(D_{\gamma, \delta}^S f(z))'}\right| < \sigma
\]
\[
-\sigma \left|z^{-1} + \sum_{l=1}^{\infty} B(\gamma, \delta, l) z^l\right|'' + 2 \left|z^{-1} + \sum_{l=1}^{\infty} B(\gamma, \delta, l) z^l\right|' \\
\leq 0
\]
\[
-\sigma \leq 0
\]
\[
\left|z^{-1} + \sum_{l=1}^{\infty} B(\gamma, \delta, l) z^l\right|'' + 2 \left|z^{-1} + \sum_{l=1}^{\infty} B(\gamma, \delta, l) z^l\right|' \\
\leq 0
\]

Let $B = B(\gamma, \delta, l)$, Then we get
\[
\left|z^{-1} + \sum_{l=1}^{\infty} B(l(1 - 1)) a_l z^{l-2}\right| + 2\left[-z^{-2} + \sum_{l=1}^{\infty} B(l) a_l z^{l-1}\right] \\
\leq 0
\]
\[
2z^{-2} + \sum_{l=1}^{\infty} B(l(1 - 1)) a_l z^{l-1} - 2 z^{-2} \\
+ \sum_{l=1}^{\infty} B(l) a_l z^{l-1} \\
-\sigma \left|2 z^{-2} + \sum_{l=1}^{\infty} B(l(l - 1)) a_l z^{l-1}\right| \\
- 2\tau(2) z^{-2} + \sum_{l=1}^{\infty} B(l(l - 1)) a_l z^{l-1} \\
\leq 0
\]
\[
\left|\sum_{l=1}^{\infty} B(l(l - 1)) a_l z^{l-1}\right| + 2 \sum_{l=1}^{\infty} B(l) a_l z^{l-1} \\
\leq 0
\]
\[
-\sigma \left|2 - 2\tau\right| z^{-2} + \sum_{l=1}^{\infty} B(l(l - 1)) a_l z^{l-1} \\
+ 2\tau \sum_{l=1}^{\infty} B(l) a_l z^{l-1} \\
\leq 0
\]

\[
\left|\sum_{l=1}^{\infty} B(l(l - 1)) a_l z^{l-1}\right| \\
\leq 0
\]
\[
\left|\sum_{l=1}^{\infty} B(l(l - 1)) a_l z^{l-1}\right| \\
\leq 0
\]
\[
-\sigma \left|2 - 2\tau\right| z^{-2} + \sum_{l=1}^{\infty} B(l(l - 1)) a_l z^{l-1} \\
+ 2\tau \sum_{l=1}^{\infty} B(l) a_l z^{l-1} \\
- 2\sigma(1 - \tau) |z|^{-2} \leq 0.
\]
Since \(|z| = 1\), we have
\[
\sum_{l=1}^{\infty} B(l(1 + \sigma) + (1 + \sigma(2\tau - 1)))a_l \leq 2\sigma(1 - \tau),
\]
thus, by the maximum modulus theorem, we get \(\in K^*(\sigma, \tau, S)\).

Conversely, if \(f(z)\) of the form (1) is in the class \(K^*(\sigma, \tau, S)\), then by (3) we get
\[
\left|z(D_{\tau, \delta} f(z))^{(n)} + 2(D_{\tau, \delta} f(z)^{(n-1)})\right| < \sigma
\]
for all \(z\) and \(l \geq 1\). Therefore, we get
\[
\sum_{l=1}^{\infty} B(l(1 + \sigma) + (1 + \sigma(2\tau - 1)))a_l \leq 2\sigma(1 - \tau),
\]
which is the Corollary 1.

**Theorem 2:**
If \(f(z) \in K^*(\sigma, \tau, S)\) of the form (1), then for \(0 < |z| = r \), we get
\[
\frac{1}{r} + \frac{\sigma(1 - \tau)}{B(1 + \tau\sigma)} \leq |f(z)| \leq \frac{1}{r} + \frac{\sigma(1 - \tau)}{B(1 + \tau\sigma)} r,
\]
with equality for \(f(z) = \frac{1}{z} + \frac{\sigma(1 - \tau)}{B(1 + \tau\sigma)} z\).

**Proof:** By hypothesis \(f \in K^*(\sigma, \tau, s)\), we get from Theorem 1
\[
\sum_{l=1}^{\infty} B(l(1 + \sigma) + (1 + \sigma(2\tau - 1)))a_l \leq 2\sigma(1 - \tau),
\]
so, for \(0 < |z| = r < 1\), we have
\[
|f(z)| \leq \frac{1}{r} + \sum_{l=1}^{\infty} a_l |z|^l,
\]
and
\[
\frac{1}{r} + \frac{\sigma(1 - \tau)}{B(1 + \tau\sigma)} \leq |z| = r
\]
and
\[
\geq \frac{1}{r} - \frac{\sigma(1 - \tau)}{B(1 + \tau\sigma)}.
\]
Thus, the proof is completed.

**Theorem 3:**
If \(f(z) \in K^*(\sigma, \tau, S)\) of the form (1), then for \(0 < |z| = r < 1\), we get
\[
\frac{1}{r^2} - \frac{\sigma(1 - \tau)}{B(1 + \tau\sigma)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{\sigma(1 - \tau)}{B(1 + \tau\sigma)},
\]
with equalit for

2. Growth and Distortion Theorems:
We derive some properties distortion and growth of \(f \in K^*(\sigma, \tau, s)\) in the next theorems.
\[ f(z) = \frac{1}{z} + \frac{\sigma(1 - \tau)}{B(1 + \sigma\tau)} z. \]

**Proof:** Utilizing Theorem 1, we get
\[
\sum_{i=1}^{\infty} B[l((1 + \sigma) + (1 + \sigma(2\tau - 1))] a_i \leq 2\sigma(1 - \tau),
\]
then
\[ a_i \leq \frac{\sigma(1 - \tau)}{B(1 + \sigma\tau)} \]

Thus, for \( 0 < |z| = r < 1 \)
\[
\left| f'(z) \right| \leq \left| \frac{-1}{z^2} \right| + \sum_{i=1}^{\infty} l a_i |z|^{l-1},
\]
\[ \leq \frac{1}{r^2} + \frac{\sigma(1 - \tau)}{B(1 + \sigma\tau)} \]

Also,
\[
\left| f'(z) \right| \geq \left| \frac{-1}{z^2} \right| - \sum_{i=1}^{\infty} l a_i |z|^{l-1},
\]
\[ \geq \frac{1}{r^2} - \frac{\sigma(1 - \tau)}{B(1 + \sigma\tau)} \]

### 3. Radii of Starlikeness and Convexity:

The radius of starlikeness and convexity for \( K^*(\sigma, \tau, s) \) is given by the following theorems:

**Theorem 4:**

If \( f(z) \in K^*(\sigma, \tau, S) \) of the form (1), then \( f \) is meromorphically starlike of order \( \lambda \) \((0 \leq \lambda < 1) \) in the disk \(|z| < r_1(\sigma, \tau, S, \lambda)\), where
\[
r_1(\sigma, \tau, S, \lambda) = \inf_{z \in 1} \left\{ \frac{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))](1 - \lambda)}{2\sigma(l + 2 - \lambda)(1 - \tau)} \right\}^{\frac{1}{l+1}}
\]

The result is sharp for \( f(z) \) is given by the following:
\[
f(z) = \frac{1}{z} + \frac{2\sigma(1 - \tau)}{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))] z^l}, \quad l \geq 1.
\]

**Proof:** It is sufficient to prove that
\[
\frac{|zf''(z)|}{f(z)} + 1 \leq 1 - \lambda,
\]
\[
\frac{|zf''(z)|}{f(z)} + 1 = \frac{\sum_{i=1}^{\infty} (l + 1) a_i z^l}{z^{-1} + \sum_{i=1}^{\infty} a_i z^l} \leq \frac{\sum_{i=1}^{\infty} (l + 1) a_i |z|^{l+1}}{1 - \sum_{i=1}^{\infty} a_i |z|^{l+1}} \leq 1 - \lambda.
\]

**Theorem 5:**

If \( f(z) \in K^*(\sigma, \tau, S) \) of the form (1), then \( f \) is meromorphically convex of order \( \phi \) \((0 \leq \phi < 1) \) in \(|z| < r_2(\sigma, \tau, S, \phi)\), where
\[
r_2(\sigma, \tau, S, \phi) = \inf_{z \in 1} \left\{ \frac{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1)](1 - \phi)}{2\sigma(l + 2 - \phi)} \right\}^{\frac{1}{l+1}}
\]

The result is sharp for \( f(z) \) is given by the following:
\[
f(z) = \frac{1}{z} + \frac{2\sigma(1 - \tau)}{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))] z^l}, \quad l \geq 1.
\]

**Proof:** It is sufficient to prove that
\[
\frac{|zf''(z)|}{(f(z))''} + 2 \leq 1 - \phi,
\]
\[
\frac{|zf''(z)|}{(f(z))''} + 2 \leq \frac{\left| z^{l-1} + \sum_{i=1}^{\infty} a_i z^l \right|}{z^{-2} + \sum_{i=1}^{\infty} (l + 1) a_i z^l} \leq 1 - \lambda.
\]
\[
\sum_{l=1}^{\infty} l(l + 1) a_l |z|^{l+1} \leq (1 - \phi) \left( 1 - \sum_{l=1}^{\infty} la |z|^{l+1} \right),
\]

that is, if
\[
\sum_{l=1}^{\infty} [l(l + 1) + l - l\phi] a_l |z|^{l+1} \leq 1 - \phi,
\]
\[
\sum_{l=1}^{\infty} l[l + 2 - \phi] a_l |z|^{l+1} \leq 1 - \phi 
\]
(10)

From (10) and Theorem 1, we obtain
\[
|z|^{l+1} \leq \frac{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))] \times (1 - \phi)}{2\sigma(1 - \tau)l[l + 2 - \phi]}.
\]
\[
|z| \leq \left[ B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))] \times (1 - \phi) \right]^{1/(l+1)}
\]

4. Convolution

**Theorem 6:**
If \( f(z) \) and \( g(z) \in K^*(\sigma, \tau, S) \), then \(( f * g)(z) \in K^*(\rho, \tau, S) \) for
\[
f(z) = \frac{1}{z} + \sum_{l=0}^{\infty} a_l z^l, \quad g(z) = \frac{1}{z} + \sum_{l=0}^{\infty} b_l z^l,
\]
where
\[
(f * g)(z) = \frac{1}{z} + \sum_{l=0}^{\infty} a_l b_l z^l, \quad \rho
\]

\[
\leq \frac{2\sigma^2(1 - \tau)(l + 1)}{2\sigma^2(1 - \tau)(l + 2 - \tau - 1) - B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}^2.
\]

**Proof:** Hence \( f, g \in K^*(\sigma, \tau, S) \), then by using Theorem 1, we get
\[
\sum_{l=1}^{\infty} B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))] a_l \leq 1
\]
and,
\[
\sum_{l=1}^{\infty} B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))] b_l \leq 1
\]
We need to find the largest \( \rho \) such that
\[
\sum_{l=1}^{\infty} \frac{B[l(1 + \rho) + (1 + \rho(2\tau - 1))]}{2\rho(1 - \tau)} a_l b_l \leq 1.
\]

By Cauchy–Schwarz inequality, we have
\[
\sum_{l=1}^{\infty} \frac{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}{2\sigma(1 - \tau)} a_l b_l \leq 1 \quad (11)
\]

To prove Theorem 6, it is sufficient to prove that
\[
\frac{2\rho(1 - \tau)}{2\sigma(1 - \tau)} a_l b_l \leq \frac{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}{2\sigma(1 - \tau)} a_l b_l,
\]

which is equivalent to
\[
\sqrt{a_l b_l} \leq \frac{\rho[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}{\sigma[l(1 + \rho) + (1 + \rho(2\tau - 1))]} \leq 1
\]
From (11), we get
\[
\sqrt{a_l b_l} \leq \frac{2\sigma(1 - \tau)}{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]} \]

We must prove that
\[
\frac{2\sigma(1 - \tau)}{B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]} \leq \frac{\rho[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}{\sigma[l(1 + \rho) + (1 + \rho(2\tau - 1))]} \]

such that
\[
2\sigma^2(1 - \tau)[l(1 + \rho) + (1 + \rho(2\tau - 1))] \leq B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))],
\]
\[
2\sigma^2(1 - \tau)(l + 1 + \rho(l + 2\tau - 1)) \leq B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))],
\]
\[
2\sigma^2(1 - \tau)(l + 1) \leq B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))],
\]
\[
2\sigma^2(1 - \tau)(l + 1) \leq \rho[2\sigma^2(1 - \tau)(l + 2\tau - 1) - B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]],
\]

which gives
\[
\rho \leq \frac{2\sigma^2(1 - \tau)(l + 1)}{2\sigma^2(1 - \tau)(l + 2\tau - 1) - B[l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}^{1/2}.
\]

**Theorem 7:**
If the functions \( f_j(z) (j = 1, 2) \) denoted by
\[
f_j(z) = \frac{1}{z} + \sum_{l=1}^{\infty} a_{l,j} z^l, \quad (a_{l,j} \geq 0, j = 1, 2)
\]
be in \( K^*(\sigma, \tau, S) \), then the function \( g(z) \) defined by
g(z) = \frac{1}{z} + \sum_{l=1}^{\infty} \left( a_{l,1}^2 + a_{l,2}^2 \right) z^l, \quad \text{in the class } K^* (\varphi , \tau , S), \text{where} \quad \varphi \leq \frac{4\alpha^2 (1 - \tau) (l + 2 \tau - 1) - Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}{4\alpha^2 (1 - \tau) (l + 2 \tau - 1) - Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}^2

**Proof:** Hence \( f_j \in K^* (\sigma , \tau , S) \), \((j = 1,2)\). Then utilizing Theorem 1, we get

\[
\sum_{l=1}^{\infty} Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))] \frac{a_{l,j}}{2\sigma (1 - \tau)} \leq 1, \ j = 1,2
\]

\[
\varphi = \sum_{l=1}^{\infty} \left( \frac{Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}{2\sigma (1 - \tau)^2} \right)^2 a_{l,j}^2 \leq 1, \ (j = 1,2)
\]

and

\[
\sum_{l=1}^{\infty} \left( \frac{Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}{2\sigma (1 - \tau)^2} \right)^2 (a_{l,1}^2 + a_{l,2}^2) \leq -1
\]

Therefore, we need to find the largest \( \varphi \) such that

\[
\varphi \leq \frac{Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))]^2}{4\alpha^2 (1 - \tau)}, \ l \geq 1.
\]

Hence

\[
\varphi \leq \frac{4\alpha^2 (1 - \tau) (l + 2 \tau - 1) - Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))]^2}{4\alpha^2 (1 - \tau) (l + 2 \tau - 1) - Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))]^2}
\]

**Theorem 8:**

Let

\[
f(z) = \frac{1}{z} + \sum_{l=1}^{\infty} a_l z^l \in K^* (\sigma , \tau , S)
\]

and

\[
g(z) = \frac{1}{z} + \sum_{l=1}^{\infty} b_l z^l
\]

with \( |b_l| \leq 1 \), is in the class \( K^* (\sigma , \tau , S) \). Then \( f(z) * g(z) \in K^* (\sigma , \tau , S) \).

**Proof:** From Theorem 1, we get

\[
\sum_{l=1}^{\infty} Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))] a_l \leq 2\sigma (1 - \tau)
\]

Hence

\[
\sum_{l=1}^{\infty} Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))] \frac{a_l b_l}{2\sigma (1 - \tau)} = \sum_{l=1}^{\infty} Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))] a_l b_l \leq \frac{Bl[(l(1 + \sigma) + (1 + \sigma(2\tau - 1))]}{2\sigma (1 - \tau)}
\]

Thus,

\( f(z) * g(z) \in K^* (\sigma , \tau , k) \).

**Corollary 2:**

Let

\[
(z) = \frac{1}{z} + \sum_{l=1}^{\infty} a_l z^l \in K^* (\sigma , \tau , S)
\]

and

\[
g(z) = \frac{1}{z} + \sum_{l=1}^{\infty} b_l z^l
\]

for \( 0 \leq b \leq 1 \)

is in the class \( K^* (\sigma , \tau , S) \). Then

\( f(z) * g(z) \in K^* (\sigma , \tau , S) \).

**5. Neighborhood on \( K^* (\sigma , \tau , S) \):**

In the section, the concept of neighborhood of analytic function was first introduced by Goodman [2] and then generalized by Ruscheweyh [7]. Lin and Srivastava [4], investigated this concept for the elements of several famous subclass of analytic function. We define the \((l, \xi)\) Neighborhood of a function \( f(z) \in \Sigma \) by

\[
N_{l,\xi}(f) = \left\{ g \in \Sigma : g(z) = z^{-1} \sum_{l=1}^{\infty} a_l z^l \right\}
\]

For the identity function \( e(z) = z \), we get

\[
N_{l,\xi}(e) = \left\{ g \in \Sigma : g(z) = z^{-1} \sum_{l=1}^{\infty} b_l z^l \right\}
\]

**Lemma:**

The function \( f(z) \in \Sigma \) is said to be in the class \( K^* (\sigma , \tau , S, \xi) \) if there exists a function \( g(z) \in K^* (\sigma , \tau , S) \), such that
\[ \left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \vartheta, \quad (13) \]
\[(z \in U, 0 \leq \vartheta < 1)\]

**Theorem 9:**
If \( g(z) \in K^*(\sigma, \tau, s) \) and
\[ \vartheta = 1 - \frac{\xi[B((1 + \sigma) + (1 + \sigma(2\tau - 1)))]}{B((1 + \sigma) + (1 + \sigma(2\tau - 1)) - 2\sigma(1 - \tau))}, \quad (14) \]
then \( N_{l, \xi}(g) \subset K^*(\sigma, \tau, S, \xi) \).

**Proof:** Assume that \( f \in N_{l, \xi}(g) \). Then we have from (12) that
\[ \sum_{l=1}^{\infty} |a_l - b_l| \leq \xi, \]
which suggests the coefficient inequality
\[ \sum_{l=1}^{\infty} |a_l - b_l| \leq \xi, \quad (l \in N) \]
Hence \( g \in K^*(\sigma, \tau, S), \) we get from corollary 1
\[ \sum_{l=1}^{\infty} b_l \leq \frac{2\sigma(1 - \tau)}{B((1 + \sigma) + (1 + \sigma(2\tau - 1)))}, \]
From (13), we get
\[ \frac{f(z)}{g(z)} - 1 \left| \frac{\sum_{l=1}^{\infty} |a_l - b_l|}{1 - \sum_{l=1}^{\infty} b_l} \leq \frac{\xi[B((1 + \sigma) + (1 + \sigma(2\tau - 1)))]}{B((1 + \sigma) + (1 + \sigma(2\tau - 1)) - 2\sigma(1 - \tau)} = 1 - \vartheta. \]
Since, by lemma \( f \in K^*(\sigma, \tau, S, \xi) \) for \( \vartheta \) given by (14).

**REFERENCES**