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The Generalized Gamma – Exponentiated Weibull Distribution with its Properties

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ABSTRACT

In this paper, we present the Generalized Gamma-Exponentiated Weibull distribution as a special case of new generated Generalized Gamma - G family of probability distribution. The cumulative distribution, probability density, reliability and hazard rate functions are introduced. Furthermore, the most vital statistical properties, for instance, the r-th moment, characteristic function, quantile function, simulated data, Shannon and relative entropies besides the stress-strength model are obtained.

KEYWORDS: Generated Family; Generalized Gamma Distribution; Exponentiated Weibull Distribution.

الخلاصة

في هذا البحث ، تم تقديم توزيع كاما المعمم- ويبل الاسي كحالة خاصة من عائلة كاما المعمم المولدة للتوزيعات الاحتمالية. تم تقديم دوال التوزيع التراكمية ، الكثافة الاحتمالية ، المعولية والمخاطرة . علاوة على ذلك ، تم تقديم أهم الخصائص الإحصائية ، على سبيل المثال ، العزم الرائي ، الدالة المميزة ، الدالة الكمية ، البيانات المولدة ، شانون وانتروبي النسبية جنبًا إلى جنب مع انموذج الإجهاد-المتانة .

INTRODUCTION

In various fields of science, it is well known that there is an ongoing need to choose a distribution that fits the data well. Numerous studies have been done to introduce new generalized distribution families. These new distributions have the common feature, see e.g [1-5], of having more parameters. One of these distributions is the Generalized Gamma (GG) that presented by [6] (sometimes called Stacy distribution). The GG distribution is an elastic family, in the set of shape and hazard rate function, for modeling data in a variety of research areas such as engineering, hydrology, reliability\survival analysis also as a statistical model of speech signals [7]. It includes special sub-models, among others, the Exponential, Gamma, Weibull, and Rayleigh distributions.

A random variable X has GG distribution if the cumulative distribution function (cdf) and its associated probability density function (pdf), with three positive parameters (a, d, p), given respectively, for $x > 0$ by [8],

$$F(x;a,d,p)_{GG} = \frac{\gamma\left[\frac{d}{p},\left(\frac{x}{a}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)} \quad (1)$$

and,

$$f(x;a,d,p)_{GG} = \frac{1}{\Gamma\left(\frac{d}{p}\right)}\left(\frac{p}{a^{d}}\right)x^{d-1}e^{-\left(\frac{x}{a}\right)^{p}} \quad (2)$$

and $f(x;a,d,p)_{GG} = 0$ otherwise, where,

$\Gamma(\cdot)$ is the gamma function, and $\gamma(\cdot,\cdot)$ is the incomplete gamma function.

In this paper, a new family based on GG distribution along with one of its special case, named Generalized Gamma-Exponentiated Weibull distribution, are proposed.

Generalized Gamma – G family

Suppose that G(x) and g(x) are any continuous baseline cdf and pdf of a random variable X. Also suppose that H(.) and h(.) represents the cdf and pdf of any continuous distribution respectively. The general formula of reliability function for this class is given by,

$$R(x)_{H-G} = \int_0^{-\ln G(x)} h(x)\, dx = H(-\ln G(x)) \quad (3)$$

Depending on (3), the general formula of cdf, $F(x)_{H-G} = 1 - R(x)_{H-G}$, and pdf, $f(x)_{H-G} = -\frac{d}{dx}[R(x)_{H-G}]$, for this class will be,

$$F(x)_{H-G} = 1 - H(-\ln G(x)) \quad (4)$$

$$f(x)_{H-G} = \frac{g(x)}{G(x)}\, h(-\ln G(x)) \quad (5)$$

Let H(.) and h(.) that mentioned in (3),(4) and (5), be the cdf and pdf of GG distribution [8], recall (1) and (2), with three positive parameters a, d, and p, as,

$$H(-\ln G(x)) = \frac{\gamma\left[\frac{d}{p}, \left(\frac{-1}{a}\ln G(x)\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)} \quad (6)$$

$$h(-\ln G(x)) = \frac{1}{\Gamma\left(\frac{d}{p}\right)} \frac{p}{a^d} [-\ln G(x)]^{d-1} e^{-\left(\frac{-1}{a}\ln G(x)\right)^p} \quad (7)$$

By substituting (6) and (7) in (4) and (5), the cdf and pdf of new family named Generalized Gamma – G (symbolized by GG-G) will be,

$$F(x)_{GG-G} = 1 - \frac{\gamma\left[\frac{d}{p}, \left(\frac{-1}{a}\ln G(x)\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)} \quad (8)$$

and,

$$f(x)_{GG-G} = \frac{1}{\Gamma\left(\frac{d}{p}\right)} \frac{p}{a^d} \frac{g(x)}{G(x)} [-\ln G(x)]^{d-1}\, e^{-\left(\frac{-1}{a}\ln G(x)\right)^p} \quad (9)$$

By using $e^{-z} = \Sigma_{i=0}^{\infty} \frac{(-1)^i}{i!} z^i$, the pdf in (9) can be re-written as,

$$f(x)_{GG-G} = \frac{\frac{p}{a^d}}{\Gamma\left(\frac{d}{p}\right)} \frac{g(x)}{G(x)} [-\ln G(x)]^{d-1} \Sigma_{i=0}^{\infty} \frac{(-1)^i}{i!} \left[\frac{-1}{a}\ln G(x)\right]^{pi}$$

$$= \frac{\frac{p}{a^d}}{\Gamma\left(\frac{d}{p}\right)} \Sigma_{i=0}^{\infty} \frac{(-1)^i}{i!a^{pi}} \frac{g(x)}{G(x)} [-\ln G(x)]^{d+pi-1}$$

For $i \geq 1$, using (see [9])

$$[-\ln z]^a = \sum_{k,l=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k+l}\, a}{a-j}\, C_k^{k-a} C_j^k C_l^{a+k} P_{j,k}\, z^l$$

we have,

$$[-\ln G(x)]^{d+pi-1} = \sum_{k,l=0}^{\infty} \sum_{j=0}^{k} (-1)^{j+k+l} \frac{(d+pi-1)}{d+pi-1-j}\, C_k^{k-d-pi+1} C_j^k C_l^{d+pi-1+k} P_{j,k}\, [G(x)]^l$$

Now $f(x)_{GG-G}$ will be,

$$f(x)_{GG-G} = \frac{\frac{p}{a^d}}{\Gamma\left(\frac{d}{p}\right)} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\, a^{pi}} \frac{g(x)}{G(x)} \sum_{k,l=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k+l}(d+pi-1)}{d+pi-1-j} C_k^{k-d-pi+1} C_j^k C_l^{d+pi-1+k} P_{j,k}\, [G(x)]^l$$

Then the expansion formula for the pdf in (9) will be,

$$f(x)_{GG-G} = \frac{\frac{p}{a^d}}{\Gamma\left(\frac{d}{p}\right)} \sum_{i,k,l=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{i+j+k+l}}{i!a^{pi}} \frac{(d+pi-1)}{(d+pi-1-j)} C_k^{k-d-pi+1} C_j^k C_l^{d+pi-1+k} \quad (10)$$

where $P_{j,0} = 1$ for $j \geq 0$ and $P_{j,k} = k^{-1} \sum_{m=1}^{k} \frac{(-1)^m [m(j+1)-k]}{m+1} P_{j,k-m}$ for $= 1,2, \dots$.

Generalized Gamma – Exponentiated Weibull Distribution

Suppose that G(x) and g(x) in (8) and (9) represents the cdf and pdf of exponentiated Weibull distribution [10][11] with three positive parameters α, β and ζ, given respectively by,

$$G(x; \alpha,\beta,\zeta) = \left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)^\zeta \quad (11)$$

$$g(x; \alpha,\beta,\zeta) = \frac{\alpha\zeta}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)^{\zeta-1} \quad (12)$$

According to (8), the cdf of new distribution named Generalized Gamma – Exponentiated Weibull (symbolized by GG-EW) distribution will be,

$$F(x)_{GG-EW} = 1 - \frac{\gamma\left[\frac{d}{p}, \left(-\frac{1}{a}\ln\left[1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^\zeta\right)^p\right]}{\Gamma\left(\frac{d}{p}\right)} \quad (13)$$

and the pdf of GG-EW distribution can be obtained, according to (9), as,

$$f(x)_{GG-EW} = \frac{\frac{p}{a^d}\alpha\zeta}{\Gamma\left(\frac{d}{p}\right)\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)^{-1} \left[-\ln\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)^\zeta\right]^{d-1} e^{-\left(\frac{-1}{a}\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)^\zeta\right)^p} \quad (14)$$

Now since,

$$g(x)[G(x)]^{l-1} = \frac{\alpha\zeta}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta l-1} \quad (15)$$

Then, according to (10) and (15), the expansion formula for the pdf of GG-EW distribution can be obtained as,

$$f(x)_{GG-EW} = \frac{\frac{p}{a^{d}}\alpha\zeta}{\Gamma\left(\frac{d}{p}\right)\beta}\sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l}}{i!a^{pi}} \frac{(d+pi-1)}{(d+pi-1-j)}C_{k}^{k-d-pi+1}C_{j}^{k}C_{l}^{d+pi-1+k}P_{j,k} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta l-1} \quad (16)$$

The reliability function of GG-EW distribution can be obtained as,

$$R(x)_{GG-EW} = \frac{\gamma\left[\frac{d}{p},\left(-\frac{1}{a}\ln\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\zeta}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)} \quad (17)$$

The hazard rate function of GG-EW distribution can be obtained as,

$$D(x)_{GG-EW} = \frac{p\alpha\zeta\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{-1}}{a^{d}\beta\,\gamma\left[\frac{d}{p},\left(-\frac{1}{a}\ln\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\zeta}\right)^{p}\right]} \left(-\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta}\right)^{d-1} e^{-\left(\frac{-1}{a}\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta}\right)^{p}} \quad (18)$$

Figures 1-4 are illustrating some of possible shapes, for some selected adoptions of the parameters, of the cdf, pdf, reliability and hazard rate functions of the GG-EW distribution.

STATISTICAL PROPERTIES OF THE GG-EW DISTRIBUTION

r-th Moment

The r-th moment of GG-EW distribution can be obtained from $\int_0^\infty x^r f(x)_{GG-EW}\,dx$. According to (10) and then (16), its simply seen that the integration will depend on $g(x)\,[G(x)]^{l-1}$ as in (15), the r-th moment of GG-EW distribution can be obtained as follows,

$$E(X^r)_{GG-EW} = \frac{\frac{p}{a^{d}}}{\Gamma\left(\frac{d}{p}\right)}\sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l}}{i!a^{pi}} \frac{(d+pi-1)}{(d+pi-1-j)}C_{j}^{k}C_{k}^{k-d-pi+1}C_{l}^{d+pi-1+k}P_{j,k}\frac{1}{l} \int_0^\infty x^r\frac{\alpha\zeta l}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta l-1}dx$$

$$= \frac{\frac{p}{a^{d}}}{\Gamma\left(\frac{d}{p}\right)}\sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l}}{i!a^{pi}} \frac{(d+pi-1)}{(d+pi-1-j)}C_{j}^{k} C_{k}^{k-d-pi+1}C_{l}^{d+pi-1+k}P_{j,k}\frac{1}{l}I^{*}$$

where,

$$I^{*} = \int_0^\infty x^r\frac{\alpha\zeta l}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta l-1}dx$$

is the r-th moment of the EW distribution (see [10][11]) with $\alpha, \beta, \zeta l$, therefor the result of I^* will be,

$$I^{*} = \begin{cases} \zeta l\,\beta^{r}\Gamma\left(\frac{r}{\alpha}+1\right)\sum_{m=0}^{\zeta l-1}C_{m}^{\zeta l-1} (-1)^{m}(m+1)^{-\frac{r}{\alpha}-1} & ;\text{ if } \zeta l\in N \\ \zeta l\,\beta^{r}\Gamma\left(\frac{r}{\alpha}+1\right)\sum_{m=0}^{\infty}\frac{{}_{\zeta l-1}P_{m}}{m!} (-1)^{m}(m+1)^{-\frac{r}{\alpha}-1} & ;\text{ if } \zeta l\notin N \end{cases}$$

where ${}_{\zeta l-1}P_m = (\zeta l-1)(\zeta l-2)\ldots(\zeta l-m)$ and N is the set of the natural number.

$$E(X^r)_{GG-EW} = \begin{cases} \frac{\frac{p}{a^{d}}}{\Gamma\left(\frac{d}{p}\right)}\zeta\beta^{r}\Gamma\left(\frac{r}{\alpha}+1\right) \sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\zeta l-1}\frac{(-1)^{i+j+k+l+m}(d+pi-1)}{i!a^{pi}(d+pi-1-j)} C_{k}^{k-d-pi+1}C_{j}^{k}C_{l}^{d+pi-1+k}C_{m}^{\zeta l-1} P_{j,k}(m+1)^{-\frac{r}{\alpha}-1} & ;\text{ if } \zeta l\in N \\ \frac{\frac{p}{a^{d}}}{\Gamma\left(\frac{d}{p}\right)}\zeta\beta^{r}\Gamma\left(\frac{r}{\alpha}+1\right) \sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+k+l+m}(d+pi-1)}{i!a^{pi}(d+pi-1-j)} C_{k}^{k-d-pi+1}C_{j}^{k}C_{l}^{d+pi-1+k}\frac{{}_{\zeta l-1}P_{m}}{m!} P_{j,k}(m+1)^{-\frac{r}{\alpha}-1} & ;\text{ if } \zeta l\notin N \end{cases} \quad (19)$$

Depending on the particular value of $E(X^r)_{GG-EW}$; $(r = 1,2,3,4)$,another properties of this distribution for instance the mean, variance, coefficients of skewness and kurtosis can be obtained.

The Characteristic Function

The characteristic function of GG-EW distribution can be obtained from

$E\left(e^{itX}\right) = \Sigma_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r)_{GG-EW}$

where $E(X^r)_{GG-EW}$ as in (19).

The Quantile Function and Simulated Data

The $100\, q^{th}$ quantile of $GG-EW$ random variable, is defined as a solution of,

$P\left(x \leq x_q\right) = F\left(x_q\right)_{GG-EW}$ with respect to x_q , where $x_q > 0$ and $0 < q < 1$.

Therefor the quantile function x_q of GG-EW random variable can be obtained by solving numerically the following non-linear equation,

$$q_{GG-EW} = 1 - \frac{\gamma\left[\frac{d}{p}, \left(-\frac{1}{a}\ln\left[1-e^{-\left(\frac{x_q}{\beta}\right)^{\alpha}}\right]^{\zeta}\right)^{p}\right]}{\Gamma\left(\frac{d}{p}\right)} \qquad (20)$$

A random variable X has the GG-EW distribution can be simulated by solving numerically the following nonlinear equation,

$$(1-U)\Gamma\left(\frac{d}{p}\right) - \gamma\left[\frac{d}{p}, \left(-\frac{1}{a}\ln\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{\zeta}\right)^{p}\right] = 0 \qquad (21)$$

where U has the standard uniform distribution**.**

Shannon Entropy

The Shannon entropy of the GG-EW distribution can be obtained from $-\int_0^{\infty} \ln(f(x)_{GG-EW})\, f(x)_{GG-EW}\, dx$.

Taking the natural logarithm of the pdf in (14), we get,

$$\ln(f(x)_{GG-EW}) = \ln\left(\frac{\frac{p}{a^d}\alpha\zeta}{\Gamma\left(\frac{d}{p}\right)\beta}\right)$$
$$+(\alpha-1)\ \ln\left(\frac{x}{\beta}\right) - \left(\frac{x}{\beta}\right)^{\alpha} - \ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)$$
$$+\ (d-1)\ln\left[-\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta}\right]$$
$$-\left(\frac{-1}{a}\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta}\right)^{p}$$

Thus, the Shannon entropy of the GG-EW distribution is given by,

$$SH_{GG-EW} = -\left(\begin{array}{c} \ln\left(\frac{\frac{p}{a^d}\alpha\zeta}{\Gamma\left(\frac{d}{p}\right)\beta}\right)+(\alpha-1)E\left(\ln\left(\frac{X}{\beta}\right)\right) \\ -\frac{1}{\beta^{\alpha}}E(X^{\alpha})-E\left(\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)\right) \\ +(d-1)E\left(\ln\left[-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)^{\zeta}\right]\right) \\ -\left(\frac{\zeta}{a}\right)^{p}E\left(\left[-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)\right]^{p}\right) \end{array}\right) \qquad (22)$$

where $E(X^{\alpha})$ as in (19) with $r = \alpha$.

Based on (16), $E\left(\ln\left(\frac{X}{\beta}\right)\right)$ can be obtained as,

$$E\left(\ln\left(\frac{X}{\beta}\right)\right) = \int_0^{\infty} \ln\left(\frac{x}{\beta}\right) \frac{\frac{p}{a^d}\ \alpha\zeta}{\Gamma\left(\frac{d}{p}\right)\beta}$$
$$\sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{i+j+k+l}\ (d+pi-1)}{i!\, a^{pi}(d+pi-1-j)}$$
$$C_k^{k-d-pi+1} C_j^k C_l^{d+pi-1+k}$$
$$P_{j,k}\ \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta l-1} dx$$

Let

$$W = \frac{\frac{p}{a^d}}{\Gamma\left(\frac{d}{p}\right)} \sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{i+j+k+l}\ (d+pi-1)}{i!a^{pi}(d+pi-1-j)}$$
$$C_k^{k-d-pi+1} C_j^k C_l^{d+pi-1+k}\ P_{j,k}$$

So that,

$$E\left(\ln\left(\frac{X}{\beta}\right)\right) = W\int_0^{\infty} \ln\left(\frac{x}{\beta}\right) \frac{\alpha\zeta}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1}$$
$$e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta l-1}\ dx$$

Let $u = \left(\frac{x}{\beta}\right)^{\alpha} \to du = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} dx$, then,

$$E\left(\ln\left(\frac{X}{\beta}\right)\right) = W\int_0^{\infty} \frac{1}{\alpha}\ln(u)\ \frac{\alpha\zeta}{\beta}\ u^{1-\frac{1}{\alpha}} e^{-u}$$
$$[1-e^{-u}]^{\zeta l-1} \frac{\beta}{\alpha}\ u^{\frac{1}{\alpha}-1} du$$
$$= \frac{W\zeta}{\alpha}\int_0^{\infty} \ln(u)\, e^{-u}(1-e^{-u})^{\zeta l-1}\, du$$

Now, for $(1-e^{-u})^{\zeta l-1}$ we have two cases,

Case one: If $\zeta l - 1 > 0$, using

$(1-z)^b = \Sigma_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{\Gamma(b+1)}{\Gamma(b-i+1)} z^i$ for

$|z| < 1, b > 0$, we get,

$$E\left(\ln\left(\frac{X}{\beta}\right)\right) = \frac{W\zeta}{\alpha}\int_0^\infty \ln(u)\ e^{-u} \Sigma_{m=0}^\infty \frac{(-1)^m\,\Gamma(\zeta l)}{m!\ \Gamma(\zeta l - m)} e^{-mu}\ du$$
$$= \frac{W\zeta}{\alpha}\ \Sigma_{m=0}^\infty \frac{(-1)^m\Gamma(\zeta l)}{m!\,\Gamma(\zeta l - m)} \int_0^\infty \ln(u)\, e^{-(m+1)u}\ du$$
$$= \frac{W\zeta}{\alpha}\ \Sigma_{m=0}^\infty \frac{(-1)^m\Gamma(\zeta l)}{m!\,\Gamma(\zeta l - m)} \frac{1}{(m+1)}\Gamma(1)\ [\psi(1) - \ln(m+1)]$$

<u>Case two:</u> If $\zeta l - 1 < 0$, using

$(1-z)^{-b} = \Sigma_{i=0}^\infty \frac{\Gamma(b+i)}{i!\,\Gamma(b)}\ z^i$; $|z| < 1, b > 0$, we get,

$$E\left(\ln\left(\frac{X}{\beta}\right)\right) = \frac{W\zeta}{\alpha}\int_0^\infty \ln(u)\ e^{-u} \Sigma_{m=0}^\infty \frac{\Gamma(\zeta l - 1 + m)}{m!\ \Gamma(\zeta l - 1)} e^{-mu}\ du$$
$$= \frac{W\zeta}{\alpha}\ \Sigma_{m=0}^\infty \frac{\Gamma(\zeta l - 1 + m)}{m!\ \Gamma(\zeta l - 1)} \int_0^\infty \ln(u)\, e^{-(m+1)u}\ du$$
$$= \frac{W\zeta}{\alpha}\ \Sigma_{m=0}^\infty \frac{\Gamma(\zeta l - 1 + m)}{m!\ \Gamma(\zeta l - 1)} \frac{1}{(m+1)}\Gamma(1)\ [\psi(1) - \ln(m+1)]$$

Therefor

$$E\left(\ln\left(\frac{X}{\beta}\right)\right) = \begin{cases} \frac{\frac{p}{a^d}}{\Gamma\left(\frac{d}{p}\right)}\ \frac{\zeta}{\alpha}\Sigma_{i,k,l,m=0}^\infty \Sigma_{j=0}^k\ \frac{(-1)^{i+j+k+l+m}\ (d+pi-1)}{i!a^{pi}(d+pi-1-j)(m+1)!} C_k^{k-d-pi+1} C_j^k\ C_l^{d+pi-1+k} P_{j,k}\ \frac{\Gamma(\zeta l)}{\Gamma(\zeta l - m)} [\psi(1) - \ln(m+1)] & ;\ \text{if } \zeta l - 1 > 0 \\ \frac{\frac{p}{a^d}}{\Gamma\left(\frac{d}{p}\right)}\ \frac{\zeta}{\alpha}\Sigma_{i,k,l,m=0}^\infty \Sigma_{j=0}^k\ \frac{(-1)^{i+j+k+l}\ (d+pi-1)}{i!a^{pi}(d+pi-1-j)(m+1)!} C_k^{k-d-pi+1} C_j^k\ C_l^{d+pi-1+k} P_{j,k}\ \frac{\Gamma(\zeta l - 1 + m)}{\Gamma(\zeta l - 1)} [\psi(1) - \ln(m+1)] & ;\ \text{if } \zeta l - 1 < 0 \end{cases} \quad (23)$$

To get $E\left(\ln\left(1 - e^{-\left(\frac{X}{\beta}\right)^\alpha}\right)\right)$, using

$\ln(1-z) = -\Sigma_{i=0}^\infty \frac{z^{i+1}}{i+1}$; $|z| < 1$ and e^{-z}, we get

$$\ln\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right) = -\Sigma_{k=0}^\infty \frac{e^{-(k+1)\left(\frac{x}{\beta}\right)^\alpha}}{k+1}$$
$$= -\Sigma_{k=0}^\infty \frac{1}{k+1}\Sigma_{r=0}^\infty \frac{(-1)^r}{r!}\left(\frac{k+1}{\beta^\alpha}\right)^r x^{\alpha r}$$
$$= \Sigma_{k=0}^\infty \Sigma_{r=0}^\infty \frac{(-1)^{r+1}(k+1)^{r-1}}{r!\beta^{\alpha r}} x^{\alpha r}$$

Then,

$$E\left(\ln\left(1 - e^{-\left(\frac{X}{\beta}\right)^\alpha}\right)\right) = \Sigma_{k=0}^\infty \Sigma_{r=0}^\infty \frac{(-1)^{r+1}(k+1)^{r-1}}{r!\,\beta^{\alpha r}} E(X^{\alpha r}) \quad (24)$$

Also, to get $E\left(\ln\left[-\ln\left(1 - e^{-\left(\frac{X}{\beta}\right)^\alpha}\right)^\zeta\right]\right)$, using

$\ln(1-z) = -\Sigma_{i=1}^\infty \frac{z^i}{i}$; $|z| < 1$

$\ln \mathrm{z} = \Sigma_{\mathrm{k}=0}^\infty \frac{(-1)^{\mathrm{k}}(\mathrm{z}-1)^{\mathrm{k}+1}}{\mathrm{k}+1}$; $0 < \mathrm{z} \le 2$

$(a+b)^n = \Sigma_{k=0}^\infty C_k^n\ a^k\ b^{n-k}$; $n \ge 0$ and e^{-z}, we have,

$$\ln\left[-\ln\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)^\zeta\right]$$
$$= \ln\left[-\zeta \ln\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)\right]$$
$$= \ln(\zeta) + \ln\left[-\ln\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right)\right]$$
$$= \ln(\zeta) + \Sigma_{n=0}^\infty \frac{(-1)^n}{n+1}\Sigma_{j=0}^{n+1} C_j^{n+1}(-1)^{n+1-j} \left(\Sigma_{k=1}^\infty \frac{e^{-k\left(\frac{x}{\beta}\right)^\alpha}}{k}\right)^j$$
$$= \ln(\zeta) + \Sigma_{n=0}^\infty \Sigma_{j=0}^{n+1}\ \frac{(-1)^{2n+1-j}}{n+1} C_j^{n+1}\left(\Sigma_{k=1}^\infty \frac{e^{-k\left(\frac{x}{\beta}\right)^\alpha}}{k}\right)^j$$
$$= \ln(\zeta) + \Sigma_{n=0}^\infty \Sigma_{j=0}^{n+1}\ \frac{(-1)^{2n+1-j}}{n+1} C_j^{n+1} \Sigma_{k_1=1}^\infty \Sigma_{k_2=1}^\infty \ldots \Sigma_{k_j=1}^\infty\ \frac{e^{-(k_1+k_2+\cdots+k_j)\left(\frac{x}{\beta}\right)^\alpha}}{k_1 k_2 \ldots k_j}$$
$$= \ln(\zeta) + \Sigma_{n=0}^\infty \Sigma_{j=0}^{n+1}\ \frac{(-1)^{2n+1-j}}{n+1} C_j^{n+1} \Sigma_{k_1=1}^\infty \Sigma_{k_2=1}^\infty \ldots \Sigma_{k_j=1}^\infty \Sigma_{i=0}^\infty\ \frac{(-1)^i}{k_1 k_2 \ldots k_j\ i!} \left((k_1+k_2+\cdots+k_j)\left(\frac{x}{\beta}\right)^\alpha\right)^i$$

Hence,

$$E\left(\ln\left[-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)^{\zeta}\right]\right)=\ln(\zeta)$$
$$+\sum_{n=0}^{\infty}\sum_{j=0}^{n+1}\sum_{k_1,k_2,\dots,k_j=1}^{\infty}\sum_{i=0}^{\infty}C_j^{n+1}\frac{(-1)^{2n+i-j+1}}{n+1}\frac{1}{k_1k_2\dots k_j\, i!}\left(\frac{k_1+k_2+\dots+k_j}{\beta^{\alpha}}\right)^{i}E(X^{\alpha i}) \quad (25)$$

Finally for $\left(\left[-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)\right]^{p}\right)$, using $[-\ln(1-z)]^{p}=\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}C_m^{p}\,b_{s,m}\,z^{p+m+s}$ and e^{-z}, we get,

$$\left[-\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)\right]^{p}$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}C_j^{p}\,b_{k,j}\left(e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{p+j+k}$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}C_j^{p}\,b_{k,j}\,e^{-(p+j+k)\left(\frac{x}{\beta}\right)^{\alpha}}$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}C_j^{p}\,b_{k,j}\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!}\left(\frac{p+j+k}{\beta^{\alpha}}\right)^{r}x^{\alpha r}$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}C_j^{p}\,b_{k,j}\frac{(-1)^{r}}{r!}\left(\frac{p+j+k}{\beta^{\alpha}}\right)^{r}x^{\alpha r}$$

So that,

$$E\left(\left[-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)\right]^{p}\right)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}C_j^{p}\,b_{k,j}\frac{(-1)^{r}}{r!}\left(\frac{p+j+k}{\beta^{\alpha}}\right)^{r}E(X^{\alpha r}) \quad (26)$$

where,

$b_{k,j}=(k\,a_0)^{-1}\sum_{l=1}^{k}[j(l+1)-k]\,a_l\,b_{k-l,j}$,

$b_{0,j}=a_0^{j}$ and $a_k=(k+2)^{-1}$

Therefor from (23), the Shannon entropy of the GG-EW distribution is given by,

$$SH_{GG-EW}=\ln\left(\frac{\Gamma\left(\frac{d}{p}\right)\beta}{\frac{p}{a^{d}}\alpha\zeta}\right)+(1-\alpha)E\left(\ln\left(\frac{X}{\beta}\right)\right)+\frac{1}{\beta^{\alpha}}E(X^{\alpha})+E\left(\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)\right)$$
$$+(1-d)E\left(\ln\left[-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)^{\zeta}\right]\right)+\left(\frac{\zeta}{a}\right)^{p}E\left(\left[-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)\right]^{p}\right) \quad (27)$$

where, the expectations can be obtained respectively from (24), (19), (25), (26) and (27).

The Relative Entropy

The relative entropy of the GG-EW distribution can be obtained from

$$\int_0^{\infty}\ln\left(\frac{f(x)_{GG-EW}}{f_1(x)_{GG-EW}}\right)f(x)_{GG-EW}\,dx\,.$$

Taking the natural logarithm of the $f(x)_{GG-EW}$ in (14) relative to the $f_1(x)_{GG-EW}$ with parameters $(a_1,d_1,p_1,\alpha_1,\beta_1,\zeta_1)$, we get,

$$\ln\left(\frac{f(x)_{GG-EW}}{f_1(x)_{GG-EW}}\right)=\ln\left(\frac{\frac{p}{a^{d}}\alpha\zeta}{\Gamma\left(\frac{d}{p}\right)\beta}\frac{\Gamma\left(\frac{d_1}{p_1}\right)\beta_1}{\frac{p_1}{a_1^{d_1}}\alpha_1\zeta_1}\right)$$
$$-(\alpha-1)\ln(\beta)+(\alpha_1-1)\ln(\beta_1)$$
$$+(\alpha-\alpha_1)\ln(x)-\left(\frac{x}{\beta}\right)^{\alpha}+\left(\frac{x}{\beta_1}\right)^{\alpha_1}$$
$$-\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)+\ln\left(1-e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}\right)$$
$$+(d-1)\left(\ln\left[-\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{\zeta}\right]\right)$$
$$-(d_1-1)\left(\ln\left[-\ln\left(1-e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}\right)^{\zeta_1}\right]\right)$$
$$-\left(\frac{-\zeta}{a}\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)\right)^{p}$$
$$+\left(\frac{-\zeta_1}{a_1}\ln\left(1-e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}\right)\right)^{p_1}$$

So that the relative entropy of the GG-EW distribution can be obtained as,

$$RE_{GG-EW}=\ln\left(\frac{\frac{p}{a^{d}}\alpha\zeta}{\Gamma\left(\frac{d}{p}\right)\beta}\frac{\Gamma\left(\frac{d_1}{p_1}\right)\beta_1}{\frac{p_1}{a_1^{d_1}}\alpha_1\zeta_1}\right)$$
$$-(\alpha-1)\ln(\beta)+(\alpha_1-1)\ln(\beta_1)$$
$$+(\alpha-\alpha_1)E(\ln(X))-\frac{1}{\beta^{\alpha}}E(X^{\alpha})+\frac{1}{\beta_1^{\alpha_1}}E(X^{\alpha_1})$$
$$-E\left(\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)\right)+E\left(\ln\left(1-e^{-\left(\frac{X}{\beta_1}\right)^{\alpha_1}}\right)\right)$$
$$+(d-1)E\left(\ln\left[-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)^{\zeta}\right]\right) \quad (28)$$
$$-(d_1-1)E\left(\ln\left[-\ln\left(1-e^{-\left(\frac{X}{\beta_1}\right)^{\alpha_1}}\right)^{\zeta_1}\right]\right)$$
$$-\left(\frac{\zeta}{a}\right)^{p}E\left(-\ln\left(1-e^{-\left(\frac{X}{\beta}\right)^{\alpha}}\right)\right)^{p}$$
$$+\left(\frac{\zeta_1}{a_1}\right)^{p_1}E\left(-\ln\left(1-e^{-\left(\frac{X}{\beta_1}\right)^{\alpha_1}}\right)\right)^{p_1}$$

where $E(\ln(X))=E\left(\ln\left(\frac{X}{\beta}\right)\right)+\ln(\beta)$ and the expectations can be obtained from (19), (24), (25), (26) and (27) with specified parameters.

The Stress Strength

Let Y and X be the stress strength random variables that independent of each other follows

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respectively GG-EW with different parameters, then the stress strength can be obtained by,

$SS_{GG-EW} = \int_0^{\infty} f_X(x)_{GG-EW}\, F_Y(x)\, dx$ where,

$$F_Y(x) = 1 - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)} \gamma\left[\frac{d_1}{p_1}, \left(-\frac{1}{a_1}\ln\left[1-e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}\right]^{\zeta_1}\right)^{p_1}\right]$$

$$= 1 - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)} \Sigma_{i=0}^{\infty} \frac{(-1)^i}{i!\left(\frac{d_1}{p_1}+i\right)} \left[\left(-\frac{1}{a_1}\ln\left[1-e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}\right]^{\zeta_1}\right)^{p_1}\right]^{\frac{d_1}{p_1}+i}$$

$$= 1 - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)} \Sigma_{i=0}^{\infty} \frac{(-1)^i \left(\frac{\zeta_1}{a_1}\right)^{d_1+ip_1}}{i!\left(\frac{d_1}{p_1}+i\right)} \left(-\ln\left[1-e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}\right]\right)^{d_1+ip_1}$$

Similarly to $\left[-\ln\left(1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)\right]^p$, we get,

$$\left(-\ln\left[1-e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}\right]\right)^{d_1+ip_1} = \Sigma_{j=0}^{\infty}\Sigma_{k=0}^{\infty}\Sigma_{r=0}^{\infty} C_j^{d_1+ip_1}\, b_{k,j}\, \frac{(-1)^r}{r!} \left(\frac{d_1+ip_1+j+k}{\beta_1^{\alpha_1}}\right)^r x^{\alpha_1 r}$$

Then,

$$F_Y(x) = 1 - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)} \Sigma_{i,j,k,r=0}^{\infty} \frac{(-1)^{i+r}}{i!\, r!} \frac{\left(\frac{\zeta_1}{a_1}\right)^{d_1+ip_1}}{\left(\frac{d_1}{p_1}+i\right)} C_j^{d_1+ip_1}\, b_{k,j} \left(\frac{d_1+ip_1+j+k}{\beta_1^{\alpha_1}}\right)^r x^{\alpha_1 r} \quad (29)$$

Now based on (30), the stress strength of the GG-EW distribution can be obtained as,

$$SS_{GG-EW} = 1 - \frac{1}{\Gamma\left(\frac{d_1}{p_1}\right)} \Sigma_{i,j,k,r=0}^{\infty} \frac{(-1)^{i+r}}{i!\, r!} \frac{\left(\frac{\zeta_1}{a_1}\right)^{d_1+ip_1}}{\left(\frac{d_1}{p_1}+i\right)} C_j^{d_1+ip_1}\, b_{k,j} \left(\frac{d_1+ip_1+j+k}{\beta_1^{\alpha_1}}\right)^r E\left(X^{\alpha_1 r}\right) \quad (30)$$

where,

$E\left(X^{\alpha_1 r}\right)$ as in (19) with $\alpha_1 r$ instead of r,

$b_{k,j} = (k\, a_0)^{-1}\Sigma_{l=1}^{k}[j(l+1)-k]\, a_l\, b_{k-l,j}$,

$b_{0,j} = a_0^j$ and $a_k = (k+2)^{-1}$.

CONCLUDING REMARKS

A new generated family of continuous probability distributions based on Generalized Gamma distribution has been proposed. The Generalized Gamma - Exponentiated Weibull (GG-EW) distribution is discussed as a special case of this new family. The cdf, pdf, reliability function (R) and hazard rate function (D) along with the most vital statistical properties of proposed distribution are derived. In addition, the stress-strength $SS = P(Y < X)$ is obtained when Y and X represents the stress and strength random variables that independent of each other follows respectively GG-EW with different parameters.

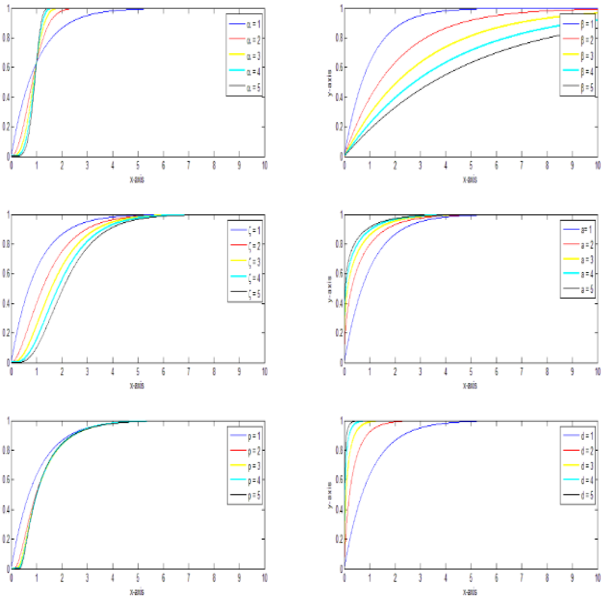

Figure 1. Plots of the cdf for some parameter values and the others equal to 1.

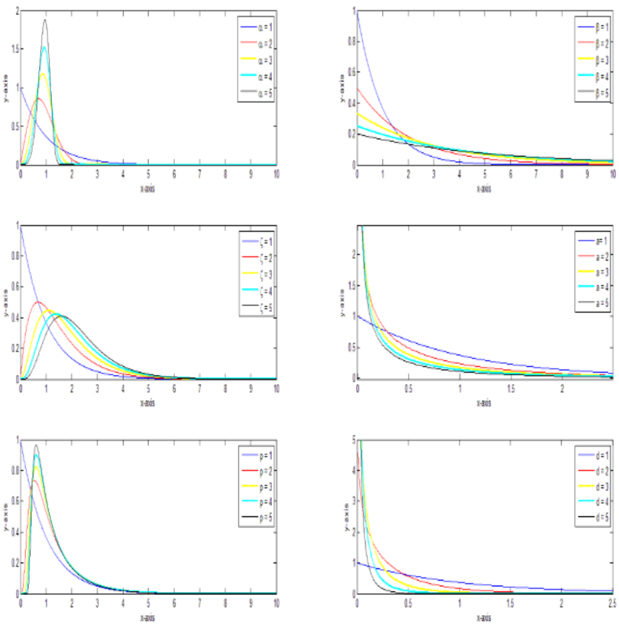

Figure 2. Plots of the pdf for some parameter values and the others equal to 1.

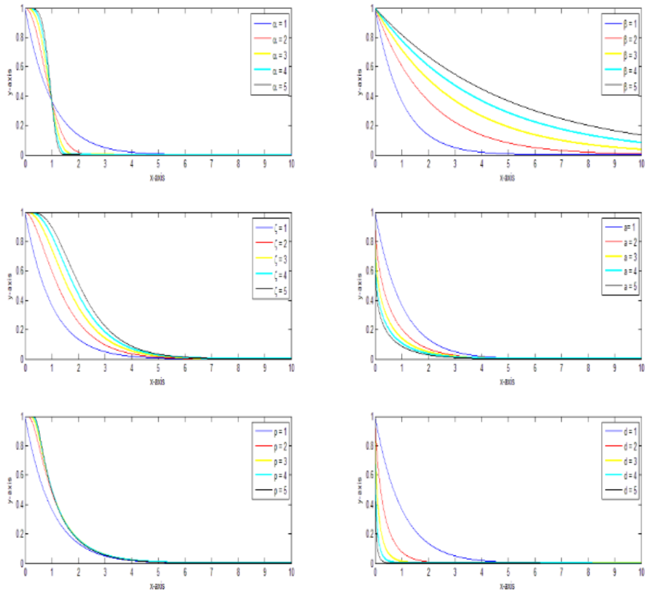

Figure 3. Plots of the R(x) for some parameter values and the others equal to 1.

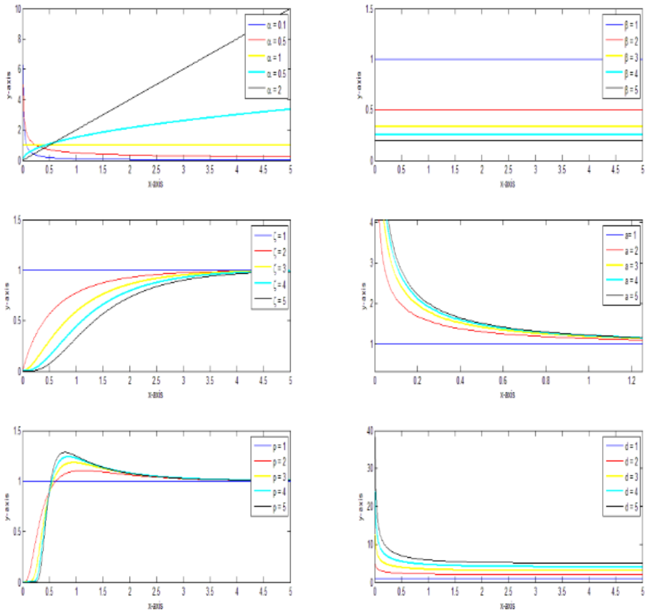

Figure 4. Plots of the D(x) for some parameter values and the others equal to 1.

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