Finding Fixed Points for Set-Valued Mappings by Graph Concepts

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ABSTRACT

The researchers have presented some theorems of the fixed points of single-valued mappings by defining known contractive conditions on those points in the same path from a given graph. Here, this procedure will be modified and used to find fixed points of order-preserve mappings in a complete partially ordered g-metric space.

KEYWORDS: Fixed point, Directed graph, g-metric space, Set-valued mappings.

INTRODUCTION

For set-valued mappings, Nadler [1] presented one of the most important research on fixed points in complete metric space. Then, fixed point theorems for set-valued mappings were established in different directions due to Reich[2 ], Many other results can see in [3-8] In 2005, Mustafa [9] introduced g- metric spaces, as, a generalization of a metric space (X, d). Subsequently, many fixed point results on such spaces appeared in [10- 12]. Recently Jachymski [13] established a result of single-valued mapping in metric spaces with a graph instead of partial ordering. Beg and Butt [4- 5] obtained sufficient conditions about the existence of fixed points by a graph. This article aims to employ previous ideas to present fixed points and common fixed points for set-valued in g – metric spaces. These results relate to the content of the references [4-5, 13]. We begin with the following definition

Definition (1.1) [8]:
Let $M$ be a nonempty set and $\omega: M^3 \to [0, \infty)$ be satisfying the following condition:

1- $\omega(p,q,e)=0$ if and only if $p = q = e$.
2- $0 < \omega(p,q,p), \forall p, q \in M$ with $p \neq q$.
3- $\omega(p,q) \leq \omega(p,q,e)$ for all $p, q, e \in M$ with $q \neq e$.
4- $\omega(p,q,e) = \omega(p,e,q) = \cdots$ (symmetry in all three variables).
5- $\omega(p,q,e) \leq \omega(p,a,a) + \omega(a,q,e)$ for all $p, q, e, a \in M$.

Then the function $\omega$ is called generalized metric on $M$ and the pair $(M, \omega)$ is called a $g$- metric space.

Example (1.2) [9]:
$M = R^+$, with usual distance $d(p,q) = |p - q|$, for all $p, q \in M$. Define $\omega: M^3 \to R^+$

$\omega(p,q,e) = |p - q| + |q - e| + |e - p|$, for all $p, q, e \in M$. Then $\omega$ is a $g$- metric on $M$.

Definition (1.3) [11]:
Let $(M, \omega)$ be a $g$- metric space, then $\omega$ is called symmetric if $\omega(p,q,q) = \omega(p,p,q)$ for all $q, e \in M$. 
Example (1.4) [9]:
Let \(\mathcal{M} = \{p,q\}\) and \(\omega(p,p,p) = \omega(q,q,q) = 0,\omega(p,p,q) = 1,\omega(p,q,q) = 2\) and by symmetry expand \(\omega\) to all of \(\mathcal{M} \times \mathcal{M} \times \mathcal{M}\). Then \(\omega\) is a \(g\) -metric, but \(\omega(p,q,q) \neq \omega(p,p,q)\).

Proposition (1.5) [12]:
Let \((\mathcal{M}, \omega)\) be a \(g\) -metric space, then the following are equivalent:
1. \((\mathcal{M}, \omega)\) is symmetric.
2. \(\omega(p,q,q) \leq \omega(p,q,a), \forall p, q, a \in \mathcal{M}\).
3. \(\omega(p,q,e) \leq \omega(p,q,a) + \omega(e,p,b), \forall p, q, e, a, b \in \mathcal{M}\).

Definition (1.6) [11-8]:
Let \((\mathcal{M}, \omega)\) be a \(g\) -metric space and \(\{r_j\}\) be a sequence of points of \(\mathcal{M}\), if there exist \(L \in \mathbb{N}\) \(\forall j, i, l \geq L\) then the sequence \(\{r_j\}\) is said to be
i) \(\omega\) - convergent to \(r\) if \(\omega(r,r_j,r_i) < \varepsilon\) for all \(i, j \geq L\) That is \(\lim_{i,j \to \infty} \omega(r,r_j,r_i) = 0\) as \(i, j \to \infty\).
ii) \(\omega\) - Cauchy if \(\omega(r_j,r_i,\varepsilon) < \varepsilon\) for all \(i, j \geq L\). That is \(\omega(r_j,r_i) \to 0\) as \(i, j \to L\).
iii) \(Ag\) - metric space \((\mathcal{M}, \omega)\) is complete if every \(\omega\)-Cauchy sequence is \(\omega\)-convergent in \((\mathcal{M}, \omega)\).

Proposition (1.7) [11]:
Let \((\mathcal{M}, \omega)\) be a \(g\) -metric space the following statements are equivalent
i) \(\{r_j\}\) is \(\omega\)-convergent to \(r\), if and only if
\(\omega(r_j,r_i,r) \to 0, as j \to \infty\).
ii) \(\omega(r_j,r_i,r) \to 0, as j \to \infty\), if and only if \(\omega(r_j,r_i,r) \to 0, as j, i \to \infty\).

Remark (1.8) [9]:
Every \(g\) -metric \((\mathcal{M}, \omega)\) on \(\mathcal{M}\) defines a metric \(d\) on \(\mathcal{M}\) given by
\[d(p,q) = \omega(p,q) + \omega(q,p)\]
\[\omega(p,q,e) = \max\{|p-q|,|q-e|,|e-p|\}\]

Proposition (1.9) [9]:
Let \((\mathcal{M}, \omega)\) be \(ag\) -metric space, then for any \(p, q, e, a \in \mathcal{M}\) is following that
1. If \(\omega(p,q,e) = 0\) then \(p = q = e\).
2. \(\omega(p,q,e) \leq \omega(p,p,q) + \omega(q,q,e)\).
3. \(\omega(p,p,q) \leq 2 \omega(q,q,q)\).
4. \(\omega(p,p,q) \leq \omega(q,a,e) + \omega(a,q,e)\).
5. \(\omega(p,q,e) \leq 2/3 (\omega(p,q,a) + \omega(a,q,e))\).
6. \(\omega(p,q,e) \leq \omega(p,a,a) + \omega(q,a,a) + \omega(e,a,a)\).

Below, if \((\mathcal{M}, \omega)\) is a \(g\) -metric space, \(2^\mathcal{M} = \{A: \emptyset \neq A \subset \mathcal{M}\}\) and \(CB(\mathcal{M}) = \{A: \emptyset \neq A \subset \mathcal{M}, A = \text{closed and bounded}\}\) and \(K(\mathcal{M}) = \{A: \emptyset \neq A \subset \mathcal{M}, A = \text{compact}\}\),
and \(\Omega = \{\text{The Hausdorff}\}.

Definition (1.10) [1]:
The point \(p\) in \(\mathcal{M}\) is called a fixed point of the set-valued mapping \(S: \mathcal{M} \to 2^\mathcal{M}\) if \(p \in Sp\) and \(p\) is a fixed point of a single mapping \(S: \mathcal{M} \to \mathcal{M}\) if \(p = Sp\).

Definition (1.11) [1]:
The mapping \(H: \mathcal{M} \times \mathcal{M} \to R^+\) is called the Hausdorff \(g\) -distance on \(CB(\mathcal{M})\), if
\[\Omega(A,B,C) = \max\{\sup_{p \in A} \omega(p,B,C), \sup_{p \in B} \omega(p,B,C)\}\]
where \(\omega(p,B,C) = d_\omega(p,B) + d_\omega(B,C) + d_\omega(C)\). If \(d_\omega(A,B) = \inf\{d_\omega(a,b), a \in A, b \in Band A,B,C \in CB(\mathcal{M})\}\).

Lemma (1.12) [1]:
i) If \(A,B \in CB(\mathcal{M})\) with \(\Omega(A,B) < \varepsilon\) then \(\forall \varepsilon \in A \exists \varepsilon \in B\) such that \(\omega(a,b) < \varepsilon\).
ii) If \(A,B \in CB(\mathcal{M})\) and \(\varepsilon \in A\), then \(\forall \varepsilon > 0, \exists \varepsilon \in B\) such that \(\omega(a,b) \leq \Omega(A,B) + \varepsilon\).

Lemma (1.13) [11]:
i) If \(A \in CB(\mathcal{M})\) and \(B \in K(\mathcal{M})\) then \(\forall \varepsilon \in A, \exists \varepsilon \in B\) such that \(\omega(a,b) \leq \Omega(A,B)\).
ii) Let \(\{A_i\}\) be a sequence in \(CB(\mathcal{M})\) and \(\lim_{j \to \infty} \Omega(A_j, A) = 0\) for \(A \in CB(\mathcal{M})\). If \(p_j \in A_j\) and \(\lim_{j \to \infty} \omega(p_j, p) = 0\), then \(p \in A\).

Definition (1.14) [4-5]:
Let \(G_r\) be a graph with finite vertices denoted by \(V(G_r)\) and finite edges \(E(G_r)\) of different pairs of different elements of \(V(G_r)\). Also, \(G_r^{-1}\) denotes the converse of \(G_r\), which is obtained by reversing the direction of its edges.
Definition (1.15) [4-5]:
A graph $G_r$ is called directed if its edges are symmetric, then $E(G_r) \subseteq E(G_r^r) \cup E(G_r^{-1})$.

Definition (1.16) [5-4]:
we say that $H$ is a subgraph of $G_r$ if $V(H) \subseteq V(G_r)$ and $E(H) \subseteq E(G_r)$.

Definition (1.17) [4-5]:
If each edge in $G_r$ has an associated weight function $W: E(G_r) \to R$ then $G_r$ is called a weighted graph.

Definition (1.18) [4-5]:
Let $p, q \in V(G_r)$. A path in $G_r$ from $p$ to $q$ of length $j \in NU[0]$ is a sequence $(p_i)_{i=0}^j \subseteq V(G_r)$ with $p_0 = p$, $p_j = q$ and $(p_{i-1}, p_i) \in E(G_r)$, $i = 1, 2, \ldots, j$.

Definition (1.19) [4-5]:
The length of the path is the number of elements in $E(G_r)$.

Definition (1.20) [4-5]:
If there is a path between any two vertices of $G_r$, then $G_r$ is called connected otherwise it is disconnected. Moreover, $G_r$ is weakly connected if $G_r^{-1}$ is connected.

Let $G_r\_p$ be the component of $G_r$, consisting of all edges and vertices which are contained in some path $\pi G_r$ beginning at $p$. Assume that $G$ is such that $E(G_r)$ is symmetric, then $G_r^{-1}$ is the equivalence class $[p]_{G_r}$, defined on $V(G_r)$ by the rule $R(u \rightarrow v)$ if there is a path from $u$ to $v$ is $V(G_r) = [p]_{G_r}$.

Jachymski [13] proved some fixed point results for the $G_r$-contraction mapping in a metric space endowed with a weighted graph, and he stated the following results,

Definition (1.21) [13]:
Let $M$ be a complete metric space. A single-valued mapping $S: M \to M$ is a Banach $G_r$-contraction if $(p, q) \in E(G_r)$ implies $(Sp, Sq) \in E(G_r)$ and $\forall (p, q) \in E(G_r)$, $\exists k \in (0, 1) \exists \omega (Sp, Sq) < k \omega (p, q)$.

property A: for any sequence $(p_j)_{j \in N}$ in $M$, if $p_j \to p$ and $(p_j, p_{j+1}) \in E(G_r)$ for $j \in N$, then $(p_j, p) \in E(G_r)$.

By using Banach $G_r$-contraction, Jachymski proved that:

Theorem (1.22) [13]:
Let $M$ be a complete metric space with property A: for any sequence $(p_j)_{j \in N}$ in $M$, if $p_j \to p$ and $(p_j, p_{j+1}) \in E(G_r)$ for $j \in N$, then there is a subsequence $(p_{k_i})_{i \in N}$ with $(p_{k_i}, p) \in E(G_r)$ for $j \in N$. Let $S: M \to M$ be a $G_r$-contraction and $M_S := \{p \in M: (p, Sp) \in E(G_r)\}$.

Then the following hold:
1. card $S = \text{card}([p]_{G_r} p \in M_p$).
2. $S \neq \emptyset$ if and only if $M_S \neq \emptyset$.
3. $S$ has a unique fixed point iff there exists $p_0 \in M_p$ such that $M_S \subseteq [p_0]_{G_r}$.
4. For any $p \in M_S$, $S([p]_{G_r})$ is a Picard operator.
5. If $M_S \neq \emptyset$ and $G_r$ is a weakly connected, then $S$ is a Picard operator.

Beg and Butt [10] presented a version of Jachymski's Theorem for set-valued mappings as the following:

Definition (1.23) [13]:
Let $M$ be a complete metric space. The mapping: $M \to CB(M)$ is said to be a $G_r$-contraction if $\exists k \in (0, 1) \exists \Omega (Sp, Sq) < k(p, q) \forall (p, q) \in E(G_r)$ and $\exists \omega (u, v) < k \omega (p, q) + \alpha, \forall \alpha > 0$ then $(u, v) \in E(G_r)$.

Theorem (1.24) [13]:
Let $M$ be a complete metric space with property A. Let $S: M \to CB(M)$ be a $G_r$-contraction and $M_S := \{p \in M: (p, u) \in E(G_r)\}$ for some $u \in Sp$, then the following hold:
1. For any $p \in M_S$, $S([p]_{G_r}, p) \cdot (Sp, S) \subseteq E(G_r)$.
2. If $M_S \neq \emptyset$ and $G_r$ is weakly connected, then $S$ has a fixed point in $M$.
3. If $S': = U([p]_{G_r}, p \in M_S)$, then $S' \cap \in S$ has a fixed point.
4. If $S \subseteq E(G_r)$ then $S$ has a fixed point.
5. $\text{Fix} S \neq \emptyset \iff M_S \neq \emptyset$.

MAIN RESULTS
Let $(M, \omega)$ is a complete $\gamma$-metric space and $G_r$ is a directed and weighted graph with $E(G_r)$ is
symmetric such that $E(G_r)$ contains all loops, i.e., $\Delta \subseteq E(G_r)$, where $\Delta$ denote the diagonal of the Cartesian product $M \times M$.

**Definition (2.1):**
Let $S : M \rightarrow CB(M)$ be a set-valued mapping. $S$ is called a $G_r$-contraction if $S$ preserves edges of $G_r$, i.e.,

$$\forall p, q \in M, (p, q) \in E(G_r) \Rightarrow (Sp, Sq) \in E(G_r)$$

And $\exists k \in (0, 1) \ni \Omega(Sp, Sq, Se) \leq k\omega(p, q, e)$ for all $(p, q, e) \in E(G_r)$.

**Definition (2.2):**
The mapping $S : M \rightarrow CB(M)$ is said to be a $G_r$-contraction if there exists a $k \in (0, 1)$ such that $\Omega(Sp, Sq, Se) \leq k\omega(p, q, e)$ for all $(p, q, e) \in E(G_r)$.

**Definition (2.3):**
Let $(M, \omega)$ be a $g$-metric space and $S, H, T : M \rightarrow CB(M)$. The mappings $S, H, T$ are said to be $G_r$-contractive if there exists a $k \in (0, 1)$ such that $\Omega(Sp, Hq, Te) < k\omega(p, q, e)$, and $u \in Sp$ and $v \in Hq, w \in Te$ with $\omega(u, v, w) \leq k\omega(p, q, e)$ then $(u, v, w) \in E(G_r)$.

The symmetry of $\Omega$ and $\omega$ implies the following:

**Proposition (2.4):**
If $S : M \rightarrow CB(M)$ is a $G_r$-contraction then $S$ is also a $G_r^{-1}$-contraction.

The following property is needed

**Property (B):**
If $u \in Sp$, $v \in Sq$ and $w \in Se \ni \omega(u, v, e) < k\omega(p, q, e) + \alpha$, $\forall \alpha > 0$ then $u, v$ and $w$ belong to the path of length 2.

**Theorem (2.5):**
Suppose that the triple $(M, G_r)$ has the properties (A-B). Let $S : M \rightarrow CB(M)$ be a $G_r$-contraction and $M_s := \{p \in M : (p, u, v) \in E(G_r)$ for some $u \in Sp, v \in Su\}$.

Then the following hold:
1. For any $p \in M_s, S[p]_{G_r}^{-}$ has a fixed point.
2. If $M_s \neq \emptyset$ and $G_r$ is weakly connected, then $S FixS \neq \emptyset$.
3. If $S' = U[p], p \in M_s$, then $FixS \mid_{p} \neq \emptyset$.
4. If $S \subseteq E(G_r)$ then $FixS \neq \emptyset$.
5. Let $FixS \neq \emptyset$ then $\exists p \in Sp, As \Delta \subseteq E(G_r)$; so, $(p, p, p) \in E(G_r) \Rightarrow p \in M_s$.

So $M_s$
\[ S \subseteq T \] for some \( p \in S \).

Next, we will show that \( \{p_j\} \) is a Cauchy sequence in \( \mathcal{M} \). Let \( i > j \). Then
\[
\omega(p_j, p_i) \leq \omega(p_j, p_{i+1}) + \omega(p_{i+1}, p_i) + \cdots + \omega(p_{i+j}, p_i) < |j - i + 1| \omega(p_0, p_1, p_1) = |j - i + 1|\omega(p_0, p_1, p_1)
\]

Because \( k \in (0, 1), 1 - k^{i-j} < 1 \).

Therefore \( \omega(p_j, p_i) \to 0 \) as \( j \to \infty \Rightarrow \{p_j\} \) is a Cauchy sequence, converges to \( p \in \mathcal{M} \).

To show that \( p \in S \) and \( p \to p \) by Lemma 1.13 \( \Rightarrow H_p, p \in T_p \). For odd \( j \)
\[ \omega(p_j, p_j) \to 0 \] as \( j \to \infty \Rightarrow \{p_j\} \) is a Cauchy sequence, converges to \( p \in \mathcal{M} \).

Hence, the same arguments as above \( p \in S \).

Next as \( p_j, p_{j+1} \) belong to the same path \( p_j, p_{j+1} \in E(G_r) \) also \( p_j, p \) belong to the same path \( p_j, p \in E(G_r) \) for \( j \in N \). We infer that \( (p_0, p_1, \ldots, p_j, p) \) is a path in \( G_r \) and so \( p \in [p_0]_{G_r} \).

2. Since \( \mathcal{M}_S \neq \emptyset \), \( \exists p_0 \in \mathcal{M}_S \) and since \( G_r \) is weakly connected then \([p_0]_{G_r} = \mathcal{M} \) by 1, mappings \( S \) and \( H, T \) have a common fixed point in \( \mathcal{M} \).

3. It follows from parts 1 and 2.

4. \( S \subseteq C E(G_r) \) if \( p \in \mathcal{M} \) be such that there exists some \( u \in S \) belong to the same path \( p, u \in E(G_r) \) so \( \mathcal{M}_S = \mathcal{M} \) and by 2 and 3, \( S, H, T \) have a fixed point.

Remark (2.8):

Replace \( \mathcal{M}_S \) by \( \mathcal{M}_S = \{p \in \mathcal{M} : (p, u, v) \in E(G_r) \} \) for some \( u \in H_p, v \in H_p \) in conditions 1-3 of Theorem 2.7, the conclusion remains true. That is if \( \mathcal{M}_S U \mathcal{M}_H U \mathcal{M}_T \) then getting \( \text{Fix} S \cap \text{Fix} H \cap \text{Fix} T = \emptyset \) which follows easily from 1-3. Similarly, in condition 4 we can replace \( S \subseteq E(G_r) \) by \( H \subseteq E(G_r) \).

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Corollary (2.9):
Let the triple \((\mathcal{M}, \omega, g)\) have the property (A). If \(G_r\) is weakly connected then \(G_r\)-contractive mappings \(S, H, T : \mathcal{M} \to CB(\mathcal{M})\) such that \((p_0, p_1, p_2)\) for some \(p_1 \in Sp_0\) has a common fixed point

Corollary (2.10):
Let \((\mathcal{M}, \omega)\) be a \(\varepsilon\)-chained complete \(g\)-metric space for some \(\varepsilon > 0\). Let \(S, H, T : \mathcal{M} \to CB(\mathcal{M})\) be such that there exists \(k \in (0, 1)\) with \(0 < \omega(p, q, e) < \varepsilon = \Omega(Sp, Hq, Te) < k\omega(p, q, e)\). Then \(S\) and \(H, T\) have a common fixed point.

Proof.
\(E(G_r) = \{(p, q) \in \mathcal{M} \times \mathcal{M} : 0 < \omega(p, q, e) < \varepsilon\}\). \(E(G_r)\) is the \(\varepsilon\)-chained complete \(g\)-metric space for some \(\varepsilon > 0\). Let \(u, v, w\) be such that there exists \(v \in Hq\) and \(w \in Te\) such that \(\omega(u, v, w) < \varepsilon\) which implies \(u, v, w\) belong to the same path, \(v, w \in E(G_r)\). Hence \(S\) and \(H, T\) are \(G_r\)-contractive. Also \((\mathcal{M}, \omega, g)\) has property (A). Indeed if \(p_j \to p\), then \(\omega(p_j, p, p) < \varepsilon\) for sufficiently \(j\). Therefore \(p_j, p\) belong to the same path \(p_j, p \in E(G_r)\). So, by Theorem 2.7 (2); \(S\) and \(H, T\) have a common fixed point.

Theorem (2.11):
Let \(S, H, T : \mathcal{M} \to CB(\mathcal{M})\) be \(G_r\)-contractive mappings and properties \(A, B\) hold. Set \(\mathcal{M}_S := \{p \in \mathcal{M} : (p, u, v) \in E(G_r)\text{ for some } u \in Sp, v \in Su\}\). Then the following hold:

1. For any \(p \in Sp, S, H, T|_{G_r}\) has a common fixed point.
2. If \(\mathcal{M}_S \neq \emptyset\) and \(G_r\) is weakly connected, then \(\emptyset \neq Fix F \cap Fix H \cap Fix T \cap \mathcal{M}\).
3. If \(\mathcal{M}' := \{[p]_{G_r} : p \in \mathcal{M}_S\}, Fix F \cap Fix H \cap Fix T|_{G}\) then \(\emptyset \neq \emptyset\).
4. If \(S \subseteq E(G_r)\) then \(Fix F \cap Fix H \cap Fix T \neq \emptyset\).
5. If \(\mathcal{M}_S \neq \emptyset\) then \(Fix S \neq \emptyset\).

Proof. The parts 1-4 can be proved by putting \(S = H\) in Theorem 2.7 and 5 obtained from the

Remark 2.8 We observe that the symmetric \(E(G_r)\) is not needed in Theorem 2.11.

REFERENCES