Research Article Open Access

Hyperfactored of Reflection Arrangement $\mathcal{A}$ (G_{25})

Rabeaa G. A. Al-Aleyawee*, Raneen S. H. Al-Muhamadi
Department of Mathematics, College of Science, Mustansiriyah University, IRAQ
*Correspondent author email: dr.rabeea@uomustansiriyah.edu.iq

Article Info

Received
25/03/2019

Accepted
29/04/2019

Published
15/10/2019

Abstract
The purpose of this paper is to study the hyperfactored of the complex reflection arrangement (G_{25}). Depending on the lattice of arrangement (G_{25}), the basis of $\mathcal{A}(G_{25})$ has been found and then partitioned. Also, showed that (G_{25}) is not hyperfactored and is not inductively factored.

Keywords: Complex reflection arrangement, nice partition, Factored arrangement, Inductively Factored.

الخلاصة
الهدف من هذا البحث هو دراسة قابلية التحليل الفوقي للترتيبة الانعكاسية المركبة $\mathcal{A}(G_{25})$. بالاعتمادعلى الشبكية للترتيبة $\mathcal{A}(G_{25})$ وجد الاساس لهذه الترتيبة ومن ثم التجزئة. وكذلك برهنت بانها غير قابلة للتحليل الفوقي والتحليل الاستقرائي.

Introduction

In Al-Aleyawee [1] found the lattice of $\mathcal{A}(G_{25})$. In this paper the basis of (G_{25}) has been found by using program. Proved that the arrangement (G_{25}) is not factored depending on lattice, and proved that (G_{25}) is not inductively factored depending on triple arrangement. The exponent vector and partition of (G_{25}) have been computed.

Throughout this paper, V is a finite dimensional complex vector space over field K. A hyperplane H in V is an affine subspace of dimension$n-1$.

A hyperplane arrangement $\mathcal{A}$ = ($\mathcal{A}$, V) is a finite set of hyperplanes in V. The product Q($\mathcal{A}$) = $\prod_{H\in\mathcal{A}}\alpha_H$ (where α_H is a linear form and H = Ker(α_H) is called a defining polynomial of $\mathcal{A}$. We agree that Q($\emptyset_n$) =1 is the defining polynomial of $\emptyset_n$, where $\emptyset_n$ is empty 1-arrangement. A reflection on V is a linear transformation on V of finite order with exactly ℓ-1 eigenvalues equal to 1. A reflection group G on V is a finite group generated by reflection on V. The lattice of $\mathcal{A}$ denoted by $L_{\mathcal{A}}$= {∩ H |H∈ $\mathcal{A}$} with the order being reverse inclusion; that is, X≤ $Y \leftrightarrow Y \subseteq X$, for each, ∈ $L_{\mathcal{A}}$. A subarrangement of $\mathcal{A}$ is $\mathcal{A}_X$ = {H∈ $\mathcal{A}$| X ⊆H}. The restriction arrangement $\mathcal{A}^X$= {X∩ H: H∈ $\mathcal{A}-\mathcal{A}_X$ and X ∩ H≠ ∅ } is the arrangement within the vector space X. A triple of arrangements ($\mathcal{A}$,$\mathcal{A}'$, $\mathcal{A}''$); that is, H ∈ $\mathcal{A}$, $\mathcal{A}'$= $\mathcal{A}$ −{ H_0} and $\mathcal{A}''$= $\mathcal{A}^{H_0}$(where H_0 distinguished hyperplane). The rank function is a function rk: $L_{\mathcal{A}} \to Z_+$ defined by rk(X) = cod(X), ∀ X∈ $L_{\mathcal{A}}$. The symmetric algebra S = S (V^*) (where V^* the duel vector space of V), which is isomorphic to the polynomial algebra K[$x_1, x_2, \dots, x_n$]. For more details on hyperplane arrangement see[2].

1: Factored and inductively factored of (G_{25})

Definition (1.1):[2][4]

Let π = ($\pi_1, \dots, \pi_s$) be partition of $\mathcal{A}$. Then π is called independent, for any choice H_i ∈ π_i, 1≤ i ≤ s, rk ($H_1 \cap \dots \cap H_s$) =s.

Definition (1.2): [2]

Let π = ($\pi_1, \dots, \pi_s$) be a partition of $\mathcal{A}$ and let x∈ $L_{\mathcal{A}}$. The induced partition π_X of $\mathcal{A}_X$ is given by the non- empty block of the form $\pi_i \cap \mathcal{A}_X$.

Definition (1.3): [2][4]

The partition π of $\mathcal{A}$ is a nice arrangement if π is independent and for each $X \in L_{\mathcal{A}} \setminus \{V\}$, π_X admits a block which is a singleton.

Definition (1.4): [2]

Let $\{e_1, e_2, \ldots, e_n\} \subset$ V be the dual basis of $\{x_1, x_2, \ldots, x_n\}$. Then define $D_i = D_{e_i}$, $1 \leq i \leq n$, to be the derivation $\frac{\partial}{\partial x_i}$, $D_i(f) = \frac{\partial f}{\partial x_i}, f \in$ S. Notice that $\{D_1, D_2, \ldots, D_n\}$ is a basis for Der_K(S) over S.

Thus, any derivation θ of S over K is $\theta = f_1 D_1 + \cdots + f_n D_n$, where $f_1, \ldots, f_n \in$ S. Therefore, Der_K(S) is free S-module of rk n.

Definition (1.5): [2]

$0 \neq \emptyset \in Der_K$(S) is homogeneous of polynomial degree p if $\theta = \sum_{j=1}^{n} f_j D_j$ and $f_j \in S_p$ for $1 \leq j \leq n$, and defined by $p$degθ=p and tdegθ= $p$deg$\theta - 1$.

Definition (1.6): [2]

Let $\mathcal{A}$ be an arrangement with defining polynomial Q($\mathcal{A}$) =$\prod_{H \in \mathcal{A}} \alpha_H$,a sub module D_S ($\mathcal{A}$) of Der_K(S) is $D_S(\mathcal{A})$ = $\{\theta \in Der_K(S) | \theta(Q) \in QS\}$. $D_S(\mathcal{A})$ is called the module of $\mathcal{A}$-derivations.

Definition (1.7): [2]

The class IFAC of inductively factored is the smallest class of pairs ($\mathcal{A}$,π) of $\mathcal{A}$ together with a partition π subject to

1. $(\emptyset_n, (\emptyset)) \in$ IFAC,$\forall$ $n \geq 0$, (where $\emptyset_n$is empty n-arrangement).
2. If there exists a partition π of $\mathcal{A}$ and H_0 the restriction map $\sigma = \sigma_\pi$,
 H_0: $\mathcal{A} \backslash \pi_1 \rightarrow \mathcal{A}''$is injective and for the induced partition π' of $\mathcal{A}'$ and π'' of $\mathcal{A}''$ both $(\mathcal{A}', \pi')$ and $(\mathcal{A}'', \pi'') \in$ IFAC, then ($\mathcal{A}$,π).

Definition (1.8): [3]

A real arrangement $\mathcal{A}$ of hyperplane is said to be factored if there exists a partition $\pi = (\pi_1, \ldots, \pi_n)$ of $\mathcal{A}$ into n disjoint subsets such that Orlik-Solomon algebra of $\mathcal{A}$ factors according to this partition.

Theorem (1.1): [3]

If $\mathcal{A}$ is a nice partition, then an arrangement $\mathcal{A}$ is factored arrangement.

2:The Complex Reflection Arrangement of $\mathcal{A}(G_{25})$

The complex Reflection Group G_{25}[1]

Let V is a finite dimensional complex vector space the defining polynomial of $\mathcal{A}$ (G_{25}) is Q ($\mathcal{A}$ (G_{25})) =$xyz \prod_{o \leq i,j \leq 2} (x_i \mp x_j)(\beta x_i \mp x_j \mp x_k)$.

The hyperplane arrangement of G_{25} [1]

The hyperplane of (G_{25}) where H_i= Kerα_{H_i}, $1 \leq i \leq 12$ are:

Table 1: The hyperplanes of (G_{25}).

H_1: $x = 0$	H_7: x+ ω y + z=0
H_2: $y = 0$	H_8: x + ωy+ ωz=0
H_3: $z = 0$	H_9: x+ ωy+ω^2z=0
H_4: $x + y + z = 0$	H_{10}: x+ω^2y+z=0
H_5: x+ y+ ωz=0	H_{11}: x+ω^2y+ ωz=o
H_6: x + y+ω^2z=0	H_{12}: x+ω^2y+ω^2z=0

Using Program (1) below, that found:

$D_1(f) = \frac{\partial f}{\partial x}$, $D_2(f) = \frac{\partial f}{\partial y}$, $D_3(f) = \frac{\partial f}{\partial z}$

of $\mathcal{A}(G_{25})$ and found degree of $\mathcal{A}(G_{25})$ is {4,7,10}. Thus, the exponent vector of (G_{25}) is $\{5, 8, 11\}$ and the partition of this arrangement is $\pi = \{\pi_1, \pi_2, \pi_3\}$ where

$$\pi_1 = \{H_1, H_2, H_3, H_4, H_5\},$$
$$\pi_2 = \{H_6, H_7, H_8, H_9, H_{10}, H_{11} H_{12}, H_{13}\},$$
$$\pi_3 = \{H_{14}, H_{15}, H_{16}, H_{17}, H_{18}, H_{19}, H_{20}, H_{21}, H_{22}, H_{23}, H_{24}\}$$

The $\mathcal{A}_{X_i}$, for each $X_i \in$ rk 2 has been found.

Theorem (2.1):

i. The induced partition π_X of $\mathcal{A}$ (G_{25}) has no singleton.

ii. $\mathcal{A}$ (G_{25})) is not factored arrangement.

Proof:

i. By the intersection of the partitions $\pi_i, i = 1,2,3$, with arrangement of rk 2 in Table (2) the result is deduced.

ii. This part is direct result from Part i.

Table 2: $\mathcal{A}_{x_i}$, for each $x_i \in$ rk 2.

$\mathcal{A}_{X_1}=\{H_1,H_4,H_8,H_{12}\}$	$\mathcal{A}_{X_{12}}=\{H_2,H_3\}$
$\mathcal{A}_{X_2}=\{H_1,H_5,H_9,H_{10}\}$	$\mathcal{A}_{X_{13}}=\{H_4,H_9\}$
$\mathcal{A}_{X_3}=\{H_1,H_6,H_7,H_{11}\}$	$\mathcal{A}_{X_{14}}=\{H_4,H_{11}\}$
$\mathcal{A}_{X_4}=\{H_2,H_4,H_7,H_{10}\}$	$\mathcal{A}_{X_{15}}=\{H_5,H_7\}$
$\mathcal{A}_{X_5}=\{H_2,H_5,H_8,H_{11}\}$	$\mathcal{A}_{X_{16}}=\{H_5,H_{12}\}$
$\mathcal{A}_{X_6}=\{H_2,H_6,H_9,H_{12}\}$	$\mathcal{A}_{X_{17}}=\{H_6,H_8\}$
$\mathcal{A}_{X_7}=\{H_3,H_4,H_5,H_6\}$	$\mathcal{A}_{X_{18}}=\{H_6,H_{10}\}$
$\mathcal{A}_{X_8}=\{H_3,H_7,H_8,H_9\}$	$\mathcal{A}_{X_{19}}=\{H_7,H_{12}\}$
$\mathcal{A}_{X_9}=\{H_3,H_{10},H_{11},H_{12}\}$	$\mathcal{A}_{X_{20}}=\{H_8,H_{10}\}$
$\mathcal{A}_{X_{10}}=\{H_1,H_2\}$	$\mathcal{A}_{X_{21}}=\{H_9,H_{11}\}$
$\mathcal{A}_{X_{11}}=\{H_1,H_3\}$	

3. Inductively Factored of (G_{25})

Let $\pi=\{\pi_1,\pi_2,\pi_3\}$. Let H_1 distinguished hyperplane then
$\pi'(\mathcal{A}'(G_{25}))=\{H_2,H_3,H_4,H_5,H_6,H_7,H_8,H_9,H_{10},H_{11},H_{12}\}$,
$\pi''(\mathcal{A}''(G_{25}))=\{y_1,y_2,y_3,y_4,y_5\}$.
To show that δ: $\mathcal{A}'\backslash\pi'_1 \to\mathcal{A}''$ is injective. Let H_6 distinguished hyperplane then by Definition (2.7) δ is not injective since $\exists\ \alpha,\beta\in\mathcal{A}'\setminus\pi'_1$ such that $\delta(\alpha)=\delta(\beta)$ and $\alpha\neq\beta$. Thus, $\mathcal{A}$ (G_{25}) is not inductively factored.

Theorem (3.1)

Every factored arrangement is a nice partition.

Proof:

Suppose that $\mathcal{A}$ is factored arrangement. Then $\exists\ \pi=(\pi_1,\ldots,\pi_n)$ of $\mathcal{A}$ such that $\pi=\bigoplus\pi_i$, $i=1,\ldots,n$. Thus, π is independent. Without loss of generality let $\pi_1=\{H_i\}$, $i=1,\ldots,n$. Then $\pi_{\mathrm{x}}=\pi_1\cap\mathcal{A}_{\mathrm{x}_k}$ is singleton $\forall\ \mathcal{A}_{\mathrm{x}_k}\in L_{\mathcal{A}}$, where x_k of rank two. Therefore, By Definition (1.3) $\mathcal{A}$ is nice arrangement.

Program (1)

```
syms x1 x2 x3 B
h1=x1
h2=x2
h3=x3
h4=x1+x2
h5=x1+x3
h6=x2+x3
h7=x1-x2
h8=x1-x3
h9=x2-x3
h10=B*x1+x2+x3
h11=B*x1-x2+x3
h12=B*x1+x2-x3
h13=B*x1+-x2-x3
h14=B*x2+x1+x3
h15=B*x2-x1-x3
h16=B*x2-x1+x3
h17=B*x2+x1-x3
h18=B*x3+x1+x2
h19=B*x3-x1-x3
h20=B*x3-x1+x2
h21=B*x3+x1-x2
H=h1*h2*h3*h4*h5*h6*h7*h8*h9*h10*h11*h12*h13
*h14*h15*h16*h17*h18*h19*h20*h21
L1=diff(H,x1)
L2=diff(H,x2)
L3=diff(H,x3)
L1=simplify(L1)
L2=simplify(L2)
L3=simplify(L2)
```

References

[1] R. Al-Aleyawee, Hypersolvable and Free Arrangements. Ph.D. Thesis, College of science, Mustansiriyah University, (2005).
[2] P. Orlik and H. Terao, Arrangement of Hyperplanes. Grundlehren Math. Wiss. vol.300, Springer-Verlag Berlin, (1992).
[3] M. Jambu and L. Paris, Combinatorics of Inductively Factored Arrangements. Euroup. J. Combinatorics, 16, 267-292, (1995).
[4] T. Hoge and G. Rohle, Nice Reflection Arrangements. Math.GR, (2015).