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Strongly Weakly Supplement Extending Modules

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Abstract

In this paper, a class of modules which are proper strong concept of weakly supplement extending modules will be introduced and studied. We call a module M is strongly weakly supplement extending, if each submodule of M is essential in fully invariant weakly supplement submodule in M. Many characterizations of strongly weakly supplement extending modules are obtained. We show that M is strongly weakly supplement extending module if and only if every closed submodule of M is fully invariant weakly supplement submodule in M. Also we study the relation among this concept and other known concepts of modules. Moreover, we give some conditions that of strongly weakly supplement extending modules is closed under direct sum property is strongly weakly supplement extending.

Keywords: Extending module, strongly extending module, weakly supplement submodule, weakly supplement extending module, strongly weakly supplement extending module.

الخلاصة

في هذا البحث, سيتم دراسة و تقديم فئة من المقاسات التي تعتبر مفهوم اقوى من مقاسات التوسع المكمل الضعيف. نسمي المقاس M هو توسع مكمل ضعيف بشدة اذا كان كل مقاس جزئي من M يكون جوهري من مركبة مقاس جزئي مكمل ضعيف ثابت في M. تم الحصول على العديد من التشخيصات للمقاسات التوسع المكمل الضعيف بشدة. نبين ان مقاس M هو التوسع المكمل الضعيف بشدة اذا وفقط اذا كل مقاس جزئي مغلق من M هو مقاس مكمل ضعيف ثابت في M. ايضا ندرس العلاقة ما بين هذا المفهوم وغيره من المقاسات المعروفة. اكثر من ذلك ندرس بعض الشروط تحت الجمع المباشر من مقاسات التوسع المكمل الضعيف بشدة هو مقاسات التوسع المكمل الضعيف بشدة.

Introduction

Following [6], a module M is extending if every submodule of M is essential in a direct summand of M. The author in [3], introduced the concept strongly extending modules that is, a module M is strongly extending if each submodule of M is essential in a stable (fully invariant) direct summand of M. Also, strongly extending modules studied in [5]. The concept of weakly supplement extending modules has been studies in [4]. A module M is weakly supplement extending if every submodule is essential in weakly supplement submodule of M. Also, a module M is weakly supplement extending if and only if every closed submodule of M is weakly supplement submodule of M [4]. In this paper, we introduce a concept stronger than weakly supplement extending modules and it considers as a generalization of strongly extending modules.

Through this paper, all ring R is associative ring with identity and all modules M are unitary left R-modules. A submodule N of M is called essential, if it has nonzero intersection with any nonzero submodule of M [8]. A submodule N of M is small (denoted by $N \ll M$), if there exists a submodule L of M such that M=N+L implies L=M [8]. A module M is called uniform if every nonzero submodule of M is essential in M [6]. A module M is called semisimple, if every submodule is a direct summand of M. A submodule N of M is called closed in M, if it has no proper essential extension in M [9]. A submodule N of M is called supplement, if there exists a submodule K of M such that M=N+K and $N \cap K \ll N$ [8]. A module M is supplemented if every submodule N of M has a supplement submodule in M [8]. A submodule N of M is called weakly supplement submodule, if there exists a submodule K in M such that M=K+N and N ∩ K≪ M. A module M is called weakly supplemented if every submodule N of M is weakly supplement in M [8]. A module M is called ⊕-supplemented, if each submodule N of M has a supplement which is direct summand [7]. The singular submodule of a module M is Z(M)={m ∈ M: Em=0 for some essential left ideal E of R}. A module M is called singular if Z(M)=M, and M is called non-singular if Z(M)=0 [10]. A module M is lifting, if for every submodule N of M there exists a direct summand K of M with K ⊆ N such that M=K⊕ K' and N∩K'≪K' (where K' is a submodule in M) [8]. A submodule N of a module M is called fully invariant, if *f(*N*)* ⊆ N for each R-endomorphism *f* of M [1]. A module M is called duo if every submodule of M is fully invariant of M [9]. Moreover, a submodule N is called stable, if *f* (N)⊆ N for each R-homomorphism *f:* N →M. It is known that, every stable submodule is fully invariant, but the converse is not true in general [3]. Also, following [3], every fully invariant direct summand is stable.

Strongly weakly supplement extending modules

Firstly, we introduce the following concept.

Definition (1):

A module M is called strongly weakly supplement extending if each submodule of M is essential in fully invariant weakly supplement submodule in M.

Firstly, we get the next characterization of strongly weakly supplement extending modules.

Proposition (2):

A module M is a strongly weakly supplement extending if and only if each closed submodule of M is a fully invariant weakly supplement.

Proof: Suppose that M is strongly weakly supplement extending module and let N be a closed submodule of M. Then there exists a fully invariant weakly supplement submodule L of M such that N is essential in L. But N is closed in M, then N=L. So N is fully invariant weakly supplement submodule in M. Conversely, let N be a submodule of M, there exists a closed submodule L of M such that N is essential in L (from Zorn's lemma). By hypothesis, L is fully invariant weakly supplement submodule in M. Then M is strongly weakly supplement extending.

Remarks and Examples (3):

1. Every strongly weakly supplement extending module is weakly supplement extending, while the convers is not true, in general. For example, M=Z_2⊕Z_2 as Z-module is weakly supplement extending (since M is weakly supplemented), but it is not strongly weakly supplement extending, since Z_2⊕(0) is closed submodule in M=Z_2⊕Z_2 but it is not fully invariant (by proposition (2.2)).
2. Every strongly extending module is strongly weakly supplement extending.
3. Every uniform module is strongly weakly supplement extending, but the converse is not true. In fact, Z_6 is strongly weakly supplement extending Z-module, but it is not uniform.
4. Following [4], every lifting module is weakly supplement extending, but strongly weakly supplement extending modules and lifting modules are different concepts, in general. For example, Z as Z-module is strongly weakly supplement extending which is not lifting. Also, M=Z_2⊕Z_2 as is lifting Z-module which it is not strongly weakly supplement extending.
5. Every ⊕-supplemented is weakly supplement extending [4], but this fact is not true for strongly weakly supplement extending modules. For example, Z as Z-module is strongly weakly supplement extending module which is not ⊕-supplemented. Moreover, M=Z_2⊕Z_2 as Z-module is ⊕-supplemented which it is not strongly weakly supplement extending.
6. Unlike weakly supplement extending modules, a semi-simple module need not be strongly weakly supplement extending, in general. For example, M=Z_2⊕Z_2 as Z-module is semi-simple module which it is not strongly weakly supplement extending. Moreover, Q is strongly weakly supplement

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extending Z-module which it is not semi-simple.

Recall that, a module M is an injective hull (denoted by E(M)) if E is injective and it essential extension of M [10].

We have another characterization of strongly weakly supplement extending modules.

Proposition (4):

For any module M, the following statements are equivalent:

1. M is strongly weakly supplement extending.
2. Each closed submodule in M is fully invariant weakly supplement submodule.
3. The intersection of M with any direct summand of injective hull of M is fully invariant weakly supplement submodule in M.

Proof: (1) $\Longrightarrow$ (2) By proposition (2.2).

(2) $\Longrightarrow$ (3) Let F be a direct summand of injective hull E(M), i.e E(M)= F$\oplus$ K, where K is a submodule of injective hull of M. Suppose that F$\cap$M is an essential in L, where L is a submodule of M and let x $\in$ L. Then x=n+k, where n $\in$ F and k $\in$ K. Suppose that x$\notin$ F, then k$\neq$0. Since M is essential in injective hull of M and 0 $\neq$ k $\in$ K$\subseteq$ E(M), so there exists r $\in$R such that 0$\neq$rk $\in$ M. Now rx=rn+rk and rn=rk-rx $\in$ F$\cap$M $\subseteq$ L. We have rk=rx-rn $\in$ L$\cap$K. But F$\cap$M is essential in L, so (F$\cap$M)$\cap$K=0 is essential in L$\cap$K, so L$\cap$K=(0). Then rk=0 which is contradiction. Thus F$\cap$M is closed of M and hence by (2) F$\cap$M is fully invariant weakly supplement submodule of M.

(3) $\Longrightarrow$ (1) Suppose that N be a submodule of M and let K be a relative complement of N in M i.e. N$\cap$K=0, then by [2] N $\oplus$ K is essential in M but M is essential in injective hull of M, then N $\oplus$ K is essential in injective hull of M, then E(M) = E(N $\oplus$K) = E(N) $\oplus$ E(K). Since E(N) is direct summand of injective hull of M, then E(N)$\cap$M is fully invariant weakly supplement submodule in M. But N is essential in injective hull of N and M is essential in M, so N$\cap$M=N is essential in E(N)$\cap$M which is fully invariant weakly supplement in M and hence M is strongly weakly supplement extending.

We observed that a strongly weakly supplement extending module is weakly supplement extending, while the convers is not true see (Remarks and Examples (2.3), (1)). The following result discusses when the converse is true.

Proposition (5):

Let M be a duo module. Then M is weakly supplement extending module if and only if M is strongly weakly supplement extending.

Recall that, a module M is supplement duo, if each supplement submodule of M is fully invariant [11].

We introduce the next concept that is as a generalization of duo modules and as a strong concept of supplement duo modules.

Definition (6):

A module M is called weakly supplement duo if each weakly supplement submodule is fully invariant.

Fully stable module and duo module are weakly supplement duo. In other direction, weakly supplement duo is supplement duo. Note that, Z is weakly supplement duo Z-module but it is not fully stable. While Q as Z-module is supplement duo which is not weakly supplement duo.

Recall that, a module M is a weak duo if every direct summand submodule of M is fully invariant [1]. Following [3], every strongly extending module is weak duo. In the following we generalized this result.

Proposition (7):

Let M be a strongly weakly supplement extending module. Then M is weak duo.

Proof: Let N be a direct summand of strongly weakly supplement extending module M. So we have N is closed submodule of M. But M is strongly weakly supplement extending. Then N is fully invariant submodule of M. Hence M is weak duo.

In fact, the concepts weakly supplement duo module and weakly supplement extending module are different. For example, Q as Z-module is weakly supplement extending module which is not weakly supplement duo.

On can arise a question: Is strongly weakly supplement extending module weakly supplement duo? The answer is not. In fact, Q as Z-module is strongly weakly supplement extending which is not weakly supplement duo.

Proposition (8):

Let M be a weakly supplement duo module. Then M is weakly supplement extending module if and only if M is strongly weakly supplement extending module.

Proof: Let N be a submodule of weakly supplement extending module M. Then there is a weakly supplement submodule K of M such that N is essential in K. By hypothesis, K is fully invariant submodule of M. So M is strongly weakly supplement extending. Converse, directly by (2.3).

Following [4], every weakly supplemented module is weakly supplement extending. This fact is not still true for strongly weakly supplement extending modules. Indeed, the concepts of weakly supplemented modules and strongly weakly supplement extending modules are different. For example, Z is strongly weakly supplement extending Z-module, but it is not weakly supplemented. Also, M=$Z_2 \oplus Z_2$ as Z-module is weakly supplemented which is not strongly weakly supplement extending module. For the relation between strongly weakly supplement extending and weakly supplemented.

Corollary (9):

Let M be a weakly supplement duo module. If M is weakly supplemented module then M is strongly weakly supplement extending.

Corollary (10):

Let M be a duo module. If M is weakly supplemented module then M is strongly weakly supplement extending.

The next result, we give a condition under which strongly weakly supplement extending module is weakly supplemented modules.

Proposition (11):

Let M be a module in which for each submodule N of M, there is a closed submodule L (depending on N) of M such that N=L+F or L=N+F for some F<<M. If M is strongly weakly supplement extending, then M is weakly supplemented.

Proof: Suppose that M is strongly weakly supplement extending and let N be a submodule of M. Then there is a closed submodule L in M such that N= L+F for some F<<M, since M is strongly weakly supplement extending. Then L is fully invariant weakly supplement submodule in M. Then M= L+H and L∩H<<M where H is a submodule in M, so we have M=L+H+F=N+H and N∩H ⊆ (L+F) ∩H<<M, then N∩H<<M. Hence M is weakly supplemented. Or, suppose that M is strongly weakly supplement extending and N be a submodule of M. There is a closed submodule L in M such that L=N+F for some F<<M, since M is strongly weakly supplement extending. Then there is a submodule K of M such that M =K+L and K∩L<<M, so M=K+L=K+N+F=K+N and K∩N ⊆ K∩L<<M. K∩N<<M. Hence M is weakly supplemented.

We asserted that, every strongly extending module is strongly weakly supplement extending, but the converse is not true. Here, we discuss the converse.

Recall that, the Jacobson Radical of a module M (denoted by J(M)) is the intersection of all maximal submodules of M [10].

Proposition (12):

Let M be a module with J(M)=0. Then M is strongly weakly supplement extending if and only if M is strongly extending.

Proposition (13):

Every direct summand of a strongly weakly supplement extending module is strongly weakly supplement extending.

Proof: Let A be a direct summand of strongly weakly supplement extending module M and let K be a closed submodule of A. Since A is a direct summand of M, then A is closed submodule of M and so K is closed submodule of M. Since M is a strongly weakly supplement extending then by proposition (2.2), K is fully invariant weakly supplement submodule of M, then M=K+H and K∩H << M, where H is a submodule of M so A∩M=A∩(K+H), then by (modular law) A=K+(A∩H) and K∩(A∩M)=(K∩H)∩A, we get K∩H << M. So we have K∩(A∩H) << M. Snce K∩(A∩H) ⊆ A ⊆ M, then K∩(A∩H) << A, hence K is a weakly supplement submodule of A. Now to prove K is fully invariant submodule in A. Let *f* :A⟶A be a homomorphism and the projection mapping π:M⟶A and the inclusion mapping i:A⟶M . Then $(if\pi)$(K) ⊆ K. So K is fully invariant submodule in A and hence A is strongly weakly supplement extending.

Proposition (14):

Let N be a submodule of a strongly weakly supplement extending module M such that the intersection of N with any fully invariant weakly supplement in M is fully invariant weakly supplement in N. Then N is strongly weakly supplement extending.

Proof: Let N be a submodule of a strongly weakly supplement extending module M and let A be a submodule of N. Then A is a submodule of M, since M is strongly weakly supplement extending then there exists a fully invariant weakly supplement submodule K in M such that A is essential in K, since A is a submodule of N and A is a submodule of K. Then A is a submodule of N∩K, so A is essential in N∩K. By hypothesis, N∩K is fully invariant weakly supplement submodule of N. Then N is a strongly weakly supplement extending.

A direct sum of strongly weakly supplement extending modules needs not to be strongly weakly supplement extending. For example, M=$Z_2 \oplus Z_2$ as Z-module is not strongly weakly supplement extending module while Z_2 is strongly weakly supplement extending Z-module.

Proposition (15):

Let M= $\oplus_{i\in I} M_i$ be a module, where M_i is submodule of M, $\forall\, i \in I,$ such that every closed submodule in M is fully invariant. If M_i is strongly weakly supplement extending module, then M is strongly weakly supplement extending.

Proof: Let N be a closed submodule in M. So by hypothesis, N is fully invariant submodule in M. Then N=⊕(N∩M_i) . So, we have (N∩M_i) is closed in N. But N is closed in M, so (N∩M_i) is closed in M. But (N∩M_i)⊆ M_i. Then (N∩M_i) is closed in M_i. Now, since M_i is strongly weakly supplement extending then (N∩M_i) is fully invariant weakly supplement submodule in M_i. Then N= (N∩M_i) is weakly supplement submodule in M=$\oplus_{i\in I} M_i$. Hence M is strongly weakly supplement extending.

The following proposition is a characterize-ation of strongly weakly supplement extending:

Proposition (16):

Let M= $\oplus_{i\in I} M_i$ be a module, where M_i is a submodules of M for each i ∈ I. Then the following statements are equivalent:

1. M is strongly weakly supplement extending.
2. Each M_i is strongly weakly supplement extending and every closed submodule of M is fully invariant weakly supplement.
3. Each M_i is weakly supplement extending and every closed submodule of M is fully invariant.

Proof: (1) ⟹ (2) Suppose that M is strongly weakly supplement extending. Then by proposition (2.13) M_i is strongly weakly supplement extending, and by proposition (2.2), we get that each closed submodule of M is fully invariant.

(2) ⟹ (3) Suppose that M_i is strongly weakly supplement extending. Then by (2.3, (1)), M_i is weakly supplement extending.

(3)⟹ (1) Let K be a closed submodule of M, then K is fully invariant of M= $\oplus_{i\in I} M_i$. Then K=$\oplus_{i\in I}$ (K∩M_i) . Now since K∩M_i is direct summand of K, then K∩M_i is closed in K. But K is closed in M, thus K∩M_i is closed in M.

Since K∩M_i ⊆ M_i ⊆ M, then K∩M_i is closed in M_i. By weakly supplement extending property of M_i. K∩M_i is weakly supplement submodule of M_i. Thus, K= ⊕ (K∩M_i) is weakly supplement submodule of M = $\oplus_{i\in I} M_i$. Then K is a weakly supplement submodule of M. Since K is closed in M then by hypothesis, K is fully invariant in M. Then by proposition (2.2), M is strongly weakly supplement extending.

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