Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 32, Issue 3, 2021 DOI: http://doi.org/10.23851/mjs.v32i3.956

Research Article Open Access

Differential Subordination and Superordination of Multivalent Functions Involving Differential Operator

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ArticleInfo

Received 25/03/2021

Accepted 07/04/2021

Published 20/06/2021

ABSTRACT

In the present work, we derive some properties of subordination and superordination results associated with the Hadamard product concept involving the composition of the differential operator.

KEYWORDS: Differential operator; Differential Subordination; Differential Superordination; Hadamard product; Multivalent function.

الخلاصة

في العمل الحالي نشتق بعض خصائص نتائج التبعية والتبعية المرتبطة بمفهوم منتج هادمرد الذي يتضمن تكوين المشغل التفاضلي

INTRODUCTION

Let $H(U)$ be the class of holomorphic function in the unit disk $U =\{z\in C: |z| < 1\}$ and $\underline{U} = \{z\in C |z| \le 1\}$,

and $H[\text{a}, n]$ *be the subclass* of $H(U)$of the from

$$f(z) = \text{a} + \text{a}_n z^n + \text{a}_{n+1}z^{n+1} + \cdots,$$

where $\text{a}\in C$, $n\in N$ with $H_0 \equiv H[0,1]$ and $H\equiv H[1,1]$. *Let* A_p be the class of all holomorphic functions of the from

$$f(z) = z^p + \sum_{n=p+1}^{\infty} \text{a}_n z^n,\ (z\in U) \quad (1.1)$$

in the unit disk U, for functions f given by (1.1) and g defined by,

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,\quad (z\in U)$$

then,

$$(f*g)(z) = z^p + \sum_{n=p+1}^{\infty}\text{a}_n b_n z^n = (g*f)(z),$$

is called the Hadamard product (convolution) of f *and* g. Let f and Fbe members of $H(U)$. Function f is Subordinate to function F or F is Superordinate to f, if there is Schwarz function ω holomorphic in U, with $\omega(0) = 0$ *and* $|\omega(z)| < 1, (z\in U)$, such that $f(z) = F(\omega(z))$. Let refer the Subordination by

$$f(z) \prec F(z) \text{ or } f \prec F.$$

Moreover, [1] if F is univalent in U then,

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(U) \subset F(U).$$

The method of differential subordinations or (the admissible functions method) is firstly introduced by Miller and Mocanu in [2]; the theory began to evolve in [3]. For more details see [4]. Let ϖ *and* Γ be sets in C, let $\pi: C^3\times U \longrightarrow C$ and ϑ be univalent in U, if p is holomorphic in U with $p(0) = \text{a}$ with generalizations of inclusion

$$\{\pi(p(z), zp'(z), z^2p''(z))\} \subset \varpi \Longrightarrow p(U) \subset \Gamma,$$

and achieves the second-order differential subordination

$$\pi(p(z), zp'(z), z^2p''(z); z) \prec \vartheta(z), (1.2)$$

then p is a solution of the differential subordination. The univalent function q is a dominant of the solution of the differential subordination, this is, if $p\prec q, \forall p$ achieved

(1.2). A dominant $\tilde{q}$ satisfying $\tilde{q} \prec q$, for all dominant (1.2), which is the best dominant of (1.2).If p and $\Pi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and achieves the second-order differential subordination

$$\vartheta(z) \prec \pi(p(z), zp'(z), z^2p''(z); z), \quad (1.3)$$

then p is a solution of the differential superordination. A holomorphic function q is subordinant of the solutions of the differential superordination this is, if $q \prec p, \forall\, p$satisfying (1.3). A univalent subordination $\tilde{q}$ that achieve $\prec \tilde{q}$,∀ subordinations q of is the better subordinat. From (1.3), we get

$$\varpi \subset \{\pi(p(z), zp'(z), z^2p''(z); z)\}.$$

The differential operator $\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)f(z)$ which has been defined by [5] and as

$$\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z) = z^p + \sum_{n=p+1}^{\infty}\left(\frac{\delta+(\mu+\lambda)(n-p)+\eta}{\delta+\eta}\right)^{\kappa} a_n b_n z^n$$

(1.4)

where δ, η, μ and λ having the same restrictions that were debated before [6]. Moreover, we can get many differential operators by direct calculation. For more details see [7].

For $\kappa, \alpha \geq 0$, we get

$$\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)\left(\psi^{\alpha}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z)\right) = \psi^{\kappa+\alpha}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z).$$

For (1.4), we get

$$z\left(\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z)\right)' = \left(\frac{\delta+\eta}{\mu+\lambda}\right)\left(\psi^{\kappa+1}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z)\right) - \left[p-\left(\frac{\delta+\eta}{\mu+\lambda}\right)\right]\left(\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z)\right). \quad (1.5)$$

We provide the following definitions and lemmas, which will help to prove our main results.

Definition 1.1.[3] Let Q the set of functions q that are holomorphic and injective on $\frac{U}{E(q)}$, where $E(q) = \{x \in \partial U, q(z) = \infty\}$, and are such that $q'(x) \neq 0$ for x$\in \frac{\partial U}{E(q)}$ .The subclass of Q for which $q(0) = \mathrm{a}$ refers by Q(a).

Definition 1.2.[3] Let ϖ is a set in $C, q(z) \in$ Q and n is a positive integer. The class of an admissible functions $\Pi_n[\varpi, q]$ made up of those functions $\pi: C^3 \times U \to C$ it achieves an admissibility condition $\pi(o, \varsigma, \tau; z) \notin \varpi$, when $o = q(z), \varsigma = yxq(z)$,

$$Re\left\{1+\frac{\tau}{\varsigma}\right\} \geq yRe\left\{1+\frac{xq''(x)}{q'(x)}\right\},$$

where $z \in U$, $x \in \frac{\partial U}{E(q)}$ and $y \geq n,$ we then $\Pi_1[\varpi, q] = \Pi[\varpi, q]$.In specially, when $q(z) = M\frac{Mz+\mathrm{a}}{M+\underline{\mathrm{a}}z}$, with $M > 0$ and $|\mathrm{a}| < M,$ then $q(U) = U_M = \{\omega: |\omega| < M\}, q(z) = \mathrm{a}, E(q) = \beta$ and q ∈Q.

Leads to$\Pi_n[\varpi, M, \mathrm{a}] = \pi[\varpi, q]$, and specially, when $\varpi = U_M$, the class is indicated by $\Pi_n[M, \mathrm{a}]$.

Definition 1.3.[4] Let ϖ is a set in C and $q \in H[\mathrm{a}, n]$ with $q'(z) \neq 0.$ The class of admissible functions $\Pi'[\varpi, q]$ made up of these functions $\pi: C^3 \times U \longrightarrow C$ this achieves the admissibility condition

$$\pi(o, \varsigma, \tau; x) \in \varpi,$$

when $o = q(z)$, $\varsigma = \frac{zP'(z)}{j}$, and

$Re\left\{1+\frac{\tau}{\varsigma}\right\} \leq \frac{1}{j}Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}$, for$z \in U, x \in \partial U$ and $j \geq n \geq 1.$ Leadsto$\Pi_1[\varpi, q] = \Pi[\varpi, q]$.

Lemma 1.4. [3] Let $\pi \in \Pi_n[\varpi, q]$ with $q(0) = \mathrm{a}$. If the holomorphic function

$$p(z) = \mathrm{a} + \mathrm{a}_n z^n + \mathrm{a}_{n+1} z^{n+1} + \cdots, \quad (z \in U)$$

This achieved the next inclusion relation$\pi(p(z), zp'(z), z^2p''(z); z) \in \varpi$, then

$$p(z) \prec q(z) \quad (z \in U).$$

Lemma 1.5. [4] Let $\pi \in \Pi_n[\varpi, q]$ with $q(0) = \mathrm{a}.$ If p ∈Q (a) and

$$\pi(p(z), zp'(z), z^2p''(z); z),$$

is univalent in U, then

$$\varpi \subset (p(z), zp'(z), z^2p''(z); z),$$

that means $q(z) \prec p(z)$.

Some results of differential subordination and superordination obtained in present work in [8] ad [9].

DIFFERENTIAL SUBORDINATION RESULTS

Definition 2.1. Let ϖ be a set in $C, q \in Q_0 \cap H[0,p]$. The class of admissible functions $\beta_n[\varpi, q]$ made up of those functions $\beta: C^3 \times U \longrightarrow C$ this achieves the admissibility condition

$$\beta(u, v, \omega, z; x) \notin \varpi,$$

whenever

$$u = q(x), v = \frac{yxq'(x) + \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right] q(x)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)}$$

and

$$Re\left\{\frac{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2 \omega - \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right]^2 u}{\left(\frac{\delta+\eta}{\mu+\lambda}\right) v - \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right] u} - 2\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right]\right\} \geq yRe\left\{1 + \frac{xq''(x)}{q'(x)}\right\},$$

for $z \in U, x \in \frac{\partial U}{E(q)}, \lambda \geq 1$ and $y \geq p$.

Theorem 2.2. Let $\beta \in B_n[\varpi, q]$. If $f \in A_p$ this achieved

$$\left\{\beta\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z), \psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f*g)(z), \psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f*g)(z); z\right); z \in U\right\} \subset \varpi, \quad (2.1)$$

then

$$\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z) \prec q(z).$$

Proof. Let $g(z) \in U$ define by

$$g(z) = \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z). \quad (2.2)$$

Using the relation (1.5) with (2.2),

$$z\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z)\right)' = \left(\frac{\delta+\eta}{\mu+\lambda}\right)\psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f*g)(z) + \left[p - \left(\frac{\delta+\eta}{\mu+\lambda}\right)\right] \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z).$$

$$\left(\frac{\delta+\eta}{\mu+\lambda}\right)\psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f*g)(z) = z\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)\right)' - \left[p - \left(\frac{\delta+\eta}{\mu+\lambda}\right)\right]\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z).$$

$$\psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f*g)(z) = \frac{z\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z)\right)'}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)} + \frac{\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right]\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)}.$$

we get

$$\psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f*g)(z) = \frac{zg'(z) + \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right] g(z)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)}, \quad (2.3)$$

$$\psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f*g)(z) = \psi_{\lambda,p}^{\kappa+1}(\psi(\delta,\eta,\mu)(f*g)(z))$$

$$\psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f*g)(z) = \frac{\left(\frac{\delta+\eta}{\mu+\lambda}\right)\left[zg''(z) + \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right] g'(z)\right]}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2}$$

$$\psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f*g)(z) = \left(\frac{\delta+\eta}{\mu+\lambda}\right)\left[\frac{zg'\left(\frac{zg'(z) + \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right] g(z)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)}\right)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2} + \frac{\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right]\left(\frac{zg'(z) + \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right] g(z)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)}\right)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2}\right],$$

$$\psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f*g)(z) = \frac{z^2g''(z) + 2\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right] zg'(z) + \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right]^2 g(z)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2}. \quad (2.4)$$

Define the transformation from $C^3 to\ C$ by

$$u(o,\varsigma,\tau) = o, v(o,\varsigma,\tau) = \frac{\varsigma + \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right] o}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)},$$

$$\omega(o,\varsigma,\tau) = \frac{\tau + 2\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right]\varsigma + \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right) - p\right]^2 o}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2}. \quad (2.5)$$

Let $\pi(o,\varsigma,\tau,;z) = \beta(u,v,\omega;z)$

$$= \beta\left(o, \frac{\varsigma+\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)p\right]o}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)}, \frac{\tau+2\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]\varsigma+\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]^2 o}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2}; z\right). \quad (2.6)$$

Applying Lemma 1.4 using relation (2.2), (2.3) and (2.4), from (2.6), we get

$$\pi(g(z), zg'(z), z^2g''(z); z) = \beta\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z), \psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f*g)(z), \psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f*g)(z); z\right). \quad (2.7)$$

Therefore, from (2.1) we get

$$\pi(g(z), zg'(z), z^2g''(z); z) \in \varpi. \quad (2.8)$$

See that

$$1+\frac{\tau}{\varsigma} = \frac{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2 \omega - \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]^2 u}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)\upsilon - \left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]u} - 2\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right],$$

And since the admissibility condition for $\pi \in \Pi_n[\varpi, q]$ and by using Lemma 1.4, we have

$$g(z) \prec q(z) \quad or \quad \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)\,(f*g)(z) \prec q(z).$$

Theorem 2.3. Let $\beta \in B_n[\vartheta, q]$ with $q(0) = 1$. If $f \in A_p$ this achieved

$$\beta\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z), \psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f*g)(z), \psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f*g)(z); z\right) \prec \vartheta(z), \quad (2.9)$$

then

$$\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z) \prec q(z). \quad (z \in U)$$

The next is a stretching of Theorem 2.2 to the case where the conduct of $q(z)$ on ∂U is unknown.

Corollary2.4. Let $\varpi \in C$ and $q(z)$ be univalent in U with $q(0) = 1$. Let $\beta \in B_n[\varpi, q_\rho]$ and $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in A_p$ and

$$\beta\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z), \psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f*g)(z), \psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f*g)(z); z\right) \in \varpi,$$

then

$$\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z) \prec q(z).$$

Proof. By Theorem 2.2, we get $\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z) \prec q(\rho z)$. From the subordination relation we deduced the following conclusion

$$q_\rho(z) \prec q(z).$$

Theorem 2.5. Let $\vartheta(z)$ and $q(z)$ be univalent in U, with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $\vartheta_\rho(z) = \vartheta(\rho z)$. Let $\beta: C^3 \times U \longrightarrow C$ it suffices achieve either 1 or 2 of the following conditions:

i. $\beta \in B_n[\vartheta, q_\rho]$, $\rho \in (0,1)$.

ii. $\exists\, \rho_0 \in (0,1)$ such that $\beta \in B_n[\vartheta_\rho, q_\rho], \forall\, \rho \in (\rho_0, 1)$. $If\ f \in A_p$ this achieved (2.9), then

$$\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z) \prec q(z).$$

Proof. (1). Applying Theorem 2.2, we get $\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z) \prec q_\rho(z)$, since $q_\rho(z) \prec q(z)$, we deduce $\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z) \prec q(z)$.

(2). If we let $g_\rho(z) = \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)_\rho(z) = \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(\rho z) = g(\rho z)$,

Then,

$$\beta\left(g_\rho(z), zg'_\rho(z), z^2g''_\rho(z), \rho z\right) = \beta\left(g(\rho z), zg'(\rho z), z^2g''(\rho z); \rho z\right) \in \vartheta_\rho(U).$$

By using Theorem 2.2 with (2.8) where $\omega(z) = \rho z$ is any mapping U in to U, we get $g_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. Thus as $\rho \longrightarrow 1^-$, to have $g(z) \prec q(z)$.

Hence, $\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f*g)(z) \prec q(z)$.

In get the better dominant from the differential subordination, we have deduced the following conclusion (2.9).

Theorem 2.6. Let $\vartheta(z)$ be univalent in U and $\beta: C^3 \times U \longrightarrow C$. Assume that the differential equation

$$\beta\left(q(z), \frac{zq'(z)+\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]q(z)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)}, \frac{z^2q''(z)+2\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]zq'(z)+\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]^2 q(z)}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2}; z\right) = \vartheta(z), \quad (2.10)$$

Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 32, Issue 3, 2021 DOI: http://doi.org/10.23851/mjs.v32i3.956

has solution $q(z)$ with $q(0)=0$ and it suffices that one of the following conditions is achieved

(1) $q(z) \in Q_0$ and $\beta \in B_n[\vartheta, q]$.
(2) $q(z)$ is univalent in U and $\beta \in B_n[\vartheta, q_\rho], \rho \in (0,1)$,
(3) $q(z)$ is univalent in U and $\exists\, \rho_0 \in (0,1)$ such that $\beta \in B_n[\vartheta_\rho, q_\rho], \forall \rho$
$(\rho_0, 1)$. If $f \in A_p$ this achieved (2.9), then $\psi^{\kappa}_{\lambda,p}(f*g)(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. By applying Theorem 2.3 and 2.5, we deduce that $q(z)$ is a dominant of (2.9). Since $q(z)$ this achieved (2.10), thus, it will be dominated by all dominants of (2.9), because it is a solution of (2.9).

Thus, $q(z)$ is the better dominant of (2.9). In special case $q(z)=Mz, M>0$, and using the Definition 1.2, a class of admissible function $B_n[\varpi, q]$ means by $B_n[\varpi, M]$ is described below.

Definition 2.7. Let ϖ be a set in $C, Re\{m\} > 0, \lambda \geq 1$ *and* $M > 0$. the class of admissible functions $B_n[\varpi, M]$ made up of those functions $\beta: C^3 \times U \longrightarrow C$ that achieve the admissibility condition:

$$\beta\left(Me^{i\theta}, \frac{y+\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]Me^{i\theta}}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)}, \frac{L+2\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]y+\left[\left(\frac{\delta+\eta}{\mu+\lambda}\right)-p\right]^2 Me^{i\theta}}{\left(\frac{\delta+\eta}{\mu+\lambda}\right)^2}; z\right) \notin \varpi, \quad (2.11)$$

whenever $\beta \in R, R\left(Le^{i\theta}\right) \geq y(y-1)M$,

$$y \geq 1 \text{ and } z \in U.$$

Corollary 2.8. Let $\beta \in B_n[\varpi, M]$. If $f \in A_p$ this achieved the next inclusion relationship

$$\beta\left(\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+1}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+2}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z); z\right) \in \varpi,$$

then

$$\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z) \prec Mz.$$

In specially $\varpi = q(U) = \{\omega: |\omega| < M\}$, the class $B_n[\varpi, M]$ is indicated by $B_n[M]$.

Corollary 2.9. Let $\beta \in B_n[M]$. If $f \in A_p$ this achieved

$$\left|\beta\left(\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+1}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+2}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z); z\right)\right| < M,$$

then

$$\left|\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z)\right| < M.$$

Theorem 2.10. Let $\beta \in B_n[\varpi, q]$. If $f \in A_p, \psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)\,(f*g)(z) \in Q_0$ and

$$\beta\left(\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+1}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+2}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z); z\right)$$

is univalent in U, then

$$\varpi \subset \beta\left(\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+1}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+2}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z); z\right), \quad 2.12)$$

means,

$$q(z) \prec \psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z). \,(z \in U)$$

Proof. By using (2.7) and (2.12), we get

$$\varpi \subset \pi(g(z), zg'(z), z^2 g''(z); z), \quad (z \in U)$$

From (2.5), a condition of admissibility for $\beta \in B'[\varpi, q]$is equivalent to a condition of admissibility for π, and this is what we deduced from above by Definition 1.3.
Therefore, by Lemma 1.5 we get

$$q(z) \prec g(z) \;\; or \;\; q(z) \prec \psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z). \,(z \in U)$$

Theorem 2.11. Let $\vartheta(z)$ is holomorphic on U

and $\beta \in B'_n[\vartheta, q]$. If $f \in A_p$

$$\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z) \in Q_0 and \;\; \beta: C^3 \times U \longrightarrow C$$

with

$$\beta\left(\psi^{\kappa}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+1}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z), \psi^{\kappa+2}_{\lambda,p}(\delta,\eta,\mu)(f*g)(z); z\right)$$

is univalent in U, then

$$\vartheta(z) \prec \beta\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f * g)(z), \psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f * g)(z), \psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f * g)(z); z\right), (2.13)$$

implies

$$q(z) \prec \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f * g)(z).$$

Proof. Using relation (2.13), we get

$$\vartheta(z) = \varpi \subset \beta\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f * g)(z), \psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f * g)(z), \psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f * g)(z); z\right),$$

and from Theorem 2.10, we obtain

$$q(z) \prec \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f * g)(z).$$

The following sandwich type Theorem by gathering Theorem 2.10, and 2.11.

Theorem 2.12. Let $\vartheta_1(z)$ and $q_1(z)$ be holomorphic functions in U, $\vartheta_2(z)$ be univalent function in U, $q_2(z) \in Q_0$ with $q_1(0) = q_2(0) = 0$ and $\beta \in B_n[\vartheta_2, q_2] \cap B_n'[\vartheta_1, q_1]$.If

$f \in A_p, \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f * g)(z) \in Q_0 \cap H[0,p]$

and

$$\beta\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f * g)(z), \psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f * g)(z), \psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f * g)(z); z\right)$$

is univalent in U, then

$$\vartheta_1(z) \prec \beta\left(\psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f * g)(z), \psi_{\lambda,p}^{\kappa+1}(\delta,\eta,\mu)(f * g)(z), \psi_{\lambda,p}^{\kappa+2}(\delta,\eta,\mu)(f * g)(z); z\right) \prec \vartheta_2(z),$$

implies,

$$q_1(z) \prec \psi_{\lambda,p}^{\kappa}(\delta,\eta,\mu)(f * g)(z) \prec q_2(z).$$

CONCLUSIONS

In this article we concluded that in this case of applying the differential operator for multivalent function using some properties of subordination and superordination results associated with the Hadamard product concept involving composition of the differential operator with remains preserving its geometric properties and to obtain results inside the unit disk.

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