Research Article

Harmonic Functions of the Class of Barzilai’s Type Related to New Derivative Operator

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Abstract
In this article, we define and investigate the class of Bazilević type harmonic univalent functions $F_{\lambda, \beta}^{(\eta, \sigma, \gamma)}$, which related with a new linear operator. We have also obtained the harmonic structures in terms of its coefficient bounds, extreme points, distortion bound, convolution and we proved the function belongs to this class be closed under linear combination.

Keywords: Harmonic, Univalent functions, Bazilević type, Derivative operator.

Introduction
Let $A$ refer to the class of functions has been the expressed as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized under two conditions with $f(0) = 0$ and $f'(0) = 1$.

In addition to that, let $P$ refer to the class of functions $h(z)$, with positive real part in $\mathcal{U}$ as follows:

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$ 

A function $f(z)$ in the form (1) is called starlike functions, if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathcal{U})$$

and denoted by $S^*$ (see [7]). From (1), we write that:

$$f(z)^\alpha = \left(z + \sum_{n=2}^{\infty} a_n z^n\right)^\alpha \quad (2)$$

By applying binomial expansion on (2), we obtain:

$$f(z)^\alpha = z^\alpha + a_2 z^{\alpha+1}$$

$$+ \left[a_3 + \frac{a(a-1)}{2!} a_2^2\right] z^{\alpha+2}$$

$$+ \left[a_4 + \frac{a(a-1)}{2!} 2a_2 a_3 + \frac{a(a-1)(a-2)}{3!} a_2^3\right] z^{\alpha+3} + \cdots.$$ 

Then we define the class of analytic functions of fractional power $A_\alpha$ as follows:

$$f(z)^\alpha = z^\alpha + \sum_{n=2}^{\infty} a_n(\alpha) z^{\alpha+n-1} \quad (3)$$

Thus, we shall define the differential operator
Let $A_H$ denote the family of all functions $f$ in the form (5) which are univalent, sense preserving function and harmonic in $\mathcal{U} = \{z : |z| < 1\}$. In the present work, we will express the functions $h^\alpha$ and $g^\alpha$ as follows:

$$
\begin{align*}
 h(z)^\alpha &= z^\alpha + \sum_{n=2}^{\infty} a_n(\alpha)z^{\alpha+n-1}, \\
g(z)^\alpha &= \sum_{n=1}^{\infty} b_n(\alpha)z^{\alpha+n-1}, \quad (z \in \mathcal{U}, 0 \leq |b_1(\alpha)| < 1)
\end{align*}
$$

Hence

$$
 f(z)^\alpha = h(z)^\alpha + g(z)^\alpha.
$$

Note that if $g$ is identically zero; that is $g = 0$, and $\alpha = 1$, then $A_H$ will generates a known class $A$.

We define our linear operator as given in (4) such that

$$
\mathcal{L}_{\lambda, \beta}^m f(z)^\alpha = L_{\lambda, \beta}^m h(z)^\alpha + (-1)^m \mathcal{L}_{\lambda, \beta}^m g(z)^\alpha,
$$

Where

$$
\begin{align*}
 L_{\lambda, \beta}^m h(z)^\alpha &= [1 + (1 - \alpha)(\lambda - \beta)]^m z^\alpha \\
 &+ \sum_{n=2}^{\infty} [1 + (\beta - \lambda)(\alpha + n - 2)]^m a_n(\alpha)z^{\alpha+n-1}, \\
 L_{\lambda, \beta}^m g(z)^\alpha &= \sum_{n=1}^{\infty} [1 + (\beta - \lambda)(\alpha + n - 2)]^m \times b_n(\alpha)z^{\alpha+n-1}
\end{align*}
$$

Now, we shall define generalization class of Bazilević type harmonic univalent functions involving new general linear operator.

**Definition 1.1** Let $f(z)$ in $A_H$, belongs to the class $\mathcal{F}_{\lambda, \beta}^m(\eta, \sigma, \gamma)$ if it satisfies the following condition:

$$
\begin{align*}
 R e \left\{ \sigma e^{i\eta} - (\sigma e^{i\eta} - 1) \frac{\mathcal{L}_{\lambda, \beta}^m f(z)^\alpha}{[1 + (1 - \alpha)(\lambda - \beta)]^m z^\alpha} \right\} &\geq \gamma, \\
\end{align*}
$$

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where $0 \leq \gamma < 1$, $\eta \in \mathbb{R}$, $\sigma \geq 0$, $\beta \geq 0$, $\lambda \geq 0$, $\alpha > 0$, $n, m \in \mathbb{N}$, $z \in \mathcal{U}$ and $\mathcal{L}^n_{\lambda, \beta} f(z)^\alpha$ is earlier defined in (8).

Furthermore, let $\mathcal{NF}_{\lambda, \beta}^m (\eta, \sigma, \gamma)$ be subclass of $\mathcal{F}_{\lambda, \beta}^m (\eta, \sigma, \gamma)$ consist of harmonic functions

$$f_m^\alpha = h^\alpha + \overline{g_m}$$

(10)

where $h^\alpha$ and $\overline{g_m}$ which has the following representation:

$$h(z)^\alpha = z^\alpha + \sum_{n=2}^{\infty} |a_n(\alpha)| z^{\alpha+n-1},$$

$$g(z)^\alpha = - (-1)^m \sum_{n=1}^{\infty} |b_n(\alpha)| z^{\alpha+n-1},$$

($z \in \mathcal{U}$, $0 \leq |b_1(\alpha)| < 1$)

Starting point in the study of functions characterized in (9) was discovered in 1955 by Bazilevi´c [2], when the Bazilevi´c function defined in $\mathcal{U}$ by the form:

$$f(z) = \left\{ \begin{array}{l}
\frac{\xi - z}{1 + \eta z} \int_{\alpha}^{z} h(t) \frac{dt}{t} \\
- \eta t^{- \frac{1 + i\eta}{1 + i\eta}} g(t)^{1+i\eta} \frac{dt}{t} \end{array} \right\}^{1+i\eta}$$

(11)

where the function $h(t)$ belongs to $P$ and $g(z) \in S^*$, $\xi, \eta \in \mathbb{R}$ with $\xi > 0$.

The class of harmonic functions have been studied by many authors for variant properties. By the earlier papers for contributors such as [3], [4], [5], [6], [8], [9] and [10] regarding of the theory of analytic functions which have a wide application in many physical problem: as electrostatic potential in heat conduction, fluid flows, and theory of fractals constitute practical examples.

The aim of this paper requests to generate class of Bazilevi´c type harmonic univalent function related to new derivative operator. Also, we obtain coefficient bounds for functions $f^\alpha$ which is define in (6) belongs in the class $\mathcal{F}_{\lambda, \beta}^m (\eta, \sigma, \gamma)$.

As well as, the distortion bounds, inclusion results and extreme points for functions in this class are also obtained.

1. **Main Results**

In this result, we present a sufficient condition for coefficient of functions in the class $\mathcal{F}_{\lambda, \beta}^m (\eta, \sigma, \gamma)$.

**Theorem 2.1.** Let $f^\alpha = h^\alpha + \overline{g^\alpha}$ be given by (7). If:

$$\sum_{n=2}^{\infty} \frac{\sigma - 1}{1 - \gamma} \frac{(1 + (\beta - \lambda)(\alpha + n - 2))}{1 + (1 - \alpha)(\lambda - \beta)} |a_n(\alpha)|$$

$$\sum_{n=1}^{\infty} \frac{\sigma - 1}{1 - \gamma} \frac{(1 + (\beta - \lambda)(\alpha + n - 2))}{1 + (1 - \alpha)(\lambda - \beta)} |b_n(\alpha)|$$

(12)

where $0 \leq \gamma < 1$, $\eta \in \mathbb{R}$, $\sigma \geq 0$, $\beta \geq 0$, $\lambda \geq 0$, $\alpha > 0$, $n, m \in \mathbb{N}$, $z \in \mathcal{U}$, then $f^\alpha$ be harmonic univalent and sense-preserving in $\mathcal{U}$ and $f^\alpha \in \mathcal{F}_{\lambda, \beta}^m (\eta, \sigma, \gamma)$.

**Proof.** First suppose that the inequality (12) holds.

If $z_1 \neq z_2$

$$\left| f^\alpha(z_1) - f^\alpha(z_2) \right| \geq 1 - \frac{\left| g^\alpha(z_1) - g^\alpha(z_2) \right|}{h^\alpha(z_1) - h^\alpha(z_2)}$$

$$= 1 - \frac{\sum_{n=1}^{\infty} b_n(\alpha)(z_1^{\alpha+n-1} - z_2^{\alpha+n-1})}{(z_1^{\alpha} - z_2^{\alpha}) + \sum_{n=2}^{\infty} a_n(\alpha)(z_1^{\alpha+n-1} - z_2^{\alpha+n-1})}$$

$$\geq 1 - \frac{\sum_{n=1}^{\infty} (\alpha + n - 1)b_n(\alpha)}{\alpha + \sum_{n=2}^{\infty} (\alpha + n - 1)a_n(\alpha)}$$

$$> 1 - \frac{\sum_{n=1}^{\infty} (\alpha + n - 1)b_n(\alpha)}{\alpha + \sum_{n=2}^{\infty} (\alpha + n - 1)a_n(\alpha)} - 1$$

$$\sum_{n=1}^{\infty} \frac{(1 + (\beta - \lambda)(\alpha + n - 1))b_n(\alpha)}{(1 + (1 - \alpha)(\lambda - \beta)) + \sum_{n=2}^{\infty} (1 + (\beta - \lambda)(\alpha + n - 1))a_n(\alpha)}$$
Hence proved the univalent, and also note that $f^\alpha$ be sense-preserving in $U$ since:

$$|h'(z)| \geq \alpha |z|^\alpha - 1 - \sum_{n=2}^{\infty} (\alpha + n - 1)|a_n(\alpha)| |z|^{\alpha + n - 2}$$

$$> \alpha - \sum_{n=2}^{\infty} (\alpha + n - 1)|a_n(\alpha)|$$

$$\geq (1 + (1 - \alpha)(\lambda - \beta)) -$$

$$\sum_{n=2}^{\infty} \frac{\sigma - 1}{1 - \gamma}(1 + (\beta - \lambda)(\alpha + n - 1))|a_n(\alpha)|$$

$$\geq \sum_{n=2}^{\infty} \frac{\sigma - 1}{1 - \gamma}(1 + (\beta - \lambda)(\alpha + n - 1))|b_n(\alpha)|$$

$$\geq \sum_{n=2}^{\infty} (\alpha + n - 1)|b_n(\alpha)|$$

$$> \sum_{n=2}^{\infty} (\alpha + n - 1)|b_n(\alpha)| |z|^{\alpha + n - 2}$$

$$\geq |g'(z)|$$

The remaining condition needs to investigate the function $f^\alpha(z)$ which belongs to the class $F_{\lambda,\beta}(\eta, \sigma, \gamma)$. By (9) and (10), we have

$$Re\left\{\sigma e^{i\eta} - (\sigma e^{i\eta} - 1)\frac{L_{\lambda,\beta}^m f(z)\alpha}{[1 + (1 - \alpha)(\lambda - \beta)]^m z^\alpha}\right\} =$$

$$Re\left\{\sigma e^{i\eta} - (\sigma e^{i\eta} - 1)\frac{L_{\lambda,\beta}^m h(z)\alpha + (-1)^m L_{\lambda,\beta}^m g(z)\alpha}{[1 + (1 - \alpha)(\lambda - \beta)]^m z^\alpha}\right\} \geq \gamma$$

By applying the fact that $Re\{w\} \geq \gamma$, if and only if $|1 - \gamma + w| \geq |1 + \gamma - w|$ for $\gamma(0 \leq \gamma < 1)$, it suffices to show that

$$|(1 - \gamma) + \sigma e^{i\eta} - (\sigma e^{i\eta} - 1)\frac{L_{\lambda,\beta}^m f(z)\alpha}{[1 + (1 - \alpha)(\lambda - \beta)]^m z^\alpha}|$$

$$|(1 + \gamma) - \sigma e^{i\eta} + (\sigma e^{i\eta} - 1)\frac{L_{\lambda,\beta}^m f(z)\alpha}{[1 + (1 - \alpha)(\lambda - \beta)]^m z^\alpha}| \geq 0.$$

That is

$$|(1 - \gamma + \sigma e^{i\eta})(1 + (1 - \alpha)(\lambda - \beta)) z^m|$$

$$- (\sigma e^{i\eta} - 1)L_{\lambda,\beta}^m f(z)\alpha$$

$$- (1 + \gamma - \sigma e^{i\eta})(1 + (1 - \alpha)(\lambda - \beta)) z^m$$

$$+ (\sigma e^{i\eta} - 1)L_{\lambda,\beta}^m f(z)\alpha \geq 0,$$

$$2(1 - \gamma)(1 + (1 - \alpha)(\lambda - \beta))^m |z|^\alpha - \sum_{n=2}^{\infty} \frac{\sigma - 1}{1 - \gamma}(1 + (\beta - \lambda)(\alpha + n - 2))|a_n(\alpha)| |z|^{\alpha + n - 1}$$

$$- (-1)^m \sum_{n=2}^{\infty} \frac{\sigma - 1}{1 - \gamma}(1 + (\beta - \lambda)(\alpha + n - 2))|b_n(\alpha)| |z|^{\alpha + n - 1} \geq 0.$$
\[
\begin{align*}
&\geq 2(1-\gamma)[1+(1-\alpha)(\lambda-\beta)]^m \left[1 - \left(\sum_{n=2}^{\infty} \frac{\sigma-1}{\sigma} \left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |a_n(\alpha)|\right) + (-1)^m \sum_{n=1}^{\infty} \frac{\sigma-1}{\sigma} \left(\frac{1+(\beta-\lambda)(\alpha+n-2)}{1+(1-\alpha)(\lambda-\beta)}\right)^m |b_n(\alpha)| \right] \geq 0
\end{align*}
\]

The expression (13) is non-negative by (12), and furthermore \( f(z)^\alpha \in F^m_{\lambda,\beta}(\eta,\sigma,\gamma) \).

The harmonic functions of the form:

\[
f^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty} \frac{1-\gamma}{\sigma-1} \left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m x_k z^{2n+1}
\]

Next theorem investigates that a condition in (12) as a necessary condition for the function \( f^\alpha_m \) which given by (10).

**Theorem 2.2.** Let \( f^\alpha_m = h^\alpha + g^\alpha_m \) be given by (10) belongs to class \( \mathcal{NF}_{(\lambda,\beta)^m} \) \( (\eta,\sigma,\gamma) \) if and only if:

\[
\begin{align*}
&\sum_{n=2}^{\infty} \frac{1-\gamma}{\sigma-1} \left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m |a_n(\alpha)| \\
&+ \sum_{n=1}^{\infty} \frac{1-\gamma}{\sigma-1} \left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m |b_n(\alpha)| \\
&\leq 1
\end{align*}
\]

where \( 0 \leq \gamma < 1, \eta \in \mathbb{R}, \sigma \geq 0, \beta \geq 0, \lambda \geq 0, \alpha > 0, n, m \in \mathbb{N}, z \in \mathcal{U}, \) and

\[
\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1
\]

Note that the coefficient bound \( s \) which given in (12) is a sharp. Therefore, functions in the form (14) belong to the class \( \mathcal{F}^m_{\lambda,\beta}(\eta,\sigma,\gamma) \), since

\[
\begin{align*}
&\sum_{n=2}^{\infty} \frac{1-\gamma}{\sigma-1} \left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m |a_n(\alpha)| \\
&+ \sum_{n=1}^{\infty} \frac{1-\gamma}{\sigma-1} \left(\frac{1+(1-\alpha)(\lambda-\beta)}{1+(\beta-\lambda)(\alpha+n-2)}\right)^m |b_n(\alpha)| \\
&= \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1
\end{align*}
\]

Next theorem investigates that a condition in (12) as a necessary condition for the function \( f^\alpha_m \) which given by (10).

**Proof.** Since \( \mathcal{N}^m_{\lambda,\beta}(\eta,\sigma,\gamma) \subset \mathcal{F}^m_{\lambda,\beta}(\eta,\sigma,\gamma) \), we just have to state the only part of theorem. To this end, suppose that \( f^\alpha_m \in \mathcal{N}^m_{\lambda,\beta}(\eta,\sigma,\gamma) \) and by virtue of (9), we get:

\[
\begin{align*}
&\text{Re} \left\{ (\sigma e^{i\eta} - \gamma) \right\} \\
&- \left( (\sigma e^{i\eta} - 1) \right) \frac{L^m_{\lambda,\beta} f^\alpha(z) - L^m_{\lambda,\beta} f(z)^\alpha}{[1+(1-\alpha)(\lambda-\beta)]^m z^\alpha} \geq 0
\end{align*}
\]

This is equivalent to:

\[
\begin{align*}
&\text{Re} \left\{ (\sigma e^{i\eta} - \gamma) [1+(1-\alpha)(\lambda-\beta)]^m z^\alpha - (\sigma e^{i\eta} - 1) [1+(1-\alpha)(\lambda-\beta)]^m z^\alpha \\
&- \sum_{n=2}^{\infty} |a_n(\alpha)||z^\alpha|^{2n+1} \\
&+ (-1)^m \sum_{n=1}^{\infty} |b_n(\alpha)||z^\alpha|^{2n+1} \right\} \geq 0
\end{align*}
\]
\[
\begin{align*}
&= \text{Re} \left\{ \frac{(1 - \gamma)(1 + (1 - \alpha)(\lambda - \beta))}{[1 + (1 - \alpha)(\lambda - \beta)]^m} \right. \\
&\quad + (-1)^m \sum_{n=1}^\infty (\sigma e^{i\eta} - 1)(\lambda - \beta) n b_n(\alpha) z^{\alpha+n-1} \left. \right\} \geq 0, \\
&= \text{Re} \left\{ \frac{(1 - \gamma)(1 + (1 - \alpha)(\lambda - \beta))}{[1 + (1 - \alpha)(\lambda - \beta)]^m} z^\alpha - \sum_{n=1}^\infty (\sigma e^{i\eta} - 1)(\lambda - \beta) n b_n(\alpha) z^{\alpha+n-1} \right\} \geq 0.
\end{align*}
\]

This condition must be true \( \forall z \in \mathcal{U} \) and for real \( \eta \). Therefore, choose \( 0 \leq |z| = r < 1 \)

\[
(1 - \gamma)(1 + (1 - \alpha)(\lambda - \beta))^{m-n} - \sum_{n=1}^\infty (\sigma e^{i\eta} - 1)(\lambda - \beta) n b_n(\alpha) z^{\alpha+n-1} \geq 0.
\]

In the following theorem, we will determine the extreme points of closed convex hulls for functions belong to the class \( \mathcal{N} \mathcal{F}^m_{\lambda, \beta}(\eta, \sigma, \gamma) \), and we refer to it with the symbol \( c\mathcal{L} \mathcal{C} \mathcal{O} \mathcal{N} \mathcal{F}^m_{\lambda, \beta}(\eta, \sigma, \gamma) \).

**Theorem 2.3.** Let \( f_{m}^f(z) \) be given by (10) belongs to class \( \mathcal{N} \mathcal{F}^m_{\lambda, \beta}(\eta, \sigma, \gamma) \) if and only if

\[
f_{m}^f(z) = \sum_{n=1}^\infty (x_n h_n(z) + y_n g_n(z)),
\]

where

\[
\begin{align*}
h_n(z) &= z^\alpha, \\
h_n^*(z) &= z^\alpha, \\
\beta_{nm}(z) &= z^\alpha, \\
\beta_{nm}^*(z) &= z^\alpha.
\end{align*}
\]

First, we have

\[
f_{m}^f(z) = \sum_{n=1}^\infty (x_n h_n(z) + y_n g_{nm}(z)),
\]

and \( \eta = 0 \), so that the above inequality reduces to:

\[
(1 - \gamma)(1 + (1 - \alpha)(\lambda - \beta))^{m-n} - \sum_{n=1}^\infty (\sigma e^{i\eta} - 1)(\lambda - \beta) n b_n(\alpha) z^{\alpha+n-1} \geq 0.
\]

In particular case, the extreme points of \( \mathcal{N} \mathcal{F}^m_{\lambda, \beta}(\eta, \sigma, \gamma) \) are \( \{h_n\} \) and \( \{\beta_{nm}\} \).

**Proof.** First, we have

\[
f_{m}^f(z) = \sum_{n=1}^\infty (x_n h_n(z) + y_n g_{nm}(z)),
\]

where

\[
\begin{align*}
x_n &= \frac{\sum_{n=1}^\infty \frac{1 - \gamma}{\sigma - 1}(1 + (1 - \alpha)(\lambda - \beta))}{\sum_{n=1}^\infty \frac{1 - \gamma}{\sigma - 1}(1 + (1 - \alpha)(\lambda - \beta)) z^{\alpha+n-1}}, \\
y_n &= \frac{\sum_{n=1}^\infty \frac{1 - \gamma}{\sigma - 1}(1 + (1 - \alpha)(\lambda - \beta))}{\sum_{n=1}^\infty \frac{1 - \gamma}{\sigma - 1}(1 + (1 - \alpha)(\lambda - \beta)) z^{\alpha+n-1}}.
\end{align*}
\]

This means that \( f_{m}^f \in \mathcal{N} \mathcal{F}^m_{\lambda, \beta}(\eta, \sigma, \gamma) \). Conversely, assume that \( f_{m}^f \in \mathcal{N} \mathcal{F}^m_{\lambda, \beta}(\eta, \sigma, \gamma) \). Putting

\[
\begin{align*}
x_n &= \frac{\sigma - 1}{1 - \gamma}(1 + (1 - \alpha)(\lambda - \beta)) z^\alpha, \\
y_n &= \frac{\sigma - 1}{1 - \gamma}(1 + (1 - \alpha)(\lambda - \beta)) z^\alpha.
\end{align*}
\]

Therefore, \( f_{m}^f(z) \) can be written as

\[
\begin{align*}
f_{m}^f(z) &= \sum_{n=1}^\infty (x_n h_n(z) + y_n g_{nm}(z)),
\end{align*}
\]

and

\[
\begin{align*}
x_n &= 1 - \left( \sum_{n=1}^\infty x_n + \sum_{n=1}^\infty y_n \right) \geq 0, \\
\end{align*}
\]

and

\[
\begin{align*}
y_n &= 1 - \left( \sum_{n=1}^\infty x_n + \sum_{n=1}^\infty y_n \right) \geq 0.
\end{align*}
\]
We give distortion bounds for functions belongs to the class $\mathcal{NF}_{\lambda,\beta}(\eta,\sigma,\gamma)$.

**Theorem 2.4.** Let $f_m^\alpha \in \mathcal{NF}_{\lambda,\beta}(\eta,\sigma,\gamma)$. Then for $|z| = r < 1$, we have

$$|f_m^\alpha(z)| \leq (1 - |b_1(\alpha)|)r^\alpha + \sum_{n=2}^{\infty} \left| a_n(\alpha) \right| z^{\alpha+n-1} + \frac{1}{1 + (\beta - \lambda)(\alpha)} \left| b_n(\alpha) \right| z^{\alpha+n-1}$$

for $|b_1(\alpha)| < 1$. This shows that the bound which given in theorem 2.4, be sharp for harmonic functions

$$f_m^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty} \left| a_n(\alpha) \right| z^{\alpha+n-1} - \left(1 + \frac{1}{\sigma - 1} \right) \frac{1}{1 + (\beta - \lambda)(\alpha)} \left| b_n(\alpha) \right| z^{\alpha+n-1}$$

The convolution of $f_m^\alpha$ and $F_n^\alpha$ is given by

$$f_m^\alpha * F_n^\alpha(z) = z^\alpha + \sum_{n=2}^{\infty} \left| a_n(\alpha) \right| z^{\alpha+n-1} - \left(1 + \frac{1}{\sigma - 1} \right) \frac{1}{1 + (\beta - \lambda)(\alpha)} \left| b_n(\alpha) \right| z^{\alpha+n-1}$$
Theorem 2.5 For $0 \leq \mu \leq \gamma < 1$, let $f_m^\alpha \in N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma)$ and $f_n^\alpha \in N^{F m}_{\lambda, \beta}(\sigma, \alpha, \mu)$. Then $f_m^\alpha * f_n^\alpha \in N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma) \cap N^{F m}_{\lambda, \beta}(\sigma, \alpha, \mu)$.

Proof. We wish to show that the coefficients of $f_m^\alpha * f_n^\alpha$ satisfy the required condition given in Theorem 2.2. For the function $f_m^\alpha \in N^{F m}_{\lambda, \beta}(\eta, \sigma, \mu)$, we note that $|A_n(\alpha)| = 1$ and $|B_n(\alpha)| \leq 1$.

Now, for the convolution function $(f_m^\alpha * f_n^\alpha)(z)$, we obtain

$$
\sum_{n=2}^{\infty} \sigma - 1 \left(1 + (\beta - \lambda)(a + n - 2)\right) m \left|a_n(\alpha)\right| |A_n(\alpha)| + \sum_{n=2}^{\infty} \sigma - 1 \left(1 + (\beta - \lambda)(a + n - 2)\right) m \left|b_n(\alpha)\right| |B_n(\alpha)| + \sum_{n=2}^{\infty} \sigma - 1 \left(1 + (\beta - \lambda)(a + n - 2)\right) n \left|a_n(\alpha)\right| + \sum_{n=2}^{\infty} \sigma - 1 \left(1 + (\beta - \lambda)(a + n - 2)\right) n \left|b_n(\alpha)\right| \leq 1.
$$

Therefore, $f_m^\alpha * f_n^\alpha \in N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma) \cap N^{F m}_{\lambda, \beta}(\sigma, \alpha, \mu)$.

Here, let $f_m^\alpha f_n^\alpha(\lambda)$ be defined as

$$
f_m^\alpha f_n^\alpha(\lambda) = z^\alpha + \sum_{n=2}^{\infty} \left|a_n(\alpha)\right| z^{\alpha + n - 1} \left(1 + (\beta - \lambda)(a + n - 2)\right) m - (-1)^m \sum_{n=1}^{\infty} \left|b_n(\alpha)\right| z^{\alpha + n - 1} \left(1 + (\beta - \lambda)(a + n - 2)\right) n \tag{18}
$$

where $\lambda = 1, 2, \ldots, k$.

Theorem 2.6 Let $f_m^\alpha f_n^\alpha(\lambda)$ which defined by (18) belongs to the class $N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma)$ for every $i = 1, 2, \ldots, k$. Then the function

$$
t^\alpha (z) = \sum_{i=1}^{k} v_i f_m^\alpha f_n^\alpha(\lambda), \quad 0 \leq v_i \leq 1
$$

are also in the class $N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma)$, where $\sum_{i=1}^{k} v_i = 1$.

Proof. According to a definition of $t^\alpha$, can be written as

$$
t^\alpha (z) = z^\alpha + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{k} v_i a_n(\alpha)\right) z^{\alpha + n - 1} - (-1)^m \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k} v_i b_n(\alpha)\right) z^{\alpha + n - 1}.
$$

Furthermore, since $f_m^\alpha f_n^\alpha(\lambda)$ belongs to the class $N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma)$ for every $\lambda = 1, 2, \ldots, k$, then by (12), we have

$$
\sum_{i=1}^{k} \sigma - 1 \left(1 + (\beta - \lambda)(a + n - 2)\right) m \left|a_n(\alpha)\right| + \sum_{i=1}^{k} \sigma - 1 \left(1 + (\beta - \lambda)(a + n - 2)\right) n \left|b_n(\alpha)\right| \leq 1.
$$

Corollary 2.7 The class $N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma)$ be closed under convex linear combination.

Proof. Let $f_m^\alpha f_n^\alpha(\lambda)$, $(i = 1, 2)$ defined by (18) belongs to the class $N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma)$. Then the function $\Phi^\alpha(\lambda)$ defined by

$$
\Phi^\alpha(\lambda) = \mu f_m^\alpha f_n^\alpha(\lambda) + (1 - \mu) f_m^\alpha f_n^\alpha(\lambda), \quad 0 \leq \mu \leq 1
$$

is in the class $N^{F m}_{\lambda, \beta}(\eta, \sigma, \gamma)$. By choosing $k = 2, v_1 = 1$ and $v_2 = 1 - \mu$ in Theorem 2.6, we obtain the above corollary.

References


